An efficient numerical technique for the Solution of Singularly Perturbed Convection Diffusion Equation with Shift Operator

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by

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Abstract

The thesis presents a numerical approach to solving second-order ordinary differential equation boundary value problems with singularly perturbed convection diffusion, where a small parameter is multiplied by the largest derivative ϵ with Dirichlet's boundary conditions. In order to solve these differential equation we use the upwind finite difference method including uniform mesh and the piecewise-uniform mesh introduced by Ivanovich Shishkin. The convergence between the analytic solution and the solution obtained from the numerical approach of the simple Convection Diffusion Problem are provided. Also we analyze this problem with delay and advance parameters. This paper presents the numerical outcomes displayed as tables and graphs, showing that our suggested approach provides a very accurate approximation of the exact solution.

Keywords: singularly perturbed, convection diffusion equation, shishkin mesh, Dirichlet boundary condition, numerical scheme, delay and advance

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Chapter 1

Introduction

1.1 Perturbation Theory

Differential equations are frequently in mathematically modelling and illustrate various physical process in science and engineering. Considering a mathematical model that describes a physical system. In many cases, finding a exact analytical solution to these equations are not possible, especially when the equations involves small parameters or the complicated boundary conditions. Perturbation theory provides a methodology to approximate solution to these equation by breaking the problem into smaller parts. By introducing a small parameter that reflects a deviation from a known solution to a similar problem, which is typically referred to as unperturbed problem.

Perturbation parameter is the study of effect of small disturbances in mathematical model of physical process and these small parameters are known as perturbation parameter. Retaining the small parameters results in difficulties that we refer to as perturbed parameter, while simplifying degraded problems to become unperturbed or reduced problems.

The perturbation problems are categorized into two

1. **Regular perturbation problems**

   Perturbation problems, denoted as L_ϵ, are the differential equations in which ϵ is a small perturbation parameter multiplied by the highest-order derivative term.If L_ϵ has a solution that uniformly converges to the reduced problem L_0 (that is formed by arranging ϵ to zero in the perturbation problem) as ϵ approaches zero, then the perturbation problem L_ϵ is referred to as regularly perturbed.

2. **Singular perturbation problems**

When some or all of the terms involving the highest-order derivatives multiplied by a small parameter ϵ, the differential equation are known as a singularly perturbed differential equation (SPDE). The mathematical characteristics of the solution to an (SPDE) often reflect physical phenomena, including the presence of boundary layer functions.

$$L_\epsilon = -\epsilon u''(x) + b(x)u'(x) + c(x)u(x) = j(x) \text{for } x \in (0,1), \quad u(0) = u(1) = 0, \quad 0 < \epsilon \tag{1.1}$$

Convection-diffusion problems occur when c(x)= 0, while reaction-diffusion problems occur when $b = 0$ and $c \neq 0$. As ϵ approaches to zero, the solution of the (3) and it's derivative will approaches to a discontinuous limit.The characteristic that divides such problems is that the solution has distinct asymptotic expansions in distinct subdomains within the given domain. They represent layers in which there are sudden changes in the solution. It is possible to solve some second-order boundary value problems using the The finite-difference technique's classical convergence theory is based on the complimentary concepts of consistency and stability. This method involves difference quotient approximations for derivatives. Now, we consider a simple model problem of [11]convection diffusion type with dirichlet condition,

$$L_\epsilon = -\epsilon u''(x) + b(x)u'(x) = j(x) \tag{1.2}$$

where both $j(x)$ and $u(x)$ are the smooth functions and $b(x) \geq \beta > 0$.

When a boundary value problem is solved, it shows boundary layer behavior if the condition $b(x) + c(x) < 0$ is satisfied. For delay and advance problems, to understand the behavior of the solution in boundary layers, we first applythe uniform mesh [3]standard upwind finite difference operator. Since an ϵ-uniform mesh often performs poorly in the boundary layer region, we utilize a Shishkin mesh, introduced by Russian mathematician Grigorii Ivanovich Shishkin in 1988. This piecewise uniform mesh is particularly effective for studying boundary layer behavior. To simplify parameters related to delay and advance, which are of order $O(\epsilon)$, we employ Taylor series expansion.

In this study, we examine a model problem of the convection-diffusion type that includes delay parameters.

1.2 Delay and Advanced Differential Equation

[3]In science and engineering, numerous mathematical models (such those used in control theory, epidemiology, and laser optics) take into account a system's past as well as its present condition. Functional differential equations called delay differential equations (DDEs) characterise these models. Delays are frequently used in the biological sciences to take into consideration unobserved factors or procedures that result in time lags. DDEs are essential in various fields, providing realistic simulations of observed phenomena.

Initially, DDEs were used in technical fields like control circuits, where delays were measurable physical quantities. Today, they are widely used in biosciences and control theory, including ecology, epidemiology, and neural networks. Hutchinson was one of the pioneers in using delays in biological models, modifying the Verhulst model to include time delays in biological processes.

DDE models are preferred over ordinary differential equations (ODEs) because they better reflect the nature of underlying processes and provide richer mathematical frameworks. Unlike ODEs, which assume instantaneous reactions to current conditions, DDEs account for after-effects. Ignoring delays in favor of ODEs can lead to significant inaccuracies, as small delays can have substantial impacts on system dynamics. This is particularly evident in applications like the chemostat, where using ODEs implies instantaneous changes, ignoring the inherent delays in nutrient supply and microbial growth.

1.3 Numerical Approach

1.3.1 Numerical Approach for Singularly Perturbation Problems

The solutions to singularly perturbed differential equation's often display boundary layer behavior, a concept from physics describing regions where the solution changes rapidly. Outside these regions, the solution changes slowly, highlighting a "two-time-scale" property. This dual behavior makes approximating solutions challenging due to the existence of both "slow" and "fast" incidents, which renders the problem stiff. To tackle these problems, methods are categorized into numerical and asymptotic approaches. The numerical approach provides detailed quantitative information about specific problems, while the asymptotic approach offers qualitative insights and semi-quantitative data for a family of problems.

The numerical solution of differential equations with singular perturbations has seen significant research progress, summarized in two notable survey papers. Kadalbajoo and Reddy reviewed numerical methods from 1968 to 1984, and Kadalbajoo and Patidar extended this review up to 1999. Since then, several studies have contributed to this field.

For instance, Farrell et al. explored fitted finite difference methods on uniform meshes for semilinear boundary value problems[2], proving the limitations of such schemes for ϵ-uniform convergence. Other researchers have developed various schemes to address different aspects of singularly perturbed problems. Methods like nonstandard upwinded difference schemes, finite element discretization, and spline approaches have been proposed and analyzed for their effectiveness and convergence properties.

Numerical techniques have been adapted to specific types of boundary value problems, such as Reaction -Diffusion and Convection -Diffusion problems[2]. Researchers like Kopteva, Stynes, and others have focused on methods that ensure uniform convergence with relation to the perturbation parameter, often using specialized meshes like Shishkin-type meshes.

In conclusion, the field has advanced through a combination of theoretical analysis and practical numerical experiments, providing robust techniques to solve differential equations that are singularly disturbed with high accuracy and convergence properties.

1.3.2 Numerical Approach for Singularly Perturbation Differential equations With both Delay and Advance

This thesis delves into a specific category of differential equation problems where the primary focus lies on equations where a small parameter scales the highest order derivative, denoted as ϵ. While there'sa rising interest in the numerical investigation of these issues because of its applicability in many other domains, including optimal control theory and neurobiology, and physiological modeling, literature on singularly perturbed differential difference equations is limited.

Previous studies on these equations mainly emphasized the existence and uniqueness of problem to the solutions, neglecting the development of approximate solutions. [4]Boundary Value Problems (BVPs) to singularly perturbed differential difference equations were first explored in 1982 by Lange and Miura, who employed asymptotic techniques to approximate solutions[4]. Their work primarily focused on linear second-order equations with shifts of both positive and negative types, discussing phenomena like turning points, resonance, and boundary layers.

In 1992, Voulov et al. studied the asymptotic stability of homogeneous systems with unbounded delay under singular perturbation. Further research by Lange and Miura examined BVPs for linear second-order equations with fixed-type shifts, extending techniques from classical perturbed ordinary differential equations.

In 2002, a numerical investigation of Problems with boundary values for differential equations of second order with small changes and unique perturbations was initiated. This study focused on boundary layer behavior and employed finite difference methods for numerical solutions, assessing stability and convergence. Numerical experiments were conducted to illustrate the influence of shifts on boundary layer behavior.

Chapter 2

Problem Description

2.1 Description of Problem

Let's consider a non-homogeneous boundary value problem involving convection-diffusion, with minor delays and advances.

$$\epsilon u''(x) + b(x)u'(x-\delta) + c(x)u(x+\eta) = j(x), \tag{2.1}$$

on $\sigma = (0, 1)$ to boundary conditions,

$$u(x) = b(x) \quad on \quad -\delta \leq x \leq 0 \tag{2.2}$$

$$u(x) = \phi(x) \quad on \quad 1 \leq x \leq (1+\eta) \tag{2.3}$$

Consider the convection-diffusion equation with a small perturbation parameter $0 < \epsilon \ll 1$, with delay and advance parameters δ and η, $b(x)$ and $f(x)$ are presumed to be sufficiently smooth functions. For the function $u(x)$ to be smooth, it must satisfy continuity on the interval $\overline{\sigma} = [0, 1]$ to be differentiably continuous on $\sigma = (0, 1)$. The behavior of the solution to exhibits layer behavior depending upon the sign of $b(x) + c(x) + d(x)$, where $c(x)$ and $d(x)$ are additional terms.

Let's now simplify by considering a basic Convection-Diffusion Problem[2].

$$L_\epsilon = -\epsilon u\text{"}(x) + b(x)u'(x) = j(x) \tag{2.4}$$

Consider the convection-diffusion equation with a perturbation parameter which is small $0 < \epsilon \ll 1$ and $b(x) \geq \beta \geq 0$. Again by taking $b(x)$ and $f(x)$ which are assumed to be sufficiently smooth functions. For the function $u(x)$ to be smooth, it must satisfy continuity on the interval $\overline{\sigma} = [0, 1]$ and be continuously differentiable on $\sigma = (0, 1)$. The solution exhibits boundary layer behavior if the condition $b(x) + c(x) + d(x) < 0$ is satisfied.

To analyze the equation's behavior in boundary layers, both standard upwind finite difference operator and ϵ-uniform mesh are initially employed. However, the ϵ-uniform mesh typically performs poorly in boundary layer regions. Therefore, the Shishkin mesh, introduced by Grigorii Ivanovich Shishkin in 1988, is used. This piecewise uniform mesh helps examine boundary layer behavior effectively. To expedite the analysis for parameters of $O(\epsilon)$, Taylor series expansion is utilized. Analytical results will investigate the stability, bounds, and convergence of the problem when small delays and advances are applied.

2.2 Analytical Results

Let's consider a model problem of convection-diffusion type that includes both delay as well as advance parameters. We use the Taylor's series expansion to analyse the boundary value issue (2) since it is expected to have a sufficiently differentiable solution.

$$u'(x-\delta) \approx u'(x) - \delta u''(x) \tag{2.5}$$

by placing equation (6) in (1) and (2), we obtain,

$$L_\epsilon = (\epsilon - \delta b(x))u''(x) + b(x)u'(x) = j(x) \tag{2.6}$$

$$u(0) = \phi_0, \quad \phi_0 = \phi(0), \quad u(1) = \alpha \tag{2.7}$$

The differential operator for problems (2.6) (2.7) above is denoted by L_ϵ. It is defined for any function $\Pi \in C^2([0,1])$as

$$L_\epsilon \Pi(x) = -(\epsilon - \delta b(x))\Pi''(x) + b(x)\Pi'(x) \tag{2.8}$$

Minimum Principle. If $\Pi(0) \geq 0$ and $\Pi(1) \geq 0$, then $L_\epsilon \Pi(x) \geq 0$ ∀ x $\in \overline{\sigma} = [0,1]$,in which Π is a smooth function.

Proof

Given that $x^* \in [0,1], Let$be such that $\pi(x^*) = min_{x\in[0,1]}\psi$, and let $\Pi(x^*) < 0$ be taken into consideration. Given that $x^* \notin \{0,1\}$, $\pi'(x^*) = 0 and \pi"(x^*) \geq 0$ have been established.

Now consider

$$L_\epsilon \pi(x) = -(\epsilon - \delta b(x))\pi''(x) + b(x)\Pi'(x) > 0, \tag{2.9}$$

This leads to an contradiction. Therefore, that follows $\pi''(x^*) \geq 0$ and $\pi(x) \geq 0$ for all $x \in \overline{\sigma}$.

Lemma-1

Assume $u(x)$ denote the solution to the problem (2.6), (2.7).

$$\|u\| \leq \theta^{-1}\|f\| + \max(|\phi_0|, |\gamma|)$$

Proof: Let us define the two Barrier Functions $\Pi^{\pm}$, now

$$\Pi^{\pm}(x) = \omega^{-1}\|\mathbf{j}\| + \max(|\phi_0|, |\alpha|) \pm u(x) \tag{2.10}$$

for $(x = 0)$ and since $u(0) = \phi_0$,We observe:

$$\Pi^{\pm}(0) = \omega^{-1}\|\mathbf{j}\| + \max(|\phi_0|, |\alpha|) \pm u(0) \tag{2.11}$$
$$= \omega^{-1}\|\mathbf{j}\| + \max(|\phi_0|, |\alpha|) \pm \phi_0 \tag{2.12}$$
$$\geq 0 \tag{2.13}$$

Now, for $x = 1 and since u(1) = \alpha$

$$\Pi^{\pm}(1) = \omega^{-1}\|\mathbf{j}\| + \max(|\phi_0|, |\alpha|) \pm u(1) = \omega^{-1}\|\mathbf{j}\| + \max(|\phi_0|, |\alpha|) \pm \alpha \geq 0$$

and we get
[3]

$$L_\varepsilon \Pi^{\pm}(x) = -(\varepsilon - \delta b(x))\,(\Pi^{\pm}(x))'' + b(x)\,(\Pi^{\pm}(x))'$$

$$= 0(\omega^{-1}\|\mathbf{j}\| + \max(|\phi_0|, |\alpha|)) \pm L_\epsilon u(x)$$

$$= 0(\omega^{-1}\|\mathbf{j}\| + \max(|\phi_0|, |\alpha|)) \pm j(x)$$

Since $-\omega \geq 0$, we can conclude that $0(-\omega) \leq -1$. Utilizing this in the previous inequality, we deduce that $L_\epsilon \pi^{\pm} \leq 0$ for all $x \in \sigma$, given $|\mathbf{j}| \geq j(x)$. Thus, from the definition of minimum principle, we terminate that $\omega^{\pm} \geq 0$ for all $x \in \overline{\sigma}$, providing the necessary evaluation.

Theorem 2.1 [3]Assume $u(x)$ to be the solution of the problem(2.6) and (2.7). Then, for $k = 1, 2, 3$, we have:

$$\|\mathbf{u^k}\| \leq P(\epsilon - \delta\beta)^{-k}$$

Proof
Create a neighbourhood $N_x = (p, p + (\epsilon - \delta\|\mathbf{b}\|))$ for each $x \in \sigma$. p to be a positive constant that is

selected in $x \in N_x$ and $N_x \subset \sigma$. Then, a point $y \in N_x$ exists such that, according to the mean value theorem:

$$u'(y) = \frac{u(p + (\epsilon - \delta\beta)) - u(p)}{(\epsilon - \delta\beta)}$$

and so

$$|(\epsilon - \delta\beta)u'(y)| \leq 2\|\mathbf{u}|$$

Integrating the differential equation problem (2.6) we get
$(\epsilon - \delta\beta)u'(x) - (\epsilon - \delta\beta)u'(y) = \int_y^x (j - b(t))u'(t)\,dt$
[10]Utilizing the fact that the maximum norm of a function is always greater than or equal to the value of the function over the domain of consideration, and taking the modulus on both sides, we obtain:

$$(\epsilon - \delta\beta)|u'(x)| \leq (\epsilon - \delta\beta)|u'(y)| + \|\mathbf{j}\||x - y| + \int_y^x |b(t)u'(t)|\,dt$$

here,
$\int_y^x b(t)u'(t)\,dt = b(t)u(t)\Big|_y^x - \int_y^x b'(t)u(t)\,dt$
By using the fact that the maximum norm and taking modulus on both sides, we obtain [3]:

$$\int_y^x |b(t)u'(t)|\,dt \leq (2\|\mathbf{b}\| + \|\mathbf{b}'\|)\|\mathbf{u}\|$$

Using the inequalities $0 \leq |x - y| \leq 1$ and Lemma 1 In the inequality above, for the bound on u, we obtain:

$$|u'(x)| \leq P(\varepsilon - \delta\beta)^{-1}$$

which implies:

$$\|u'\| \leq P(\varepsilon - \delta\beta)^{-1}$$

where $P = \|j\| + (2 + 2\|\beta\| + \|\beta'\|)\left(\sigma^{-1}\|j\| + \max\left(|\phi_0|, |\alpha|\right)\right)$ is not influenced by ε. The bounds on u'' and u''' may also be readily found [10]using the differential equation and the bounds on u' and u'.

Precise details regarding the behaviour of the specific solution to the problem are required to illustrate the ϵ-uniformity with relation to the numerical technique. This is achieved by breaking down the solution u into two parts, the singular component z and the smooth component w:

$$u = w + z$$

in which v, the smooth component, can be expressed as:

$$w(x) = w_0(x) + (\epsilon - \delta\beta)w_1(x) + (\epsilon - \delta\beta)^2 w_2(x)$$

and the following problems' solutions are what these smooth components, w_0, w_1, and w_2, are defined as, respectively:

$$b(x)(w_0)'(x) + c(x)w_0(x) = j(x), \quad x \in \sigma, \quad w_0(1) = u(1)$$

$$b(x)(w_1)'(x) = -\frac{(\epsilon - \delta b(x))(w_0)''(x)}{(\epsilon - \delta\beta)}, \quad x \in \sigma, \quad w_1(1) = 0$$

$$L_\epsilon w_2(x) = -\frac{(\epsilon - \delta b(x))(w_1)''(x)}{(\epsilon - \delta\beta)}, \quad x \in \sigma, \quad w_2(0) = 0, \quad w_2(1) = 0$$

The component $w(x)$ which are smooth to be obtain solution of:

$$L_\epsilon = j(x), \quad x \in \sigma, \quad w(0) = w_0(0) + (\epsilon - \delta\beta)w_1(0), \quad w(1) = u(1)$$

As so, [1] $z(x)$ singular component represents the homogenous problem's solution:

$$L_\epsilon z(x) = 0, \quad x \in \sigma, \quad z(0) = u(0) - w(0), \quad z(1) = 0$$

Lemma 2: Bounds on w_0 and its derivatives for $0 \le k \le 3$ satisfy the following conditions:

$$\left\| w_0^{(k)} \right\| \le P$$

Proof The problem from the Theorem 2.1 that we have used

$$b(x)(w_0)'(x) + c(x)w_0(x) = j(x), \quad x \in \sigma, \quad w_0(1) = u(1)$$

[6]which can be written as

$$w_0'(t) + q(t)w_0(t) = j(t)/b(t), \quad w_0(1) = \alpha \tag{2.14}$$

with $q(t) = c(t)/b(t)$.

The equation (2.14) is a 1st-order linear differential equation in (w_0). To solve this problem, we multiply Eq.(2.14) by (exp($\int q(t), dt$), which, upon simplification, gives[6]:

$$\left(\exp\left(\int q(t)dt\right) w_0\right)' = j(t)\exp\left(\int q(t)dt\right)/b(t) \tag{2.15}$$

Now integrating the above equation from x to 1 for some x $\in (0,1)$, we get

$$(w_0(t)\exp(\int q(t)dt))\Big|_{t=x}^{1} = \int_x 1[j(t)\exp(\int q(t)dt)/b(t)]dt, \tag{2.16}$$

Let R= $\exp(\int q(t)dt)\Big|_{t=1}$ and T(x)= $\exp(\int q(t)d(t))\Big|_{t=x}$, then on simplification equation(15) reduces to

$$w_0(x) = \alpha R/T(x) - \int_x^1 [j(t)\exp(\int q(t)d(t))/b(t)]/T(x), \tag{2.17}$$

For each $t \in [0,1]$, $q(t) = c(t)/a(t) < 0$ and $j(t)$ are bounded since $b(t)$ is smooth, that is, bounded for all $t \in [0,1]$. Furthermore, for each $t \in [0,1]$, we have $b(t) \geq \beta > 0$ and $b(t) \neq 0$ respectively. By combining these facts, we can determine that the second term the right-hand side and the terms R and $T(x)$ are bounded, meaning that it is necessary for w_0 to be bounded.

Using equation (2.14), we now have: $j(x)/b(x) - q(x)v_0(x) = (w_0)'(x)$ Since w_0 is bounded, its derivative $(w_0)'$ must likewise be bounded. The bounds for these two can be obtained by differentiating equation (2.14) and applying the boundedness of w_0 and $(w_0)'$. We have for $0 \leq k \leq 3$:

$$\|(\mathbf{w_0})^{(\mathbf{k})}\| \leq C \tag{2.18}$$

Theorem- 2.2 [3]Suppose $u(x)$ be the solution to the equations (2.6) and (2.7), and let $u = v + z$ for $0 \leq k \leq 3$. For very small ϵ, w and z and their derivatives need to satisfy the following requirements:

$$\|(\mathbf{w})^{(\mathbf{k})}\| \leq P(\epsilon - \delta\beta)^{2-k} \tag{2.19}$$

$$\|\mathbf{f(x)}\| \leq \mathbf{P}\exp\left(-\beta\mathbf{x}/(\epsilon - \delta\beta)\right), x \in \bar{\sigma} \tag{2.20}$$

$$\|\mathbf{g(x)}\| \leq C(\epsilon - \delta\beta)^{-k}\exp\left(-\frac{\beta x}{\epsilon - \delta\beta}\right), \quad x \in \bar{\sigma} \tag{2.21}$$

Proof As per equation (2.18), it's clear that w_1 is the first-order linear differential equation's solution, and all the terms on the right-hand side are bounded over the interval $[0,1]$. Consequently, the entire right-hand side term is bounded. Using Lemma 2.2, we can infer:

$$\|\mathbf{w_1}\| \leq C\ ,$$

[10]Where P is a constant. Similarly, using the bound on v_1 and equation (2.18), we obtain $\|(\mathbf{w_1})'\| \leq P$. After differentiating (18) and using w_1 and $(w_1)'$, we can easily derive the bounds on $(w_1)''$ and $(w_1)'''$.

[10]The terms $(w_1)''$ and $(\epsilon - \delta b(x))/(\epsilon - \delta\beta)$ are constrained by a constant separate from ϵ, hence the term on the right side of equation (18) is bounded by ϵ. Thus, w_2 is the solution of the problem (2.6) with similar boundary value conditions. Hence, by Theorem 2.1, we have for $0 \leq k \leq 3$:

$$\|(\mathbf{w_2})^{\mathbf{k}}\| \leq P(\epsilon - \delta\beta)^{-k} \tag{2.22}$$

This provides the necessary estimate for the regular component w. Now, we introduce two barrier functions $\Pi^{\pm}$ to determine the required bounds on the singular component z and its derivative. These barrier functions are defined by:

$$\Pi^{\pm}(x) = (|\mathbf{u(0)}| + |(\mathbf{w_0})|)\exp(-\mathbf{x}\beta/(\epsilon - \delta\beta)) \pm \mathbf{z(x)} \tag{2.23}$$

Then we have

$$\Pi^{\pm}(x) = (|u(0)| + |w_0|)\exp(-x\beta/(\epsilon - \delta\beta)) \pm z(x) \tag{2.24}$$

then we have $\Pi \pm (0) = |u(0)| + |w_0(0)| \pm (u(0) - w_0(0)) \geq 0$
since $z(0) = (u(0) - w_0(0))$ and

$$\Pi \pm (1) = (|u(0)| + |w_0(0)|)\exp(-\beta(\epsilon - \delta\beta)^{-1}) \geq 0$$

since z(1) =0
and

$$\begin{aligned} L_\epsilon \Pi^{\pm}(x) &= (\epsilon - \delta b(x))\Pi^{\pm}(x) + c(x)\Pi^{\pm}(x) \\ &= (|u(0)| + |v_0(0)|)[\beta^2(\epsilon - \delta\beta)^{-1} - b(x)\beta(\epsilon - \delta\beta)^{-1} + c(x)].\exp(-x\beta(\epsilon - \delta\beta)^{-1}) \\ &\pm L_\epsilon z(x) \\ &\leq 0 \end{aligned}$$

Therefore by minimum principle, we have

$$\Pi^{\pm}(x) = (|y(0)| + |w_0|)\exp(-x\beta/(\epsilon - \delta\beta)) \pm z(x) \geq 0, x \in \bar{\sigma}$$

which, when simplified, yields

$$|z(x)| \leq C\exp(-x\beta(\epsilon - \beta)^{-1}), x \in \bar{\sigma} \tag{2.25}$$

where $C = (|u(0)| + |v_0(0)|$. Let's find out the bounds on the derivative of the singular component z of [1]solution u, using the technique of Theorem 2.1 construct a neighbourhood $N_x = (p, p + (\epsilon - \delta\beta))$, for some x $\in \bar{\sigma}$ where p is constant and x $\in N_x \subset \sigma$, so by mean value theorem, there exists a point y$\in N_x$ such that

$$z'(y) = \frac{z(c + (\epsilon - \delta\beta)) - z(c)}{(\epsilon - \delta\beta)},$$

which implies that,

$$|(\epsilon - \delta\beta)z'(y)| \leq 2\|\mathbf{z}\|. \tag{2.26}$$

[1]Now we have,

$$\int_y^x z"(t)dt = z'(x) - z'(y),$$

i.e,

$$z'(x) = z'(y) + \int_y^x z"(t)dt$$

Using $L_\epsilon z(x) = 0, x \in \sigma, z(0) = u(0) - w(0), z(1) = 0$
from Theorem 2.1 for z"(t) in above equation, we obtain

$$\begin{aligned} |z'(x)| &\leq |z'(y)| + |\int_y^x (\epsilon - \delta b(t))^{-1}(b(t)z'(t) + b(t)z(t))dt| \\ &\leq |z'(y)| + (\epsilon - \delta\beta)^{-1}(|\int_y^x b(t)z'(t)dt| + \int_y^x |b(t)z(t)|dt) \\ &\leq (\epsilon - \delta\beta)^{-1}(|\int_y^x b(t)z'(t)dt| + \|\mathbf{b}\|\|\mathbf{z}\||x-y|). \end{aligned}$$

[10]Maximum norm of a function is always greater than the value of the function over the domain of consideration and the inequality $0 < |x - y| \leq 1$ followed by

$$|\int_y^x b(t)z'(t)dt| \leq (2\|\mathbf{b}\| + \|\mathbf{b}'\|)\|\mathbf{z}\| \tag{2.27}$$

using these inequalities we get,

$$|z'(x)| \leq P(\epsilon - \delta\beta)^{-1}\|\mathbf{z}$$

For x $\in N_x$,

$$\begin{aligned} \|\mathbf{z}\| &= \sup_{x \in N_x} |z(x)| \\ &\leq P\exp(-\beta/(\epsilon - \delta\beta)) \end{aligned}$$

Using this we obtian

$$|z'(x)| \leq P(\epsilon - \delta\beta)^{-1}\exp(-\beta/(\epsilon - \delta\beta))$$

which gives the required result. It is simple to determine the estimate for $z"$ using the differential equation and the z and z' bounds.

Chapter 3

The Difference Scheme

3.1 Standard Finite Difference Scheme

The scenarios where identifying an analytical solution of a problem proves challenging, the finite difference method serves as a numerical scheme to approximate the solution of the differential equation. In this method, finite difference formulas are substituted for the derivative terms in the differential equation, providing approximations. This leads to an algebraic equation system derived from the differential equation using these approximated finite difference formulas. This system of algebraic equations can be represented as $AU = B$, where U is the set of solutions to the equations and A is a tridiagonal matrix.

When shifting from a differential operator to a difference operator, the error that occurs is determined by the difference between the numerical solution and the exact solution. The error, often referred to as "truncation error" or "discretization error," arises from the use ofa Taylor series' finite portion for approximation.

In equations (2.6) and (2.7) for discrete optimization, by defining a uniform mesh of size $h = \frac{1}{N}$, and by replacacing u'' and u' with central and forward difference approximations. Here, $x_i = \frac{i-1}{h}$ represents the values of mesh points, where $i = 1, 2, \ldots, N + 1$.

$$L_1^N u_i = g(x_i)$$

$$u_0 = \phi(0) u_N = \alpha(1)$$

$$u_0 = \phi(0) u_N = \alpha(1)$$

[9]The operator is defined as:

$$L_1^N = (\epsilon - \delta b(x_i)) D_+ D_- u_i + b(x_i) D_+ u_i = g(x_i)$$

where:

$$D_+D_-u_i = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}$$

$$D_+u_i = \frac{u_{i+1} - u_i}{h}$$

$$D_-u_i = \frac{u_i - u_{i-1}}{h}$$

[5]

Upon simplification, this yields a three-point difference scheme.

$$L_n^N u_i = -E_i u_(i-1) + F_i u_i + G_i u_(i+1) = H_i \tag{3.1}$$

where

$$\begin{aligned} E_i &= (\epsilon - \delta b(x)) \\ F_i &= -2\epsilon + b(x)(2\delta - h) \\ G_i &= -\epsilon - b(x)(h - \delta) \\ H_i &= h^2 J(x_i), \quad i = 1, 2, ...N + 1. \end{aligned}$$

The system of equations represented by (E_i, F_i, G_i, H_i) will constitute a system of tridigonal matrix $N+1$ equations with $N+1$ unknowns $u_0, u_1, \ldots, u_N$.

3.1.1 Finite Difference Discretization

Let us examine a convection-diffusion situation that is singularly perturbed.

$$L_\epsilon = -\epsilon u"(x) + b(x)u(x) = j(x), \quad for \quad x \in (0, 1) \tag{3.2}$$

with given boundary conditions

$$u(0) = 0, \quad u(1) = 0$$

in which $b(x) > 0 and \epsilon << 1$ Suppose $b(x)$ and $f(x)$ are divided into n subintervals using an equidistant mesh $x_i = a + (i-1) \cdot h$ for $i = 0, 1, 2, \ldots, N+1$ where $h = 1/N$. We employ central difference formulas to approximate the solution at equidistant points.

$$u"_(x) = \frac{u_(i-1) - 2u_i + u_(i+1)}{h^2}, \quad u'(x) = \frac{u_(i+1) - u_(i-1)}{2h}$$

For $i = 1, 2, ...N$ by using these central difference approximations, we get

$$u_(i-1)(-\epsilon - hb(x)) + u_i(2\epsilon) + u_(i+1)(-\epsilon + hb(x)) = h^2 j(x)$$

and further we can express it in the form of tridiagonal matrix of $N + 1 * N + 1$ in $AU = D$ form

$$\text{A} = \begin{bmatrix} 1 & 0 & & & \\ -\epsilon - hb(x) & 2\epsilon & -\epsilon + hb(x) & & \\ 0 & -\epsilon - hb(x) & 2\epsilon & -\epsilon + hb(x) & \\ & & & & \\ & & \epsilon - hb(x) & \ddots & \ddots \\ & & & & \\ & & -\epsilon - hb(x) & 2\epsilon & -\epsilon + hb(x) \\ & & & 0 & 1 \end{bmatrix}, U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n+1} \end{bmatrix}$$

$$and$$

$$\text{D=} \begin{bmatrix} h^2 j(x_1) \\ h^2 j(x_2) \\ \vdots \\ h^2 j(x_(n+1) \end{bmatrix}$$

3.1.2 Upwind Finite Difference Method

The Upwind Finite Difference Method is commonly employed to mitigate unnecessary oscillations in the solutions obtained. Oscillations occur when a one-sided difference is taken on the side that is not part of the layer, i.e.,

$$u'(x) = \frac{u_i - u_(i-1)}{h}$$

In this paper, the difference approach will be utilized to further estimate this solution. The following graphs depict the disparities between the standard finite difference approach using the upwind scheme and the standard finite difference method employing forward difference approximation with a uniform mesh.

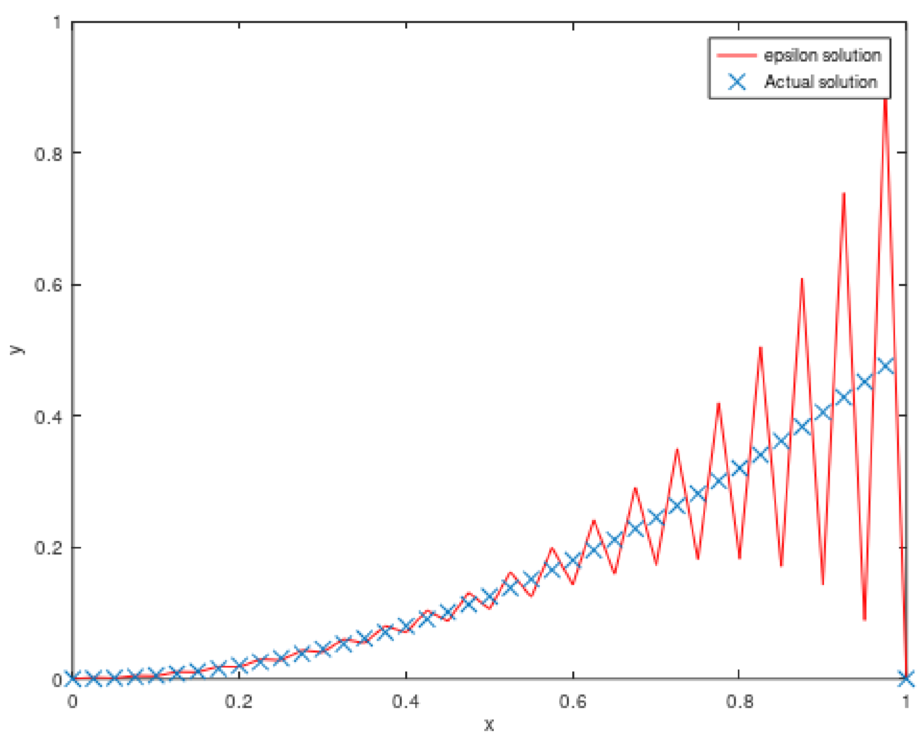


Figure 3.1: Finite Difference Method with Forward Difference

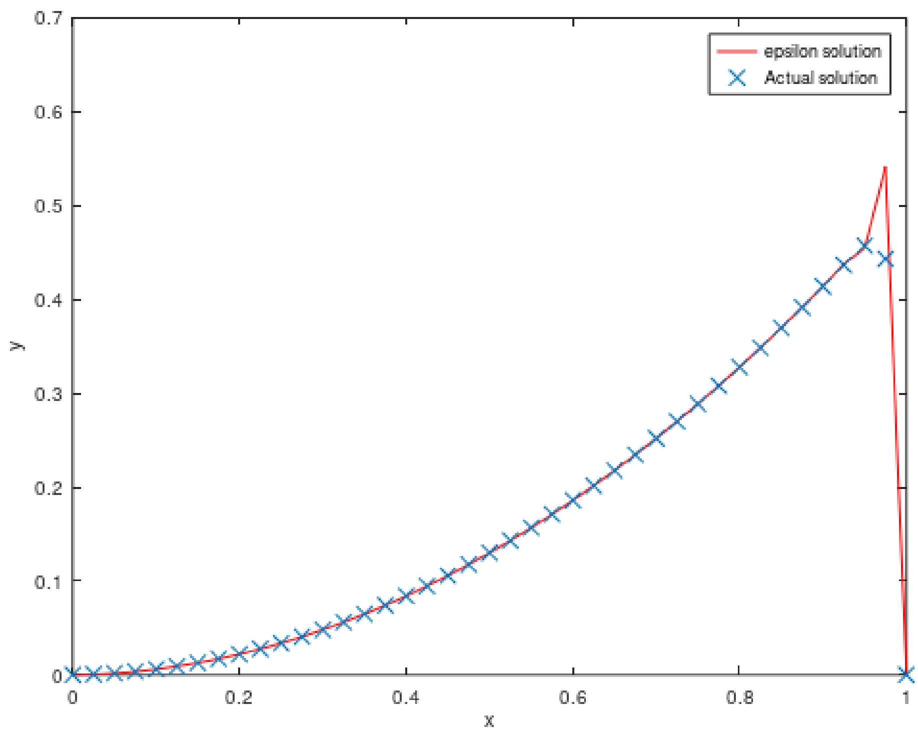


Figure 3.2: Finite Difference Method with upwind Scheme

3.1.3 Stability Convergence Analysis

Theorem 3.1 Based on the presumptions that $(\epsilon - \delta b(x)) > 0$, $b(x) \geq \beta > 0$, and $c(x) \leq -\gamma < 0$ for all $x \in [0,1]$, the system of difference equations (3.1), together with the designated boundary conditions, exists, is unique, and meets the requirements[9].

$$\|\mathbf{u}\|_{h,\infty} \leq C^{-1}\|\mathbf{j}\|_{h,\infty} + (\|\phi\|_{h,\infty} + |\alpha|), \tag{3.3}$$

taking $C = \beta$ or $\|\mathbf{b}\|_{h,\infty}$. Here $\|.\|_{h,\infty}$ is the discrete l_∞-norm, further given by

$$\|\mathbf{x}\|_{h,\infty} = \max_{0\leq i\leq N} |x_i|$$

Proof
Let y_i be any mesh function that satisfy the following:

$$(L_1)^N y_i = j_i$$

Combining this with equation (3.1) and on rearranging gives:

$$F_i y_i = -f_i + E_i y_{i-1} + G_i y_{i+1}$$

By using the non-negativity of the coefficients (E_i, F_i, G_i), and taking the modulus on both sides, we arrive at:

$$F_i|y_i| \leq |f_i| + E_i|y_{i-1}| + G_i|y_{i+1}|$$

Substituting the values of E_i, F_i, and G_i into the above defined inequality, we get:

$$(-2\epsilon + b(x)(2\delta - h))|y_i| \leq |j_i|(\epsilon - \delta b(x))|y_{i-1}| - (\epsilon - b(x)(h - \delta))|y_{i+1}|$$

for $i = 1, 2, \ldots, N-1$. Rearranging terms in the inequality gives:

$$\frac{(\epsilon - \delta b(x))}{h^2}(|y_{i-1}| - 2|y_i| + |y_{i+1}|) + \frac{b(x)}{h}(|y_{i+1}| - |y_i|) + |j_i| \geq 0$$

Now, to replace the coefficients $(\epsilon - \delta b_i)$ and b_i with constants, we consider the signs of the expressions $(|y_{i-1}| - 2|y_i| + |y_{i+1}|)$ and $(|y_{i+1}| - |y_i|)$. If $(|y_{i-1}| - 2|y_i| + |y_{i+1}|) \geq 0$, we use the inequality $0 < (\epsilon - \delta b_i) \leq (\epsilon - \delta\beta)$, and if $(|y_{i-1}| - 2|y_i| + |y_{i+1}|) < 0$, we use the inequality $(\epsilon - \delta b_i) \geq (\epsilon - \delta\|b\|_{h,\infty}) > 0$. Similarly, if $(|y_{i+1}| - |y_i|) \geq 0$, we use the inequality $0 < b_i \leq \|b\|_{h,\infty}$, and if $(|y_{i+1}| - |y_i|) < 0$,Using the inequality $b_i \geq \beta > 0$.

Thus, using these facts in the above inequality, gives:

$$C_1(|y_{i-1}| - 2|y_i| + |y_{i+1}|)/h^2 + C_2(|y_{i+1}| - |y_i|)/h + |j_i| \geq 0$$

where C_1 and C_2 are positive constants, $C_1 = (\epsilon - \delta b(x))$ or $(\epsilon - \delta\beta)$, and $C_2 = b(x)$ or β depending on the signs of the expressions $(|y_{i-1}| - 2|y_i| + |y_{i+1}|)$ and $(|y_{i+1}| - |y_i|)$, respectively.

On rearranging the terms in the inequality and by using this inequality $|j_i| \leq \|j\|_{h,\infty}$ yields:

$$C_1(|y_{i+1}| - |y_i|)/h^2 - C_1(|y_i| - |y_{i-1}|)/h^2 + C_2(|y_{i+1}| - |y_i|)/h + \|j\|_{h,\infty} \geq 0$$

Let $\langle u_i \rangle_{i=0}^N, \langle v_i \rangle_{i=0}^N$ be two sets of solutions of the difference equations that meet the boundary requirements in order to demonstrate uniqueness and existence.

Suppose $y_i = u_i - v_i$, then z_i fulfill:

$$L_1^N y_i = j_i$$

$$j_i = -C_1 |y_1| / h^2 - C_1 |y_{N-1}| / h^2 - C_2 |y_1| / h \geq 0$$

For inequality continue, we need to have $|y_i| \geq 0 \forall i, i = 1, 2, \ldots N-1$, and $C_1 > 0, C_2 \geq 0,$.

$$y_i = 0 \quad \forall i, i = 1, 2, \ldots N-1$$

This demonstrates that the tridiagonal system of difference equations has a unique solution(3.1). Uniqueness implies existence for linear equations. Presently, in order in order to get the necessary constraint on the distinct issue $\langle w_i \rangle_{i=0}^N$, we set:

$$y_i = w_i - l_i$$

where y_i fulfills the difference equations (3.1), the boundary conditions and

$$L_i = (1 - ih)\phi_0 + (ih)\alpha,$$

also

$$y_0 = 0 = y_N$$

and

$$y_i, i = 1, 2, \ldots N-1, \text{which satisfies}$$
$$L_1^N y_i = f_i$$

Now let

$$|y_n| = \|y\|_{h,\infty} \geq |y_i|, \quad i = 0, 1, 2, \ldots N \tag{3.4}$$

and

$$z_i, i = 1, 2, \ldots N-1, \text{ satisfies}$$
$$L_1^N z_i = f_i$$

Now let

$$|y_n| = \|y\|_{h,\infty} \geq |y_i|\,, \quad i = 0, 1, 2, \ldots N \tag{3.5}$$

Adding up equation (3.4) to $i = n$ to $N-1$ and utilizing the inequalities $b(x) \geq \beta > 0$ and $0 < b(x) \leq \|b\|_{h,\infty} > 0$ yields:

$$-C_1\left(|y_n| - |y_{n-1}|\right)/h^2 - C_1\,|y_{N-1}|\,/h^2 - C_2\,|y_n|\,/h + \sum_{i=n}^{N-1} b_i\,|y_i| + (N-n-1)\|j\|_{h,\infty} \geq 0 \tag{3.6}$$

From inequality (3.6) and the fact that $b(x) < 0$, we obtain $(|y_n| - |y_{n-1}|) \geq 0$. As a result, the left side of inequality (3.6) has negative first, second, and third terms. The inequality simplifies to: when these terms are removed.

$$C_2|y_n| \quad \leq (N-n-1)h\|\mathbf{j}\|_{h,\infty} \quad \leq \|\mathbf{j}\|h,\infty,$$

$$since(N-n-1)h \leq 1$$

i.e., we have

$$|y_n| \leq C_2^{-1}\|j\|_{h,\infty} \tag{3.7}$$

we get,

$$w_i = y_i + L_i$$

$$\begin{aligned} \|w\|_{h,\infty} &= \max_{0\leq i\leq N} |w_i| \\ &\leq \|y\|_{h,\infty} + \|L\|_{h,\infty} \\ &\leq |y_n| + \|L\|_{h,\infty} \end{aligned} \tag{3.8}$$

We now need to determine the bound on L_i in order to finish the estimate.

$$\begin{aligned} \|L\|_{h,\infty} &= \max_{0\leq i\leq N} |L_i| \\ &\leq \max_{0\leq i\leq N} \left[(1-ih)\,|\phi_0| + ih|\alpha|\right] \end{aligned}$$

i.e., we have

$$\begin{aligned} \|L\|_{h,\infty} &\leq |\phi_0| + |\alpha| \\ &\leq \|\phi\|_{h,\infty} + |\alpha| \end{aligned} \tag{3.9}$$

From Eq. (3.7) – (3.9), we obtain the required estimate.

This theorem states that, for any mesh size h and singular perturbation parameter ϵ, to be the solution to the system of differential equations is uniformly limited. As such, all step sizes show stability in the system.

Corollary-1

The error $e_i = w(x_i) - w_i$ in between the solution $w(x)$ of the continuous problems (2.6) and (2.7) and the solution w_i of the discretized problem would meet the following under the presumptions of Theorem 3.1:

$$L_1^N = (\epsilon - \delta b(x_i))D_+D_-u_i + b(x_i)D_+u_i = j(x_i)$$

which satisfies

$$\|e\|_{h,\infty} \leq C^{-1}\|R\| \tag{3.10}$$

where $C = \beta$ or $\|b\|_{h,\infty}$ and

$$\begin{aligned} |R_i| \leq & \max_{x_{i-1}\leq x\leq x_{i+1}} \left[(h/2)b(x)\left|w''(x)\right|\right] + \max_{x_{i-1}\leq x\leq x_{i+1}} \left[(h/6)\left|b(x)\right|\left|w'''(x)\right|\right] \\ & + \max_{x_{i-1}\leq x\leq x_{i+1}} \left[\left(h^2/24\right)\left\{2(\varepsilon - \delta b(x)) + hb(x)\right\}\left|w^{iv}(x)\right|\right] \end{aligned}$$

Proof R_i be the truncation error which is given by

$$\begin{aligned} R_i &= (\varepsilon - \delta b_i)\left[\left(w_{i-1} - 2w_i + w_{i+1}\right)/h^2 - w''\left(x_i\right)\right] \\ &+ b_i\left[\left(w_{i+1} - w_i\right)/h - w'\left(x_i\right)\right] \\ R_i &= (h/2)b_i\left[w''(ih) + hw'''(ih) + h^2 w^{iv}(ih)/12\right] + (\varepsilon - \delta b_i)\,h^2 w^{iv}(ih)/12 \end{aligned}$$

$$\begin{aligned} |R_i| \leq & \max_{x_{i-1}\leq x\leq x_{i+1}} \left[(h/2)b(x)\left|w''(x)\right|\right] + \max_{x_{i-1}\leq x\leq x_{i+1}} \left[(h/6)|b(x)|\left|w'''(x)\right|\right] \\ & + \max_{x_{i-1}\leq x\leq x_{i+1}} \left[\left(h^2/24\right)\left(2(\varepsilon - \delta b(x)) + hb(x)\right)\left|y^{iv}(x)\right|\right] \end{aligned} \tag{3.11}$$

It is simple to demonstrate that the mistake e_i satisfies

$$L_1^N e\left(x_i\right) = L_1^N w\left(x_i\right) - L_1^N w_i = R_i, \quad i = 1, 2, \ldots N-1$$

$e_0 = 0 = e_N$, as well. The mesh function e_i can then be used to apply Theorem 3.4, which results in

$$\|\mathbf{e}\|_{h,\infty} \leq C^{-1}\|\mathbf{R}\|_{h,\infty}$$

The estimate determines the convergence of the difference scheme for fixed values of the parameter

3.2 Method of Fitted Mesh Finite Difference

3.2.1 Piecewise Uniform Shishkin Mesh

The layer behavior cannot be accurately captured by a uniform mesh $x_i = a + (i-1)h$. To address this, The Shishkin mesh which is a piecewise-uniform mesh was developed by the Russian mathematician G.I. Shishkin. This mesh adapts its width based on the solution's characteristics. By concentrating additional mesh points in the sections of the layer where the solution changes rapidly, the Shishkin mesh provides a more detailed examination of these regions. This technique is particularly useful for solutions with sharp gradients, enhancing the accuracy within the layer regions of the solution.

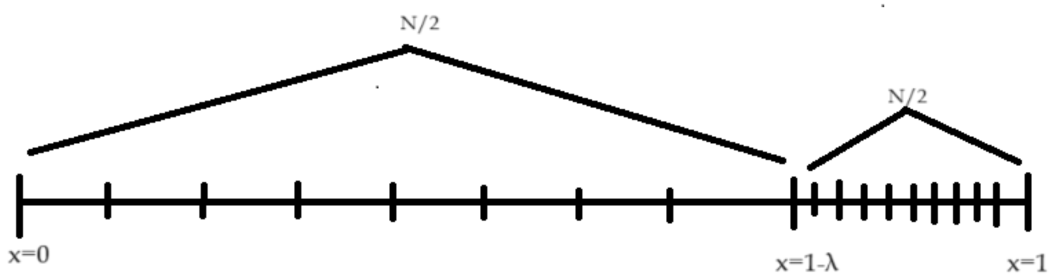
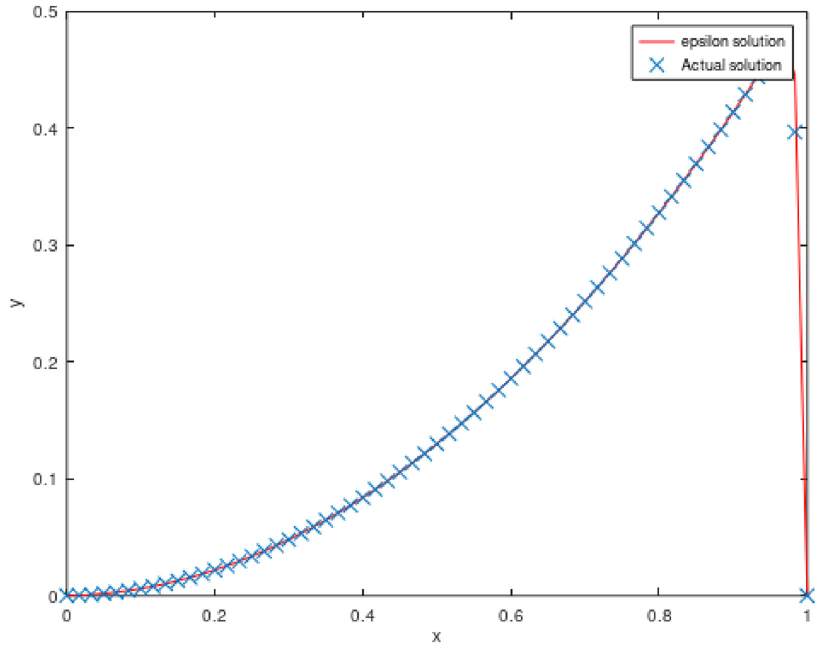


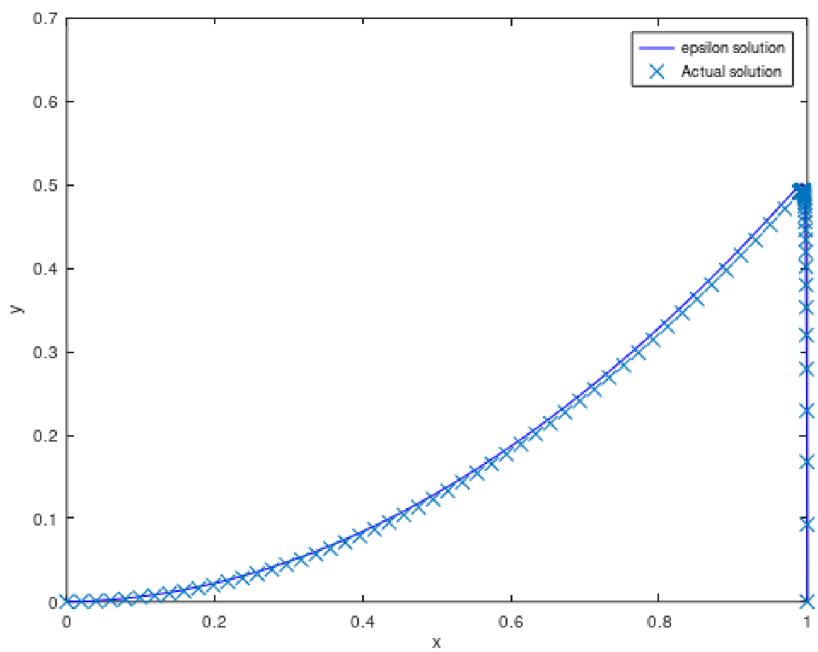
Figure 3.3: Figure:1 Interval Distribution of Shishkin Mesh for convection diffusion
In this section, we employ uniform piecewise mesh and a conventional upwind finite difference operator by applying the finite difference method under fitted mesh. By discretize the boundary value problems (8) and (9), we condense the mesh at the boundary points $x = 0$ and $x = 1$. There are three subintervals divided within the interval $[0, 1]$: $(0, \lambda)$, $(\lambda, 1-\lambda)$, and also $(1-\lambda, 1)$. These subintervals create a fitted piecewise uniform mesh x_i on the $[0, 1]$.

The intervals $(0, \lambda)$ and $(1-\lambda, 1)$ each contain $\frac{N}{4}+1$ equally spaced mesh points, while the interval $(\lambda, 1-\lambda)$ contains $\frac{N}{2}$ equally spaced mesh points. The parameter for transition λ defined to be $\lambda = \min\left[\frac{1}{4}, \left(\frac{2}{\alpha}\right)\epsilon \log N\right]$. This parameter ensures that the mesh density increases near the boundary layers, where sharp changes in the solution occur. Additionally, it is assumed that $N = 2^r$ with $r > 3$, to ensure the border layer contains a enough number of points.

ε.



(a) Mesh space under Standard Finite Difference Method



(b) Mesh space Under Shishkin Mesh

3.2.2 Discrete Minimum Principle

Assume $\Pi_0 \geq 0$ with $\Pi_N \geq 0$. let $(L_3)^N \Pi_i \leq 0$ for all $x_i \in \Xi$

Proof

Suppose m to be9 $Pi_k = \min_{0 \leq i \leq N} \Pi_i and assume \psi_m < 0$. Then we have $\Pi_m - \psi - m - 1 \leq 0, \Pi_{m+1} - \Pi_m \geq 0$ and

$$\begin{aligned} L_1^N &= (\epsilon - \delta b(x_m)) D_+ D_- \Pi_m + b(x_m) D_+ \Pi_m \\ &= 2(\epsilon - \delta b(x_i)) \left(\frac{(\Pi_{m+1} - \Pi_m)}{h_{m+1}} - \frac{(\Pi_m - \Pi_{m-1})}{h_m} \right) \Big/ (h_m + h_{m+1}) \\ &\quad + b(x_m) \left(\frac{(\Pi_{m+1} - \Pi_m)}{h_{m+1}} \right) \Pi_m \\ &\geq 0 \end{aligned}$$

This is contradictory. Consequently, $\Pi_m \geq 0$ since our presumption that $\Pi_m < 0$ is incorrect. Because m has been set but left open-ended, $\Pi_i \geq 0 \forall i, 0 \leq i \leq N$.

Lemma

Any mesh function Q_i such that $Q_0 = Q_N = 0$ can be considered. When $0 \leq i \leq N$, for every i,

$$|Q_i| \leq \theta^{-1} \max_{1 \leq j \leq N-1} \left| L_3^N Q_j \right|$$

Proof Consider the two auxiliary mesh functions $\Pi_i^{\pm}$ defined as:

$$\begin{aligned} \Pi_i^{\pm} &= \theta^{-1} \max_{1 \leq j \leq N-1} \left| L_3^N Q_j \right| \pm Q_i \\ \Pi_0^{\pm} &= \theta^{-1} \max_{1 \leq j \leq N-1} \left| L_3^N Q_j \right| \pm Q_0 \\ &\geq 0, \quad \text{since } Q_0 = 0 \\ \Pi_N^{\pm} &= \theta^{-1} \max_{1 \leq j \leq N-1} \left| L_3^N Q_j \right| \pm Q_N \\ &\geq 0, \quad \text{since } Q_N = 0 \end{aligned}$$

For $1 \leq i \leq N-1$:

$$\begin{aligned} L_3^N \Pi_i^{\pm} &= (\varepsilon - \delta a(x_i)) D^+ D^- \Pi_i^{\pm} + a(x_i) D^+ \Pi_i^{\pm} + b(x_i) \Pi_i^{\pm} \\ &= b(x_i) \theta^{-1} \max_{1 \leq j \leq N-1} \left| L_3^N Q_j \right| \pm L_3^N Q_i \\ &\leq 0, \quad \text{since } b(x_i) \theta^{-1} \leq -1 \end{aligned}$$

[3]Based on the concept of discrete minimum, we have:

$$\Pi_i^{\pm} \geq 0 \quad \forall i, \quad 0 \leq i \leq N$$

This implies:

$$-\theta^{-1} \max_{1 \le j \le N-1} \left| L_3^N Q_j \right| \le Q_i \le \theta^{-1} \max_{1 \le j \le N-1} \left| L_3^N Q_j \right|$$

Thus, we conclude:

$$|Q_i| \le \theta^{-1} \max_{1 \le j \le N-1} \left| L_3^N Q_j \right|$$

which completes the proof.

Chapter 4

Error Analysis

Error Approximate

Let $U(x)$ denote continuous solution described by equations (8) and (9), and let $U^M = \langle u_i \rangle_{i=0}^{M}$ represent the solution of the corresponding discrete problem.

$$L_1^M u_i = g(x_i)$$

The adapted mesh finite difference technique, employing the conventional upwind operator for finite differences on a piecewise uniform mesh$\bar{\alpha}^M$ which becomes denser near the boundary layer at $x = 0$, achieves ε-uniformity. Additionally, for the solution y and its discrete counterpart $U^M = \langle u_i \rangle_{i=0}^{M}$, the subsequent error estimation is applicable.

$$\sup_{0<\varepsilon\leq 1} \left\| U^M - u \right\| \leq KM^{-1}(\ln M)^2$$

where K is a constant that is unaffected by ε.

Proof

Similar to the continuous solution $U^M = \langle u_i \rangle_{i=0}^{M}$ of the discrete problem can be split into the regular and the singular components:

$$U^M = V^M + Z^M$$

V^M defines the solution of the non-homogeneous problem:

$$L_3^M V^M(x_i) = f(x_i) \quad \forall x_i \in \alpha^M, \quad V^M(0) = v(0), \quad V^M(1) = v(1)$$

and Z^M is the solution of the homogeneous problem [3]:

$$L_3^M Z^M(x_i) = 0 \quad \text{for all } x_i \in \alpha^M, \quad Z^M(0) = z(0), \quad Z^M(1) = z(1)$$

This is one way to express the error:

$$U^M - u = (V^M - v) + (Z^M - z) \tag{4.1}$$

Therefore, it is possible to estimate the errors in the solution's regular and singular components separately. We examine both the differential and difference equationsin order to calculate the regular component's error. We acquire:

$$L_\varepsilon^M (V^M - v)(x_i) = j(x_i) - L_3^M v(x_i)$$

This simplifies to:

$$L_\varepsilon^M (V^M - v)(x_i) = (L_\varepsilon - L_3^M) v(x_i)$$

Substituting L_ε and L_3^M:

$$L_\varepsilon^M (V^M - v)(x_i) = (\varepsilon - \delta b(x)) \left(\frac{d^2}{dx^2} - D^+ D^- \right) v(x_i) + b(x) \left(\frac{d}{dx} - D^+ \right) v(x_i) \tag{4.2}$$

For $x_i \in \alpha^M$ and operator $\Pi \in C^2(\bar{\alpha})$, we have:

$$\left| \left(D^+ - \frac{d}{dx} \right) \Pi(x_i) \right| \leq (x_{i+1} - x_i) \left\| \Pi^{(2)} \right\| /2$$

and for operator $\Pi \in C^3(\bar{\alpha})$,

$$\left| \left(D^+ D^- - \frac{d^2}{dx^2} \right) \Pi\left(x_i\right) \right| \leq (x_{i+1} - x_{i-1}) \left\| \Pi^{(3)} \right\| /3$$

For the proof of these results, one can see Lemma 1. Using these results, we obtain [4]

$$\left| L_3 \left(V^M - v \right) (x_i) \right| \leq (x_{i+1} - x_{i-1}) \left(\frac{(\varepsilon - \delta b\left(x_i\right))}{3} \left\| v^{(3)} \right\| + \frac{b\left(x_i\right)}{2} \left\| v^{(2)} \right\| \right)$$

[10]Using Theorem 3.1 for bounds on $v^{(2)}$ and $v^{(3)}$, we obtain that $x_{i+1} - x_{i-1} \leq 2M^{-1}$.

$$\left|L_3^M\left(V^M-v\right)(x_i)\right| \leq KM^{-1}, \quad x_i \in \sigma^M \tag{4.3}$$

Now an application of Lemma 2 for the mesh function $\left(V^M-v\right)(x_i)$ gives

$$\left|\left(V^M-v\right)(x_i)\right| \leq \theta^{-1} \max_{1\leq j\leq M-1}\left|L_3^M\left(V^M-v\right)(x_j)\right| \tag{4.4}$$

$$\left|\left(V^M-v\right)(x_i)\right| \leq KM^{-1} \tag{4.5}$$

The reasoning behind estimating the singular component's error $L_3^M(Z^M-z)$ is contingent upon the value assigned to the Whether $\gamma=1/2$ or $\gamma=K(\varepsilon-\delta\beta)\ln M$, where $K=1/\theta$, is the transition parameter γ.

Case i) When $K(\varepsilon-\delta\beta)\ln M \geq 1/2$, namely, when the mesh is uniform, we follow the same line of reasoning as we did when estimating the regular part of the error, leading to:

$\left|L_3^M\left(Z^N-z\right)(x_i)\right| \leq K\left(x_{i+1}-x_{i-1}\right)\left(\left(\varepsilon-\delta b\left(x_i\right)\right)\left\|z^{(3)}\right\|+b\left(x_i\right)\left\|z^{(2)}\right\|\right)$

Using Theorem 2 for bounds on $z^{(2)}$ and $z^{(3)}$ and the fact that $(x_{i+1}-x_{i-1})=2M^{-1}$ for the uniform mesh, we obtain

$$\left|L_3^M\left(Z^M-z\right)(x_i)\right| \leq K(\varepsilon-\delta\beta)^{-1}M^{-1} \tag{4.6}$$

In this case, we have $(\varepsilon-\delta\|b\|)^{-1} \leq 2K\ln M$. Using this inequality in above inequality (4.6), we obtain

$$\left|L_3^M\left(Z^M-z\right)(x_i)\right| \leq KM^{-1}(\ln M)^2 \tag{4.7}$$

Now an application of Lemma 2.4 for the mesh function $\left(Z^N-z\right)(x_i)$ gives

$$\left|\left(Z^M-z\right)(x_i)\right| \leq \left|L_3^M\left(Z^M-z\right)(x_i)\right| \quad \forall x_i \in \sigma^M$$

Using (4.7) in (4.8), we obtain

$$\left|\left(Z^M-z\right)(x_i)\right| \leq KM^{-1}(\ln M)^2 \quad \forall x_i \in \sigma^M$$

[10]Case ii) $K(\varepsilon-\delta\beta)\ln M < 1/2$, i.e., when the mesh is piecewise uniform with mesh

spacing $2\gamma/N$ in the subinterval $[0, \gamma]$ and $2(1-\gamma)/M$ in the subinterval $[\gamma, 1]$

We give separate proofs for the bounds on the singular component of the error in the coarse and fine mesh subintervals.The bound on the singular component in the outer area, or in the subinterval, is first obtained. $[\gamma, 1]$. Using the triangular inequality, we have [10]

$$|(Z-z)(x_i)| \leq |Z(x_i)| + |z(x_i)|$$

From inequality (22), we have

$$|z(x_i)| \leq K \exp\left(-\beta x_i/(\varepsilon - \delta\beta)\right)$$

for all $x_i \in [\gamma, 1]$. $\operatorname{Exp}\left(-\beta x_i/(\varepsilon - \delta\beta)\right)$ is a decreasing function and $x_i \geq \gamma$. [10]Using these facts in above inequality (4.11) we have

$$|w(x_i)| \leq K \exp(-\beta\gamma/(\varepsilon - \delta\beta))$$

In this case we have $\gamma_i = K(\varepsilon - \delta\beta)\ln N$. [10]Using this value of γ in the above inequality, we get

$$|z(x_i)| \leq KM^{-1}$$

Now to obtain the bound on Z^M, we construct a mesh function $\hat{Z}_\varepsilon^M$ defined as the solution of the following problem

$$(\varepsilon - \delta b(x_i)) D^+ D^- \hat{Z}^M(x_i) + \beta D^+ \hat{Z}^M(x_i) = 0$$

$1 \leq i \leq M-1$, under the same boundary conditions as for Z

$$\left|Z^M(x_i)\right| \leq \left|\hat{Z}^M(x_i)\right|, \quad 0 \leq i \leq M$$

$$\hat{Z}_\varepsilon^M$$

gives

$$\left|\hat{Z}^M(x_i)\right| \leq M^{-1}, \quad M/2 \leq i \leq M$$

Using this estimate for $\hat{Z}_\varepsilon^M$ in inequality (4.14), we obtain

$$\left|Z^M(x_i)\right| \leq KM^{-1}, \quad N/2 \leq i \leq M$$

Thus from inequalities (4.12) and (4.15), we obtain the bound on the singular component of error in the outer region $[\gamma, 1]$

$$\left|Z^M - z(x_i)\right| \leq KM^{-1}, \quad M/2 \leq i \leq M$$

[10]Now it remains to prove the result for $x_i \in [0, \gamma]$, i.e., in the boundary layer region. For $i = 0$, there is nothing to shown. For $x_i \in (0, \gamma)$ the proof follows on the same lines as for the case i) except that we use the discrete minimum principle on $[0, \gamma]$ and the already established bounds $Z\left(x_{M/2}\right) \leq KM^{-1}$ and $z\left(x_{M/2}\right) \leq KM^{-1}$. [10]Consequently, by applying the same reasoning as we did to estimate the error's regular component, we obtain

$$\begin{aligned} &\left|L_3^M\left(Z^M - z\right)(x_i)\right| \leq 2\gamma M^{-1}(\varepsilon - \delta\|b\|)^{-2} \\ &\left|Z^M(0) - z(0)\right| = 0 \end{aligned}$$

and

$$\begin{aligned} \left|Z^M\left(x_{M/2}\right) - z\left(x_{M/2}\right)\right| &\leq \left|Z^M\left(x_{M/2}\right)\right| + \left|z\left(x_{M/2}\right)\right| \\ &\leq KM^{-1} \end{aligned}$$

Now let us introduce comparison functions $\Pi_i^{\pm}$defined by

$$\Pi_i^{\pm} = (\alpha - x_i) K_1(\varepsilon - \delta\|b\|)^{-2}\alpha M^{-1} + K_2 M^{-1} \pm \left(Z^M - z\right)(x_i)$$

where K_1 and K_2 are arbitrary constants. Then we have[3]

$$\begin{aligned} \Pi_0^{\pm} &= C_1\alpha M^{-1}(\varepsilon - \delta\|a\|)^{-2} + K_2 M^{-1} \geq 0 \\ \Pi_{M/2}^{\pm} &= K_2 M^{-1} \pm \left(Z^M - z\right)\left(x_{M/2}\right) \end{aligned}$$

We chose K_2 such that, in the equation above, the first term predominates over the second term, resulting in $\Pi^{\pm}_{M/2} \geq 0$ and consider[1]

$$\begin{aligned} L_3^M \Pi_i^{\pm} &= \alpha M^{-1} K_1 (\varepsilon - \delta\|b\|)^{-2} L_3^M (\gamma - x_i) \\ &= -\gamma M^{-1} (\varepsilon - \delta\|b\|)^{-2} (b_i C_1 \mp 2) + b_i \gamma M^{-1} (\varepsilon - \delta\|b\|)^{-2} (\gamma - x_i)(K_1 + K_2) \end{aligned}$$

Now we choose the constant K_1 so that $(b_{i1} \mp 2) \geq 0$, Consequently, in the inequality above, every term on the right side is negative, giving[10]

$$L_3^M \Pi_i^{\pm} \leq 0, \quad 1 \leq i \leq M/2 - 1$$

Then by the defination of discrete minimum principle, we get

$$\Pi_i^{\pm} \geq 0, \quad 0 \leq i \leq N/2$$

which on simplification gives[1]

$$\left| \left(Z^M - z \right)(x_i) \right| \leq K_1 (\varepsilon - \delta\beta)^{-2} \gamma^2 M^{-1} + K_2 M^{-1}$$

Since $\gamma = K(\varepsilon - \delta\beta) \ln M$, where $K = 1/\theta$, we obtain

$$\left| \left(Z^M - z \right)(x_i) \right| \leq K M^{-1} (\ln M)^2 \tag{4.17}$$

Now combining inequalities (4.16) and (4.17) to obtain the bound on the singular component of error throughout the interval $[0, 1]$, we obtain

$$\left| \left(Z^M - z \right)(x_i) \right| \leq K M^{-1} \varepsilon (\ln M)^2, \quad 0 \leq i \leq M \tag{4.18}$$

[10]Combining the two inequalities—inequality (4.5) to bound the regular error component and inequality (4.18) to bound the singular error component—allows us to obtain the required error estimate.[10].

Chapter 5

Numerical Results

5.1 Numerical Computations

Let us examine a basic convection diffusion model problem.

$$\epsilon u''(x) - (1+x)u'(x-\delta) - e^{-x}u(x) = 1 \tag{5.1}$$

under the interval with boundary constraints

$$u(x) = 1, -\delta \le x \le 0, \quad and \quad u(1) = -1$$

Utilizing Taylor's series expansion, a function can be approximated as a sum of terms derived from its derivatives at a specific point.

$$u(x-\delta) \approx u(x) - \delta u'(x)$$

we get,

$$(\epsilon + (x+1)\delta)u''(x) - u'(x)(1+x) - u(x)e^{-x} = 1 \tag{5.2}$$

actual solution of the equation(5.1) is given by,

$$u(x) = c_1 e^{(r+s)x} + c_2 e^{(r-s)x} - e^x \tag{5.3}$$

where,

$$r = \frac{(x+1)}{2(\epsilon + (x+1)\delta)}, s = \frac{\sqrt{(x+1)^2 + 4e^{-x}(\epsilon + (x+1)\delta)}}{2(\epsilon + (x+1)\delta)} \tag{5.4}$$

$$c_2 = \frac{e - 2e^{(r+s)} - 1}{e^{(r-s)} - e^{(r+s)}}, \quad c_1 = 2 - c_2$$

Now, we will show the boundedness, stability and convergence of the problem(37)

5.1.1 5.1.1 Boundedness

Here we consider 2 cases and by using lemma1,
Case 1:- When $\delta = 0$, then using this in equation (5.4), we get,

$$r - s < 0$$

Case 2:- When $\delta \neq 0$ then again using equation (5.4) since magnitude of 'r' is less than 's' so we get,

$$r - s < 0$$

So we get that $r - s < 0$ for each case, then the term $e^{r-s} \to 0 as \epsilon \to 0$. Therfore,(5.4) becomes

$$\begin{aligned} u(x) &\leq 2e^{(r+s)x} - 2e^{s+r} - 1 - e^x + e \\ &\leq 2e^{(r+s)x} - e^x + 2 \leq 2 \end{aligned}$$

so, we get

$$u(x) \leq 2$$

Then, Lemma 1 proved and u(x) is bounded.

5.1.2 Boundedness of Derivatives

Theorem 5.1: Suppose u be the solution of the problem described by equations (2.6) and (2.7). It states that $\left\|u^{(k)}\right\| \leq C\epsilon^{-k}$ for $k = 1, 2, 3$. **Proof**:A neighbourhood $N_x = (c, c + \epsilon)$ for $x \in \sigma$ can be defined as follows: c is a positive constant defined so that $x \in N_x$ and $N_x \subset \sigma$. The Mean Value Theorem states that $z \in N_x$ exists such that

$$u'(v) = \frac{u(c + \epsilon) - u(c)}{\epsilon}$$

This simplifies to:

$$|u'(v)| \leq 2\epsilon^{-1}\|u\|$$

By applying Lemma 1 to bound u in the inequality above, we obtain:

$$|u'(v)| \leq 2\epsilon^{-1}\left(\|j\|\theta^{-1} + \max\left(|\phi_0|, |\alpha_1|\right)\right)$$

Using Equation (5.3), we have:

$$|u'(v)| \leq 2\epsilon^{-1}(e) \leq (2e)\epsilon^{-1}$$

Then, by integrating u'' and We obtain the following by taking the modulus on both sides and substituting the differential equation's value for $u''(t)$ (5.3).

$$|u'(x)| \leq C(\epsilon + (x+1)\delta)^{-1}, \quad x \in \sigma$$

If $\delta = 0$, then:

$$|u'(x)| \leq C\epsilon^{-1}$$

This yields $\|u'\| \leq C\epsilon^{-1}$, where C is a constant. Similarly, by differentiating Equation (5.3) and utilizing bounds on $u(x)$ and $u'(x)$, we can derive The necessary bounds for the second and third derivatives of the solution u [7].

5.1.3 Stability and Convergence

On comparing equation (3.3) with equation (2.6) and (2.7), we get,

$$b(x) = -(x+1) \quad and \quad -\exp -x = 0, j(x) = 1$$

and $u(x)$ is given by equation (38).Now,using $\|\cdot\|$ as The discrete l_∞-norm, denoted as $\|x\|_{h,\infty}$, is defined as $\|x\|_{h,\infty} = \max_{0\leq i\leq N}|x_i|$.

$$\begin{aligned}\|\mathbf{u}\|_{h,\infty} &\leq C^{-1}\|\mathbf{j}\|_{h,\infty} + (\|\phi\|_{h,\infty} + \|\alpha\|_{h,\infty}) \quad \text{while} \quad C = \|\beta\| \\ &\leq C^{-1}(1) + (1+1) \\ &\leq C^{-1}(1) + 2\end{aligned} \tag{5.5}$$

Since, the solution exist uniquely with the boundary conditions and satisfies (4.17).

Therefore, the solution of difference equation is "uniformly bounded" indicates that the values remain constrained regardless h or the parameter ϵ of the mesh size. As such, this technique is stable for arbitrary step sizes.

and the Corollary1 we have discused states the convergence of this problem type.

5.2 Error and Order of Convergence

If we assume $u(x_i)$ as the exact solution and u_i as the numerical solution, the error at each mesh point can be calculated as:

$$r_i = |u(x_i) - u_i|$$

The maximum norm error is given by,

$$R_N = \max \|u(x_i) - u_i\|$$

[8]and the Order of Convergence is given by,

$$O_N = \frac{\log\left(\frac{R_N}{R_N+1}\right)}{\log 2}$$

Let's examine the numerical problem defined by equation (5.1), considering the provided boundary conditions and the true solution. Initially, we'll create tables illustrating the maximum norm error and convergence order for various combinations of ϵ, N, and δ.

Further, we will draw graphs showing convergence between actual solution and numerical solution. in the table each box has two parts for the value of ϵ and N. The upper part of the box represents the Max. norm error (R_N) and the lower part of the box represents the Order of Convergence (O_N)

Table 1: The maximum norm error and order of convergence for δ=0.5ε under standard finite difference method.

ε/N	32	64	128	256	512
2^{-1}	0.0025	0.0013	0.00064077	0.00032158	0.00016109
	0.94	1.02	0.99	1.00	1.00
2^{-2}	0.0072	0.0037	0.0019	0.00093669	0.00046979
	0.96	0.96	1.02	1.00	1.00
2^{-3}	0.0168	0.0088	0.0045	0.0023	0.0011
	0.93	0.97	0.97	1.06	1.06
2^{-4}	0.0303	0.0184	0.0096	0.0049	0.0025
	0.72	0.94	0.97	0.97	0.97
2^{-5}	0.0414	0.0314	0.0246	0.0186	0.0120
	0.40	0.35	0.30	0.63	0.63

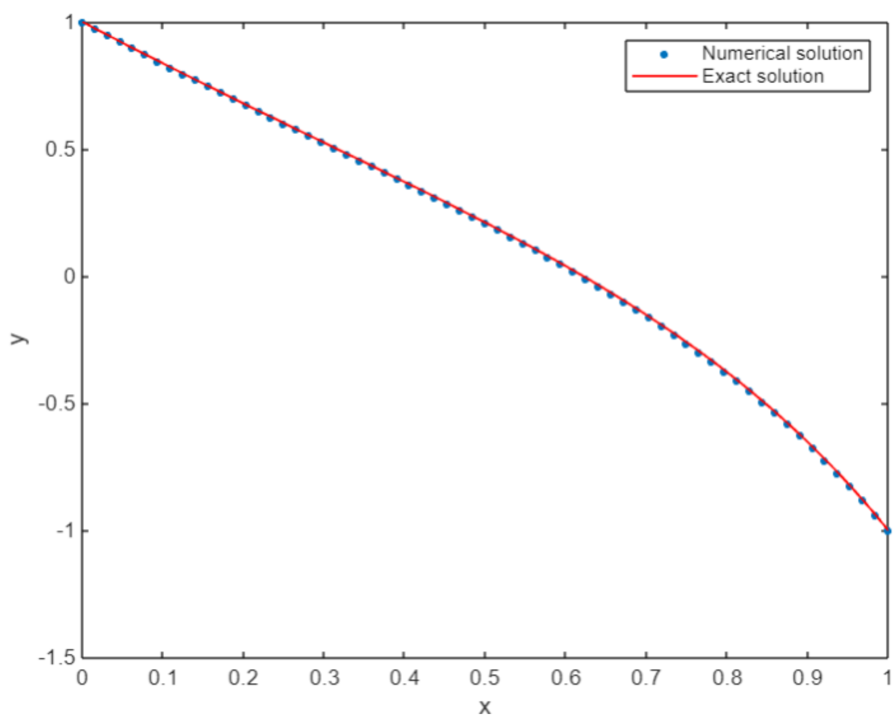


Figure 5.1: Solution Plot For $\epsilon = 2^{-4}$ and $N = 64$

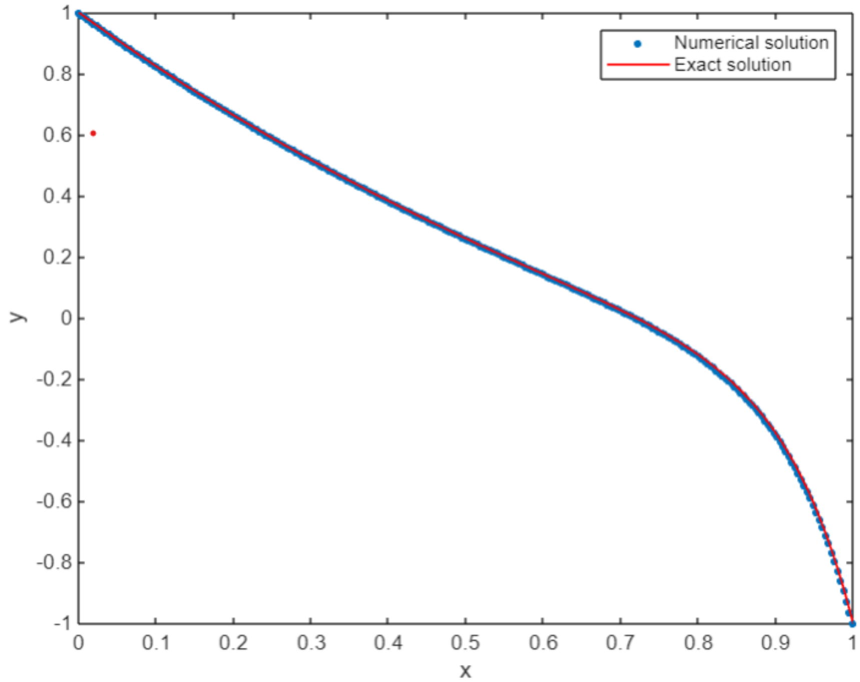
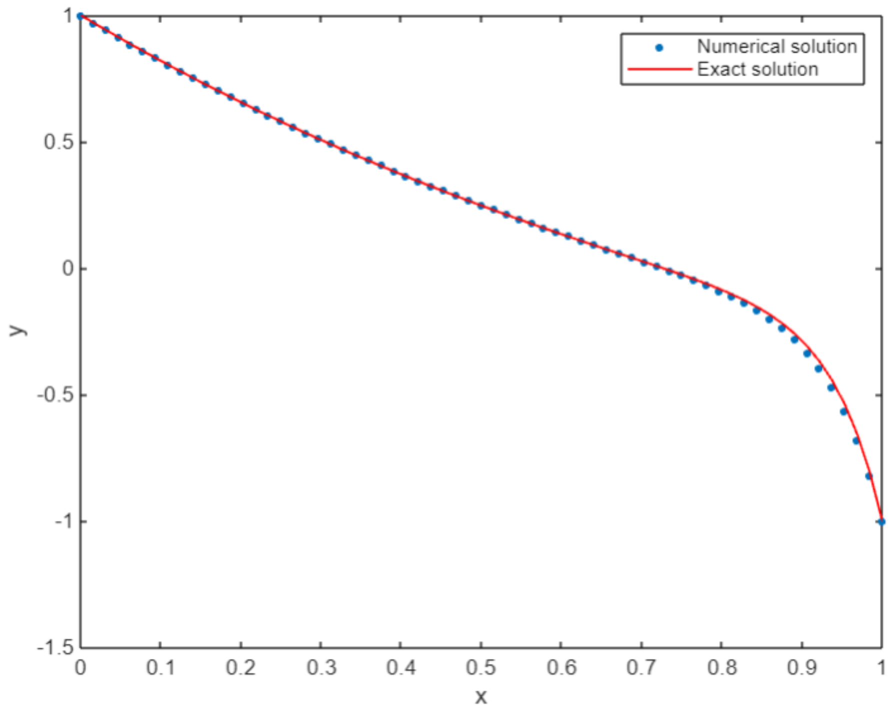


Figure 5.2: Solution Plot For $\epsilon = 2^{-4}$ and $N = 256$

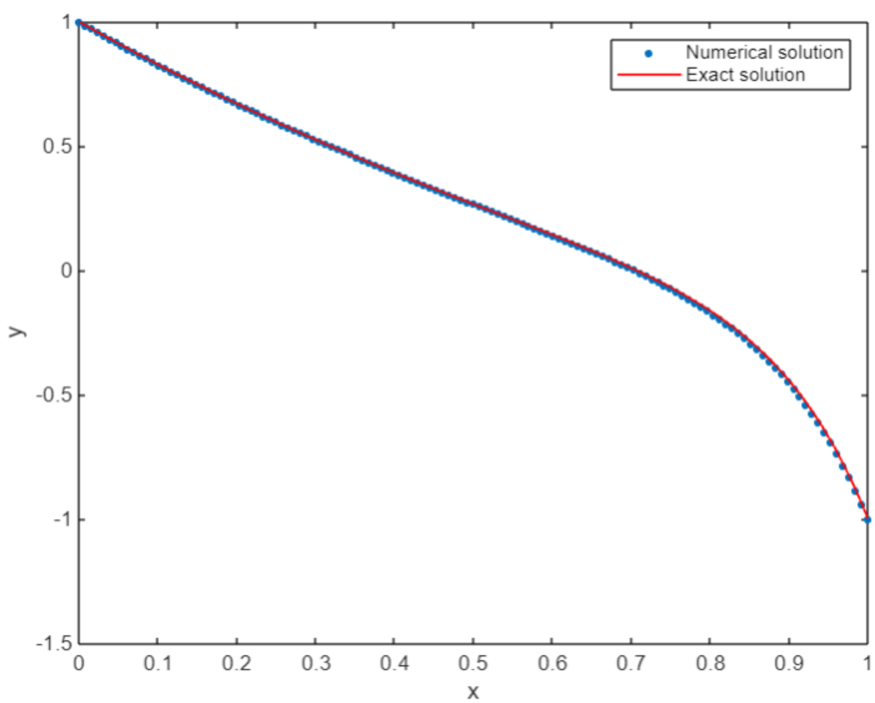
Table 2: The maximum norm error and order of convergence for δ=0 under standard finite difference method.

ε/N	32	64	128	256	512
2^{-1}	**0.0047**	**0.0024**	**0.0012**	**0.00061499**	**0.00030827**
	0.97	**1.00**	**0.96**	**1.00**	**1.00**
2^{-2}	**0.0121**	**0.0062**	**0.0032**	**0.0016**	**0.00080282**
	0.96	**0.95**	**1.00**	**0.99**	**0.99**
2^{-3}	**0.0257**	**0.0137**	**0.0070**	**0.0036**	**0.0018**
	0.91	**0.97**	**0.96**	**1.00**	**1.00**
2^{-4}	**0.0473**	**0.0278**	**0.0146**	**0.0075**	**0.0038**
	0.77	**0.93**	**0.96**	**0.98**	**0.98**
2^{-5}	**0.0448**	**0.0320**	**0.0225**	**0.0179**	**0.0173**
	0.49	**0.50**	**0.33**	**0.05**	**0.05**



(a) Solution Plot For $\epsilon = 2^{-4}$ and $N = 64$

Figure 5.3: Solution Plots for Table-2 for N=64



(a) Solution Plot For $\epsilon = 2^{-3}$ and $N = 128$

Figure 5.4: Solution Plots for Table-2 for N=128

Chapter 6

Conclusion

In this study we have research on the numerical analysis of singular perturbation boundary value issues for differential difference equations with delay. To solve these boundary value problems, we have use the delay shift operator with layer behaviour, [7]a standard finite difference method with uniform mesh and an ϵ piecewise uniform fitted mesh approach is approved. These kinds of problems with boundary values are often seen in control theory and the biosciences when modelling a variety of processes in real life mathematically. Specifically, a general boundary value issue can be presented to determine the expected time for random synaptic inputs in the dendrites to generate action potentials in nerve cells. Now, let's summarize the working of two numerical methods we have used in this paper.

1. Standard Finite difference Method with uniform mesh

[10]The standard finite difference method is based on the technique of mesh spacing with evenly spaced mesh points all the way across the interval. Bias towards the boundary layer region does not exist in the mesh spacing. In this study, we analyse the boundedness of the solution, its derivatives, stability and convergence analysis. The standard finite difference method's order of convergence and error estimate are displayed using a number of tables and graphs.

2. Fitted mesh finite difference operator with piecewise uniform mesh The fitted mesh approach consists of a standard finite difference operator and a piecewise uniform mesh condensing in the boundary layer regions to reflect the singularly perturbed nature of the solution[3].The conventional upwind finite difference operator and a specific type of mesh are used in this method. We study a piecewise uniform fitted mesh in this case, which works well for developing the ρ −uniform method. Although more complex meshes can be used, [4]the piecewise uniform mesh's simplicity is intended to be one of its key selling points. The established error estimation shows that the approach is ρ-uniform. We solve several numerical illustrations to display how little changes impact the boundary layer solution. Numerical data reported in terms of maximum errors and solution graphs are provided to demonstrate the approach's efficiency[4].

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