Estimates of Second Hankel Determinant of Logarithmic Coefficients

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE AWARD OF THE DEGREE
OF

MASTER OF SCIENCE
IN
Applied Mathematics

Submitted by

Snehal (2K22/MSCMAT/43)

Arya Tripathi (2K22/MSCMAT/56)

Under the supervision of

Prof. S. Sivaprasad Kumar

Applied Mathematics
DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College of Engineering)
Bawana Road, Delhi 110042

JUNE, 2024

DEPARTMENT OF APPLIED MATHEMATICS
DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College of Engineering)
Bawana Road, Delhi-110042

CANDIDATE'S DECLARATION

We, Snehal and Arya Tripathi, Roll No's –2K22/MSCMAT/43 and 2K22/MSCMAT/56 students of M.Sc. (Applied Mathematics), hereby declare that the project Dissertation titled **Estimates of Logarithmic Coefficient of Second Hankel Determinant** which is submitted by us to the Department of Applied Mathematics, Delhi Technological University, Delhi in partial fulfilment of the requirement for the award of degree of Master of Science, is original and not copied from any source without proper citation. This work has not previously formed the basis for the award of any Degree, Diploma Associateship, Fellowship or other similar title or recognition.

Place: Delhi — Snehal

Date: 06-06-2024 — Arya Tripathi

DEPARTMENT OF APPLIED MATHEMATICS
DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College of Engineering)
Bawana Road, Delhi-110042

CERTIFICATE

I hereby certify that the Project Dissertation titled **Estimates of Logarithmic Coefficient of Second Hankel Determinant** which is submitted by Snehal and Arya Tripathi, Roll No's – 2K22/MSCMAT/43 and 2K22/MSCMAT/56, Applied Mathematics, Delhi Technological University, Delhi in partial fulfilment of the requirement for the award of the degree of Master of Sciencs, is a record of the project work carried out by the students under my supervision. To the best of my knowledge this work has not been submitted in part or full for any Degree or Diploma to this University or elsewhere.

Place: Delhi **Prof. S. Sivaprasad Kumar**

Date: 06-06-2024 **SUPERVISOR**

DEPARTMENT OF APPLIED MATHEMATICS
DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College of Engineering)
Bawana Road, Delhi-110042

ACKNOWLEDGEMENT

We wish to express our sincerest gratitude to our Supervisor, Prof. S. Sivaprasad Kumar for his continuous guidance and mentorship that he provided us during the project. He showed us the path to achieve our targets by explaining all the tasks to be done and explained to us the importance of this project as well as its industrial relevance. He was always ready to help us and clear our doubts regarding any hurdles in this project. Without his constant support and motivation, this project would not have been successful.

Place: Delhi — Snehal

Date: 06-06-2024 — Arya Tripathi

Abstract

Logarithmic and inverse logarithmic coefficients are crucial concepts in univalent functions theory. The aim of this work is to provide bounds of the second Hankel determinant for some starlike functions associated with the petal-shaped domain with respect to the logarithmic and inverse logarithmic coefficients.

Contents

Content **v**

List of Symbols **vi**

1 Introduction **2**

1.1 Classes of univalent and starlike functions 4

2 Hankel-determinant **10**

2.1 Hankel and Toeplitz-Matrix 13

2.1.1 Topology of Hankel-matrix 13

2.2 Hankel-determinant for sub-class of analytic funtions 15

3 Petal-shaped Domain **17**

3.0.1 Certain properties around $\rho(z)$: 18

3.0.2 Inclusion Relations 19

3.0.3 Sharp Coefficient Problems for the Class S^*_ρ 19

3.0.4 Application 20

4 Logarithmic Coefficient and Inverse Logarithmic Coefficient **24**

4.1 Estimation of the Hankel-determinant on Logarithmic coefficients 26

4.2 Estimation of the Hankel-determinant on Inverse Logarithmic coefficients . . 29

5 Conclusion **32**

List of Symbols

$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\Re$	Real part of a complex number
$\mathbb{C}$	Set of complex numbers
$\mathbb{D}$	Open unit disk
$\mathbb{D}^*$	Open punctured unit disk
$\mathcal{A}$	The class of functions being analytic in $\mathbb{D}$
$\mathcal{S}$	Subclass of $\mathcal{A}$ consisting of univalent function
$\mathcal{H}_0$	The class of harmonic functions in $\mathbb{D}$
$\mathcal{S}^*$	The class of starlike functions in $\mathcal{S}$
$\mathcal{S}^*(\alpha)$	The class of starlike functions of order α in $\mathcal{S}$
$\mathcal{CV}$	The class of convex functions in $\mathcal{S}$
$\mathcal{CV}(\alpha)$	The class of convex functions of order α in $\mathcal{S}$
$\mathcal{CCV}$	The class of close-to-convex functions
$\mathcal{CV}(\varphi)$	The class of Ma-minda convex functions where $\psi \in \mathcal{P}$
$\mathcal{S}^*(\varphi)$	The class of Ma-Minda starlike functions, where $\varphi \in \mathcal{P}$
$\mathcal{S}_s^*$	The class of starlike functions wrt to symmetric points
$\mathcal{K}_s$	The class of convex functions wrt to symmetric points
$\rho_(\mathbb{D})$	Petal-shaped domain
$\mathcal{S}_\rho^*$	The class of starlike functions under petal-shaped domain.
$\mathcal{R}$	The class of bounded turning functions in $\mathcal{S}$
$\mathcal{P}$	The class of Carathéodory functions
$H_q(n)$	Hankel-determinant
$T_q(n)$	Toeplitz-determinant
$k(z)$	Koebe function
$\prec$	Subordination symbol i.e., $g \prec f$ implies g is subordinate to f
$\mathcal{F}(\psi)$	The class of analytic functions
$f * g$	Convolution $f * g(z)$ of two analytic functions f and g in $\mathcal{A}$
$M_r^N(f)$	N^{th}-tail sum $\sum_{n=N}^{\infty} \|a_n\|\|z^n\|$ of an analytic function
$\sum$	Class of all meromorphic functions f of the form $f(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + \dots$
$\sum_{n,b}$	Class of all meromorphic function f of the form $f(z) = \frac{1}{z} + bz^n + a_{n+1}z^{n+1} + \dots (b \leq 0)$
Y_n	Logarithmic coefficients
Γ_s	Inverse logarithmic coefficient

List of Figures

3.1 Petal shaped domain $\rho(\mathbb{D})$ 18

Chapter 1

Introduction

This chapter covers the definition of several classes of analytic functions as well as some fundamental terms and ideas that will be used in later chapters.Some basic notations have also been made along with a snapshot view of the thesis indicating some significant findings of the present study.

Definition 1.0.1. (Univalent Function) [10] In a domain $\mathcal{D} \subset \mathbb{C}$, a function $f(z)$ is considered *univalent* if it is one-to-one; that is, if $z_1, z_2 \in \mathcal{D}$, then $f(z_1) = f(z_2)$ implies that $z_1 = z_2$.

Geometrically, this implies that various points in the domain will correspond to various spots or points on the image domain [14].

Definition 1.0.2. (Analytic Function) [10] When a function $f(z)$ is differentiable in a neighborhood of z_0, i.e., that $f'(z_0) \neq 0$, it is considered *analytic* at a point $z_0 \in \mathcal{D}$. If $f(z)$ is analytic at every point in $\mathcal{D}$, then it is considered *analytic* in a domain $\mathcal{D} \subset \mathbb{C}$.

When $z \in \mathcal{D}$, the Taylor series expansion of an analytic function $f \in \mathcal{D}$ is as follows:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, a := \frac{f^{(n)}(z_0)}{n!}$$

. Let the class of analytic functions defined on $\mathbb{D}$ be $\mathcal{H}(D)$ [14]. Let $\mathcal{H}[a, n]$ be the sub-class of $\mathcal{H}(D)$ that is made up of the following types of functions:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots.$$

Additionally, the class $\mathcal{A}$ denotes all functions f that are analytic in the open unit disk $\mathbb{D}$, which is normalized by the factor $f'(0) = 1$ and $f(0) = 0$ [11]. The Taylor series expansion of a function $f \in \mathcal{A}$ [10] is as follows:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

This function is defined on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ [10]. Furthermore, the presence of a solution to the coefficient-related problem and its relationship to the compactness of a certain function space demonstrate the significance of normalization. Since analytic univalent functions in the domain $\mathcal{D}$ maintain angles (both in magnitude and direction), they are also known as *conformal mappings* in $\mathcal{D}$.

$\mathcal{S}$ is the sub-class of univalent functions within $\mathcal{A}$. The *Koebe function* [14], which maps $\mathbb{D}$ onto the complex plane with the exception of a slit along the half-line $(-\infty, -1/4]$, is the function k given by:

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n$$

defined on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ [10]. A surprising conclusion known as the Riemann Mapping Theorem [11] was announced by Riemann in 1851. It states that every simply-connected domain that is not the whole complex plane $\mathbb{C}$ may be conformally transferred onto the unit disk $\mathbb{D}$.

Theorem 1.0.3. ***(Riemann Mapping Theorem)*** *[11] Suppose $b \in \mathcal{D}$, $\mathcal{D} \subset \mathbb{D}$ be a simply-connected domain. A distinct analytic function [14] $g : \mathcal{D} \to \mathbb{C}$ exists, such that*

(A.) $g(b) = 0$ and $g'(b) > 0$;

(B.) g is univalent;

(C.) $g(\mathcal{D}) = \Omega$, where Ω is also a simply-connected domain.

This theorem allows one to restrict the study of analytic univalent functions on a simply-connected domain to the open unit disk. $\mathbb{D}$ [10].

Consequently, it is easy to convert the properties of a univalent function defined on the simply-connected domain $\mathcal{D}$ into the properties of the original function defined on the open unit disk $\mathbb{D}$ [10]. Studying analytic functions inside the unit disk $\mathbb{D}$ is therefore adequate.

Since

$$f_1(z) = \frac{f(z) - f(0)}{f'(0)}, \quad f'(0) \neq 0$$

symbolizes the image domain's $f(\mathbb{D})$ contraction and shifting with rotation and any property of the function $f_1(z)$ is immediately translated into a corresponding property of $f(z)$. Furthermore, the presence of a solution to the coefficient-related problem and its relationship to the compactness of a certain function space demonstrate the significance of normalization.

Theorem 1.0.4. ***(Bieberbach's Conjecture)*** *[35] If $f \in S$, then $|a_n| \leq n$, for $n \geq 2$ and equality holds if and only if f is the rotation of the Koebe function k.*

Löwner, Garabedian, and Schier, respectively, proved the conjecture for the instances $n = 3$ and $n = 4$. The hypothesis was later proven by Pederson and Schi er for $n = 5$, then by Pederson and Ozawa separately for $n = 6$. For all coeffcients n, Louis de Branges proved the Bieberbach's conjecture in 1985 [14].

Theorem 1.0.5. ***(de Branges Theorem or Bieberbach's Theorem)*** *[19] If f belongs to S i.e. $f \in S$ then for $n \geq 2$, $|a_n| \leq n$.*

The equality occurs only for the Koebe function and it's rotation. Several important features of univalent functions are covered by Bieberbach's theorem. The well-known covering theorem is one noteworthy characteristic: If f belongs to the class of S, then the image of unit disk $\mathbb{D}$ under f includes a disk of radius $1/4$.

Theorem 1.0.6. ***(Koebe One-Quarter Theorem)*** *[11] Every function f belongs to the class of S i.e. $f \in S$ has an image that includes the disk $z \in \mathbb{C} : z < 1/4$ [14, 15].*

As an additional result of the Bieberbach theorem, the Distortion theorem provides precise upper and lower bounds for $|f'(z)|$ [10].

Theorem 1.0.7. ***(Distortion Theorem)*** *[11] If a function f belongs to the class of S i.e. $f \in S$, then*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}; \quad |z| = r < 1.$$

Growth Theorem can be obtained by applying the Distortion Theorem to determine exact upper and lower bounds for $f(z)$ [10, 14].

Theorem 1.0.8. ***(Growth Theorem)*** *[11] If a function f belongs to the class of S i.e. $f \in S$, then*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}; \quad |z| = r < 1.$$

The Rotation Theorem is another outcome of Bieberbach's theorem..

Theorem 1.0.9. ***(Rotation Theorem)*** *[11] If a function f belongs to the class of S i.e. $f \in S$, then for $|z| = r < 1$,*

$$|argf'(z)| \leq \begin{cases} 4sin^{-1}r, & r \leq 1/\sqrt{2} \\ \pi + log\frac{r^2}{1-r^2}, & r \geq 1/\sqrt{2} \end{cases}$$

The bound is sharp.

Univalency of analytic functions is also studied using the Fekete-Szegö coefficient functional.

Theorem 1.0.10. ***(Fekete-Szegö Theorem)*** *[11] If a function f belongs to the class of S i.e. $f \in S$, then*

$$|a_3 - \alpha a_2^2| \leq 1 + 2e^{-2\alpha/(1-\alpha)}; \quad \alpha \in (0, 1).$$

1.1 Classes of univalent and starlike functions

Consider $\mathcal{S}$ as the sub-class of $\mathcal{A}$ consisting of univalent functions [10]. If $f \in \mathcal{S}$, then the Taylor Series expansion of f is given by [14]:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + a_3 z^3 + \dots. \tag{1.1.1}$$

In the year 1907, Koebe proved that for the class $\mathcal{S}$, there exists an absolute constant $k > 0$ such that boundary of the image $f(\mathbb{D})$ can not be distorted so far as to come within a distance less than k of the origin [14, 15]. In 1916, Bieberbach established the beautiful result that $|a_2| \leq 2$ for every function $f \in \mathcal{S}$ and using this, determined the value of k as 1/4. This shows the geometrical connection of coefficient bounds on the geometry of functions. Shortly, we shall see the importance of coefficients bounds in the concept of Bohr phenomenon. Bieberbach also conjectured that $|a_n| \leq n$. Meanwhile, the validity of this conjecture was found true for many sub-classes of $\mathcal{S}$. In 1925, J. E. Littlewood proved that $|a_n| \leq en$ for all n, showing that Bieberbach conjecture is true up to a factor of $e = 2.718\cdots$ [11,14]. Finally in 1985, Louis De Branges [19] proved this conjecture, by using special functions. Before the proof of Bieberbach conjecture, several sub-classes and other fascinating results appeared to solve it. A systematic study in this direction can be seen in some known standard books. Books by Nehari [22], Pommerenke [33], Goodman [14, 15] these are excellent sources of information on univalent function theory.

Coming back, we first describe a geometrical property, which further leads to an important sub-class of univalent functions.

Definition 1.1.1. (Starlike Function) [10, 11] A domain D is considered *starlike* with respect to a point $w_0 \in D$ if every ray from point w_0 crosses the interior of D in a set that is either a line segment or a ray i.e.

$$(1-t)w + tw_0 \in D; \quad t \in [0, 1].$$

If a function $f(z)$ maps the unit disk $\mathbb{D}$ onto a starlike domain w.r.t a point $w_0 = 0$ [10], we classify $f(z)$ as a *starlike function.*

From an analytical perspective, if $f(z) \in \mathcal{A}$ and $\Re(zf'(z)/f(z)) > 0$ then the function $f(z)$ is starlike with respect to origin. The class of starlike functions represented as $\mathcal{S}^*$ [10] and defined as

$$\mathcal{S}^* := \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \right\}.$$

Definition 1.1.2. (Starlike Function of order α) [10, 11] A function f belongs to class of S i.e. $f \in \mathcal{S}$ is said to be *Starlike of order* α if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (0 \leq \alpha < 1, z \in \mathbb{D}).$$

The class of starlike functions of order α represented as $\mathcal{S}^*(\alpha)$ [10]. If we take $\alpha = 0$ then $\mathcal{S}^*(0) = \mathcal{S}^*$, the class of starlike functions [14].

Definition 1.1.3. (Convex Function) [10, 11] A set D in the plane $\mathbb{C}$ is considered as *convex* if for every pair of points w_1 and w_2, the line connecting w_1 and w_2 lies entirely within in D i.e.

$$tw_1 + (1-t)w_2 \in D; t \in [0, 1].$$

If a function $f(z)$ maps unit disk $\mathbb{D}$ onto a convex domain, then the function $f(z)$ is said to be a *convex function* [14].

From an analytical perspective, if $f(z) \in \mathcal{A}$ and $\Re(1 + zf''(z)/f'(z)) > 0$ then the function $f(z)$ is said to be a convex function [14]. The class of convex functions is represented as $\mathcal{CV}$ or sometimes as $\mathcal{K}$ [10, 14] and defined as

$$\mathcal{CV} := \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\}.$$

It can be seen that for any point in the region $f(\mathbb{D})$, every convex function is starlike, therefore every convex function is a starlike function though the converse might not always be true. For this take the example of $f(z) = z + z^2/2$. Later in 1936, Robertson [34] extended the classes $\mathcal{S}^*$ and $\mathcal{CV}$.

Definition 1.1.4. (Convex Function of order α) [10, 11] A function f belongs to class of S i.e. $f \in \mathcal{S}$ is said to be *Convex function of order* α if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (0 \leq \alpha < 1, z \in \mathbb{D}).$$

The class of Convex function of order α is denoted by $\mathcal{CV}(\alpha)$ [14]. If we take $\alpha = 0$ then $\mathcal{CV}(0) = \mathcal{CV}$, the class of convex functions.

The well-known Alexander's transformation given below establishes a two-way bridge between the classes $\mathcal{CV}(\alpha)$ and $\mathcal{S}^*(\alpha)$ [11, 14] defined as

$$f \in \mathcal{CV}(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{S}^*(\alpha).$$

Definition 1.1.5. Bounded turning [10, 11] A function f belongs to the class of S i.e. $f \in \mathcal{S}$ is said to be bounded turning, $\mathcal{R}$ if and only if

$$\left\{ f'(z) \prec \frac{1+z}{1-z}; \quad z \in \mathbb{D} \right\}.$$

A significant connection between convex and starlike functions was initially identified by Alexander in 1915 and subsequently recognized as Alexander's theorem [14].

Theorem 1.1.6. ***(Alexander's theorem)*** *[11] Let f belongs to the class of S i.e. $f \in \mathcal{S}$. Then $f \in \mathcal{CV}$ if and only if $zf' \in \mathcal{S}^*$.*

From this theorem we can directly prove that $f \in \mathcal{CV}^*(\alpha)$ if and only if $zf' \in \mathcal{S}^*(\alpha)$ [14].

The class of close-to-convex functions, which Kaplan established in 1952, is another subclass of S that is essential to the study of univalent functions.

Definition 1.1.7. (Close-to-convex) [11] A function f belongs to the class of $\mathcal{A}$ i.e. $f \in \mathcal{A}$ is close-to-convex in a unit disk $\mathbb{D}$ if there $\exists$ a convex function g and a real number $\theta \in (-\pi/2, \pi/2)$, such that

$$\Re\left(e^{i\theta}\frac{f'(z)}{g'(z)}\right) > 0; \quad z \in \mathbb{D}.$$

The class of all close-to-convex functions is represented as $\mathcal{CCV}$. The sub-classes of $\mathcal{S}$ namely convex, starlike and close-to-convex functions are interconnected [11, 14]in the following manner:

$$\mathcal{CV} \subset \mathcal{S}^* \subset \mathcal{CCV} \subset \mathcal{S}$$

The renowned Noshiro-Warschawski theorem asserts that a function $f \in \mathcal{A}$ with positive derivative in $\mathbb{D}$ is univalent [14].

Theorem 1.1.8. ***(Noshiro-Warschawski theorem)*** *[18] For a real value of α ($\alpha \in \mathbb{R}$), if a function f is analytic in a convex domain D and it satisfies*

$$\Re\left(e^{i\alpha} f'(z)\right) > 0,$$

then f is univalent in D.

Kaplan utilized the Noshiro-Warschawski theorem to demonstrate that every close-to-convex function is univalent [14, 18].

Definition 1.1.9. (Class of starlike functions wrt to symmetric points) [10] A function f is said to be in the class of starlike function w.r.t. ssymmetric points if for every Y on the circle $|z| < 1$, at Y, the angular velocity of $f(z)$ around the point $f(-Y)$ is positive as z moves in a positive direction along the circle $|z| < r$, i.e.,

$$Re\left(\frac{2zf(z)}{(f(z) - f(-Y))}\right) > 0; \ |z| = r, \ z = Y. \tag{1.1.2}$$

and we represent this class as $\mathcal{S}_s^*$. Sakaguchi [38] established and studied $\mathcal{S}_s^*$ in 1959. This class is made up of functions that are starlike with respect to symmetric points [10] and are distinguished by the following

$$Re\left(\frac{2zf(z)}{f(z) - f(-z)}\right) > 0; \ z \in \mathbb{D}. \tag{1.1.3}$$

Definition 1.1.10. (Carathéodory class) [10, 27] An analytic function $p(z)$ in an open disk $\mathbb{D}$ is said to be in the Carathéodory class $\mathcal{P}$, if it satisfies

$$p(0) = 1 \text{ and } \Re p(z) > 0$$

and $p(z)$ is represented as: $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$.

A function in $\mathcal{P}$ is referred to as a function with positive real part, also recognized as Carathéodory function [11, 14]. The subsequent lemma is well known for functions in $\mathcal{P}$.

Thus, the classes $\mathcal{P}$ and $\mathcal{S}^*$ can now be related to each other as follows [10, 14]:

$$f \in \mathcal{S}^* \Longleftrightarrow \frac{zf'(z)}{f(z)} \in \mathcal{P}.$$

Lemma 1.1.11. *If a function p belongs to the Carathéodory class [14] i.e. $p \in \mathcal{P}$ is defined by the following series*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots .$$

then the subsequent precise estimation is valid:

$$|p_n| \leq 2; \ n \in \mathbb{N}.$$

For $\alpha \in [0, 1)$, the class of analytic functions $p \in \mathcal{P}$ [14] with

$$\Re p(z) > \alpha; \; z \in \mathbb{D}.$$

represented by $\mathcal{P}(\alpha)$. In the context of subordination, the analytic criterion for the function $p(z)$ with positive real part can be expressed as

$$p(z) \prec \frac{1+z}{1-z}; \quad z \in \mathbb{D}$$

[10,11]. This occurs because the function $q(z) = (1+z)/(1-z)$ maps $\mathbb{D}$ onto the right-half plane [14].
Ma and Minda have provided a comprehensive approach to various sub-classes containing starlike and convex functions by substituting the superordinate function $q(z) = (1+z)/(1-z)$ by a more general analytic function [11, 14]. Beacause of this reason, they examined an analytic function φ with positive real part on $\mathbb{D}$ where $\varphi(0) = 1, \varphi'(0) > 0$ and φ maps symmetric with respect to the real axis, $\mathbb{D}$ onto a region starlike with respect to 1, The class of Ma-Minda starlike functions is represented by $\mathcal{S}^*(\varphi)$ containing of functions $f \in \mathcal{A}$ which satisfies

$$\frac{zf'(z)}{f(z)} \prec \varphi(z); \quad z \in \mathbb{D}$$

and likewise the class of Ma-Minda convex functions represented by $\mathcal{CV}(\varphi)$ containing of functions $f \in \mathcal{A}$ satisfies the subordination

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z); \quad z \in \mathbb{D}.$$

Definition 1.1.12. (Subordination) [10] An analytic function f is subordinate to another analytic function g, denoted by $f \prec g$, if there is an analytic function w with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. Further, If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$ [11].

The basic defnitions and theorems in the theory of subordination and certain applications of differential subordinations are stated in this section [10]. The theory of differential subordination were developed by Miller and Mocanu [25].

let $\eta(r, s, t; z) : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ and h be univalent in $\mathbb{D}$. If p is analytic in $\mathbb{D}$ and satisfies the second order differential subordination

$$\eta(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.1.4}$$

then p is called a solution of the differential subordination [25]. The univalent function q is called a dominant of the solution of the differential subordination or more simply dominant, if $p \prec q$ for all satisfying above condition (1.1.4) [25]. A dominant q_1 satisfying $q_1 \prec q$ for all dominant q of (1.1.4) is said to be best dominant of (1.1.4). The best dominant is unique up to a rotation of $\mathbb{D}$ [25].

Definition 1.1.13. (Convolution of Functions) [10, 11]

Let f and g be functions belonging to $L(-\infty, +\infty)$. The function h defined by the convolution [15] of f and g is given by:

$$h(x) = \int_{-\infty}^{+\infty} f(x-y)g(y)\,dy = \int_{-\infty}^{+\infty} f(y)g(x-y)\,dy.$$

This operation is denoted by the symbol $f * g$ and is defined almost everywhere and also belongs to $L(-\infty, +\infty)$ [11, 25].

The convolution has the basic properties of multiplication [15, 25], namely:

$$f * g = g * f,$$

$$(\alpha_1 f_1 + \alpha_2 f_2) * g = \alpha_1 (f_1 * g) + \alpha_2 (f_2 * g), \quad \alpha_1, \alpha_2 \in \mathbb{R},$$
$$(f * g) * h = f * (g * h).$$

The convolution of generalized functions also has the commutativity property and is linear in each argument; it is associative if at least two of the three generalized functions have compact supports [25].

Chapter 2

Hankel-determinant

This chapter explores the definitions of the Hankel- and Toeplitz-determinants as well as how they differ across various analytic function sub-classes. It also looks at how these determinants alter when logarithmic and inverse logarithmic coefficients are used in lieu of their original entries. In addition, the discussion of Lie groups and algebra pertains to the features of Hankel and Toeplitz-matrices.

The rationality of a function with restricted characteristic in $\mathbb{D}$, or a function that is the ratio of two bounded analytic functions with integral coefficients in its Laurent series around the origin, may be shown, for example, using Hankel-determinant [10, 31]. Regarding the application of Hankel-determinant to meromorphic function analysis we can refer to [43]. Several authors discussed Hankel-determinant $H_{q,n}(f)$, in Fekete-Szegö problem, they discussed for $q = 2$ and $n = 1$ i.e., $H_{2,1} = a_3 - a_2^2$, which is further generalised to $H_{2,1} = a_3 - \mu a_2^2$ where $\mu \in \mathbb{C}$. Pommerenke's study [31] states that the Hankel-determinant of univalent functions must satisfy the subsequent conditions: $|H_{q,n}(f)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}}$, $\beta > 1/4000$, where k only relies on q. Later, Hayman [16] proved that $|H_2(n)| < A_n^{1/2} (n = 1, 2, \ldots)$; an absolute constant for areally mean univalent functions. In 1967, Pommerenke [32] investigated the Hankel-determinant of areally mean p-valent functions, univalent functions, and starlike functions. ElHosh discovered bounds on the Hankel-determinant of univalent functions with a positive Hayman index α and $k-$fold symmetric, close-to-convex functions [11].

The q_{th} Hankel-determinant for $q, n \in \mathbb{N}$, or $H_{q,n}(f)$, for a function $f \in \mathcal{A}$ is as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix} \tag{2.0.1}$$

The Fekete–Szegö problem is regarded as one of the most significant results concerning univalent functions [17], [26], [20]. It relates to the coefficients of a function's Taylor series [14] and was introduced by Fekete and Szegö [20]. In Fekete-Szegö problem the optimization of the absolute value of the functional $a_3 - \mu a_2^2$ is our goal. Numerous researchers have carefully examined and analyzed this outcome. For the Koebe function, the equality is valid. Keogh and Merkes [13] discovered the sharp upper bound of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ in 1969 for a few univalent function sub-classes.

The Fekete–Szegö functional is obtained for $q = 2$ and $n = 1$ in (2.0.1);

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2.$$

Further, sharp bounds for the functional $|a_2a_4 - a_3^2|$ are obtained in (2.0.1) for $q = 2$ and $n = 2$, the second order Hankel-determinant [32]:

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

Many writers have concentrated their attention in recent years on estimating an upper bound for $|H_{2,2}(f)|$. The precise approximations of $|H_{2,2}(f)|$ for the family of univalent functions, represented by $\mathcal{R}, \mathcal{S}^*$, and $\mathcal{K}$, respectively, are *bounded turning*, *starlike*, and *convex.*

It was demonstrated by Ye and Lim in 2016 that every $n \times n$ matrix over $\mathcal{C}$ may be expressed generically as the product of certain Toeplitz- or Hankel-matrices. With their many applications, Hankel-matrices and determinants are fundamental elements in many mathematical fields [8, 32]. Toeplitz-matrices and Toeplitz-determinants are also important in practical and pure mathematics. They can be found in a variety of fields, including analysis, quantum physics, image processing, integral equations, and signal processing. The survey paper and its references are consulted for additional applications. The diagonals of Toeplitz-matrices have identical elements.

For a function $f \in \mathcal{A}$, the q_{th} Toeplitz-determinant, $T_{q,n}(f)$, as:

$$T_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n)} \end{vmatrix}$$

Bieberbach approximated $H_{2,1}(f)$ for the class $\mathcal{S}$ [31, 32]. For $f \in \mathcal{A}$, the q_{th} Hankel-determinant $H_{q,n}(F_f)$, where $q, n \in \mathbb{N}$, and entries are logarithmic coefficients [27](refer Chapter 4) is expressed as:

$$H_{q,n}(F_f) = \begin{vmatrix} Y_n & Y_{n+1} & \cdots & Y_{n+q-1} \\ Y_{n+1} & Y_{n+2} & \cdots & Y_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n+q-1} & Y_{n+q} & \cdots & Y_{n+2(q-1)} \end{vmatrix}$$

and $T_{q,n}(F_f)$ having logarithmic coefficients as:

$$T_{q,n}(F_f) = \begin{vmatrix} Y_n & Y_{n+1} & \cdots & Y_{n+q-1} \\ Y_{n+1} & Y_n & \cdots & Y_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n+q-1} & Y_{n+q-2} & \cdots & Y_n \end{vmatrix}$$

Kowalczyk *et al.* [31] studied about the Hankel-determinant with entries of logarithmic coefficients. In this we're going to study about $H_{2,1}(F_f)$ which can be find this with the help of $H_{2,1}(f) = a_2a_4 - a_3^2$, where $f \in \mathcal{S}$ (following logarithmic function methodolgy).

Realizing the wide use of these coefficients, Kowalczyk and Lecko recently suggested a Hankel-determinant whose constituents are the logarithmic coefficients of $f \in \mathcal{S}$ [27, 31]. Motivated by these concepts, we begin the study of the Hankel-determinant $H_{q,n}(F_{f^{-1}}/2)$ and the Toeplitz-determinant $T_{q,n}(F_{f^{-1}}/2)$, in which the logarithmic coefficients of inverse functions [37] of $f^{-1} \in \mathcal{S}$ are the elements. (see Chapter 4).
The determinant $H_{q,n}(F_{f^{-1}}/2)$ is expressed as follows:

$$H_{q,n}(F_{f^{-1}}/2) = \begin{vmatrix} Y_n & Y_{n+1} & \dots & Y_{n+q-1} \\ Y_{n+1} & Y_{n+2} & \dots & Y_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ Y_{n+q-1} & Y_{n+q} & \dots & Y_{n+2(q-1)} \end{vmatrix}$$

and, the determinant $T_{q,n}(F_{f^{-1}}/2)$ is expressed as follows:

$$T_{q,n}(F_{f^{-1}}/2) = \begin{vmatrix} Y_n & Y_{n+1} & \dots & Y_{n+q-1} \\ Y_{n+1} & Y_n & \dots & Y_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ Y_{n+q-1} & Y_{n+q-2} & \dots & Y_n \end{vmatrix}$$

Extensive research has been conducted on the topic of Hankel and Toeplitz-determinants for various function classes, including starlike, convex, and others, resulting in the establishment of sharp bounds [28, 31], and [9]. More recently, investigations have focused on Hankel-determinants with logarithmic coefficients for specific sub-classes of starlike, convex, close-to-convex, univalent, strongly starlike, and strongly convex functions [10]. Notwithstanding the importance of these problems, the sharp bounds of Toeplitz-determinants with logarithmic coefficients for inverse functions remain unexplored. Some progress has been made in this area, particularly with regard to Toeplitz-determinants for univalent functions exhibiting specific symmetries or confined to particular domains. A notable contribution was made in 2021 by Zaprawa, who obtained the sharp bounds for the initial logarithmic coefficients Y_n for functions belonging to the classes $\mathcal{S}_S^*$ and $\mathcal{K}_S$ [11, 27].

With a wide range of useful applications, Toeplitz-matrices and their accompanying determinants have a prominent place in many mathematical fields [8]. For those interested in a thorough summary of the various uses of Toeplitz-matrices in practical and pure mathematics, the scholarly survey paper by Ye and Lim [44] is a good resource.

2.1 Hankel and Toeplitz-Matrix

Definition 2.1.1. [12] A *Hankel-matrix* is a particular kind of matrix in which every element is the same along a line that runs parallel to the main anti-diagonal. Alternatively, if and only if a series of values $s_1, s_2, \ldots$, exists such that each element$h_{i,j}$ is equal to s_{i+j-1}, where i and j are positive integers, then a matrix $H = (h_{i,j})$ can be defined as a Hankel-matrix. A block Hankel-matrix is the matrix H that results when the sequence s_k is made up of square matrices. Moreover, the representation of Hankel operators, which work on the Hilbert space of complex sequences that are square summable, is intimately associated with infinite Hankel-matrices.

Mathematically, A square matrix with constant skew-diagonals, denoted as $b_{i,j} = b_{i+1,j-1}$ $\forall \;\; 1 \leq i,j \leq n$, is known as a *Hankel-matrix*. The space of Hankel-matrices of size $n \times n$ can be parameterized as follows by $(b_1, b_2, \ldots, b_{2n-1}) \in \mathbb{R}^{2n-1}$:

$$b(b) = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ b_2 & b_3 & b_4 & \cdots & b_{n+1} \\ b_3 & b_4 & b_5 & \cdots & b_{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ b_n & b_{n+1} & b_{n+2} & \cdots & b_{2n-1} \end{bmatrix}$$

Hankel-matrices are often used in situations where very efficient numerical solution methods are created by taking advantage of the intimate relationship between matrix and polynomial calculations. The Hilbert-Hankel operator, defined as $s_{i+j-1} = (i+j-1)^{-1}$, and the Hilbert matrix are well-known examples of special instances. These operations are crucial to the study of the spectral characteristics of integral operators of Carleman type [11] [16].

2.1.1 Topology of Hankel-matrix

Definition 2.1.2. When a group G is both a differentiable manifold and its group operations (multiplication and inversion) are smooth mappings on G, the group is called a *Lie group* [12, 32].

Definition 2.1.3. The following conditions of a *Lie algebra* are satisfied by a vector space v over a field $\mathbb{F}$ that is furnished with a bilinear operation $[.,.] : v \times v \to v$, also known as the *Lie bracket* [12, 32]: $\forall l, m, n \in v$ and $\alpha_1, \alpha_2 \in F$

- Skew-symmetry: $[l, m] = -[m, l]$.
- Jacobi identity: $[l, [m, n]] + [m, [n, l]] + [n, [l, m]] = 0$.
- Bilinearity: $\alpha_1 l + \alpha_2 m, n] = \alpha_1 [l, n] + \alpha_2 [m, n] and [n, \alpha_1 l + \alpha_2 m] = \alpha_1 [n, l] + \alpha_2 [n, m]$.

Theorem 2.1.4. *Over $\mathbf{R}^+$, the set of Hankel-matrices under the Hadamard product forms a Lie group [12].*

Theorem 2.1.5. *With dimension $2n-1$, the set $\mathcal{H}$ of Hankel-matrices over $\mathbf{R}^+$ is an open connected differentiable manifold [12].*

(Which can be easily shown by proofing that$\mathcal{H}$ is smooth submainfold of $\mathbb{R} \times \mathbb{R}$, its tangent space at any point will have $2n-1$ dimension and lastly by showing connectedness and open nature of $\mathcal{H}$).

Corollary 2.1.6. *Over the Hadamard product, the set of Hankel-matrices over $\mathbf{R}^+$ is unbounded [12].*

Corollary 2.1.7. *Over the Hadamard product, the set of Hankel-matrices over $\mathbf{R}^+$ is not compact [12].*

Theorem 2.1.8. *The vector space of Hankel-matrices $\mathcal{H}_n$ over $\mathbb{R}$ is isomorphic to the tangent space at I_n of $\mathcal{H}_n$, $\mathcal{H}_n(\mathcal{H}_n)$ [31]. The commutator operation $[A, B] = A \circ B - B \circ A = 0$ is a $2n-1$ dimensional Lie algebra. $\circ$ denotes Hadamard pairwise component product [12].*

Rebuilding a quantum state from measurements is the goal of quantum state tomography. Understanding the interrelations and symmetries of these observations in quantum mechanics depends on knowledge of the Lie group and algebra structure of certain transformation groups [12].

Definition 2.1.9. An infinite matrix $(\omega_{mk}), m, k = 1, 2, \ldots$ satisfying the following conditions is called *Toeplitz-matrix* (T - matrix):

$$\sum_{k=1}^{\infty} |\omega_{mk}| \leq L, \quad m = 1, 2, \ldots$$

where L have no relation with m;

$$\lim_{m \to \infty} \omega_{mk} = 0, \quad k = 1, 2, \ldots;$$

$$\lim_{m \to \infty} \sum_{k=1}^{\infty} \omega_{mk} = 1.$$

The matrix summation technique described by transmitting a sequence $\{v_m\}$ to a sequence $\{\tau_m\}$ via the matrix (ω_{mk}) is considered regular if these requirements are satisfied;

$$\tau_m = \sum_{k=1}^{\infty} \omega_{mk} v_k.$$

O. Toeplitz demonstrated the adequacy and need of these requirements for regularity in the context of triangular matrices.

When referring to (finite or infinite) matrices (ω_{jk}) with the feature that ω_{jk} relies exclusively on the difference $j - k$, i.e., $\omega_{jk} = \beta_{j-k} \forall j$ and k, the phrase *Toeplitz-matrix* is also used in the literature. Similar to the Hankel-matrix, we may examine the relationships and characteristics between Toeplitz and Hankel-matrices. Applications of finite Toeplitz-matrices in systems theory, signal processing, and statistics are significant.

2.2 Hankel-determinant for sub-class of analytic funtions

In this section we will look up to the variations in the Hankel-determinant wrt to the sub-classes of analytic funtions which are $M(\alpha)$ and $M_\lambda(\eta, \phi_{\eta,m})$.

- [9] For $z \in \mathbb{D}$ and $0 \le \alpha < 1$, let $f \in A$ be locally univalent, if and only if

$$Re\left(\frac{(1-z^2)f(z)}{z}\right) > \alpha, \quad z \in \mathbb{D},$$

then $f \in M(\alpha)$.

This class has a great influence on the theory of geometric functions because of its geometrical features [9]. For each w_1, $w_2 \in f(\mathbb{D})$, a function $f \in M(\alpha)$ maps univalently $\mathbb{D}$ onto a domain $f(\mathbb{D})$ convex in the imaginary axis direction [36]. $Re(w_1) = Re(w_2)$, where $[w_1, w_2]$ is the line segment.lies in $f(\mathbb{D})$, with the extra property that there are two points on the border of $f(\mathbb{D})$, namely $\{w_1 + it : t > 0\} \subset C\backslash f(\mathbb{D})$ and $\{w_2 - it : t > 0\} \subset C\backslash f(\mathbb{D})$ [1].

Theorem 2.2.1. *If $f \in M(\alpha)$, $0 \le \alpha < 1$, then*

$$H_2(2)| \le \frac{4(1-\alpha)(64-37\alpha)+27}{27}.$$

Also the above inequality is sharp.

Corollary 2.2.2. *If $f \in M(1/2)$, then*

$$H_2(2)| \le \frac{118}{27} \approx 4.3703.$$

Also the above inequality is sharp.

-

Definition 2.2.3. Let $\eta \in \mathbb{C} \setminus \{0\}$ and the class $M_\lambda(\eta, \phi_{\eta,m})$ denote the sub-class of $\mathcal{A}_p$ [11], consisting of functions f of the form (1.1), and satisfying the following subordination condition:

$$\frac{1 + \frac{1}{\eta}\left(zf'(z) + \lambda z^2 f''(z)\right)}{1 - \lambda\left(f(z) + \lambda z f'(z)\right)} - 1 \le \phi_{\eta,m}$$

for $0 \le \lambda \le 1$ and $\phi_{\eta,m}$ is a simple logistic Sigmoid activation function [26].

Definition 2.2.4. Let $\eta \in C \setminus \{0\}$ and the class $M_{1(*)}(\eta, \phi_{n,m})$ denote the sub-class of $\mathcal{A}_p$ [11], consisting of functions f of the form (1.1), and satisfying the following subordination condition:

$$\frac{1 + \frac{1}{\eta}\left(z(f*g)'(z) + \lambda z^2 (f*g)''(z)\right)}{(1-\lambda)(f*g)(z) + \lambda (f*g)'(z)} - 1 < \phi_{n,m} = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1}{2^m} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n!} z^{-n}$$

for $0 \le \lambda \le 1$ and $\phi_{n,m}$ is a simple logistic Sigmoid activation function [26].

Theorem 2.2.5. *Let*

$$\phi_{n,m}(z) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1}{2^m} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n!} z^{-n}$$

, where $\phi_{n,m}(z) \in A$ is a modified logistic Sigmoid activation function and $\phi_{n,m}(0) > 0$. If $F(z) = (f * g)(z)$ given by (1.1) belongs to the class $M_{\lambda,(*)}(\eta, \phi_{n,m})$ [26], then,

$$a_{p+1} + b_{p+1} = \frac{(1-\lambda+\lambda p)\eta}{2p(1+\lambda p)}$$

$$a_{p+2}b_{p+2} = \frac{(1-\lambda+\lambda p)^2\eta^2}{4p(p+1)(1+\lambda(p+1))}$$

$$a_{p+3}b_{p+3} = \frac{\eta(1-\lambda+\lambda p)(3\eta^2 - p(\eta+1))}{24p(p+1)(p+2)(1+\lambda(p+2))}$$

Corollary 2.2.6. *For coefficient a_{p+1}, b_{p+1},*

$$|a_{p+1}b_{p+1}| = \frac{(1-\lambda+\lambda p)|\eta|}{2p(1+\lambda p)}$$

is written and since $\phi(\lambda) = \frac{(1-\lambda+\lambda p)}{(1+\lambda p)}$, $\phi'(\lambda) < 0$ in the interval $0 \le \lambda \le 1$ and $\phi(\lambda)$ is decreasing, it will be

$$\frac{|\eta|}{2(p+1)} \le |a_{p+1}b_{p+1}| \le \frac{|\eta|}{2p}$$

for $\frac{1}{2} \le \frac{(1-\lambda+\lambda p)}{(1+\lambda p)} \le 1$ [26].

Similarly, since the coefficients $a_{p+1}b_{p+1}$, $a_{p+2}b_{p+2}$ [26] and $a_{p+3}b_{p+3}$ depend on λ and are decreasing with respect to λ, the following inequalities can be written easily:

$$\frac{|\eta|^2}{4(p+1)(p+2)} \le |a_{p+2}b_{p+2}| \le \frac{|\eta|^2}{4p(p+1)}$$

$$\left|\frac{(\eta^3 - p(p+1)\eta)}{24(p+1)(p+2)(p+3)}\right| \le |a_{p+3}b_{p+3}| \le \left|\frac{(\eta^3 - p(p+1)\eta)}{24p(p+1)(p+2)}\right|$$

Similarly, we can derive analogous results for the Toeplitz-determinant.

Chapter 3

Petal-shaped Domain

The chapter focuses on the introduction of the petal-shaped domain and its distinctive geometric properties and results. Additionally, sharp coefficient bounds are examined. The application of the petal-shaped domain is explored to enhance understanding through practical examples.

A family of star-like functions connected with the petal-shaped domain $\rho(\mathbb{D})$ was introduced by Arora and Kumar [6]. They are stated as follows [21]:

$$\mathcal{S}_\rho^* = \{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \sinh^{-1}(z), z \in \mathbb{D}\}. \tag{3.0.1}$$

Clearly, with its branch cuts around the line segments $(-i\infty, -i) \cup (i, i\infty)$ on the imaginary axis, the function ρ is obviously a multivalued function and holomorphic in $\mathbb{D}$ [2]. $\Omega_\rho := \{w \in \mathbb{C} : |sinh(w-1)| < 1\}$ [21, 42], the petal-shaped area that defines our class, characterized as the Ma-Minda function $\rho(z) = 1 + sinh^{-1}(z)$. An analytic univalent function f is said to be a Ma-Minda function with $f'(0) > 0$ such that $Ref(x) > 0$ where $x \in \mathbb{D}$ and $f(x)$ symmetric about the real axis and starlike with respect to $f(0) = 1$ [36].
Refer Fig. 1 for better understanding of the domain of $\rho(\mathbb{D})$.

From the definition of the class of starlike functions associated with the petal-shaped domain, we concluded that $f \in \mathcal{S}_\rho^*$ if and only if we obtain a regular function $r(z)$ where $r(z) \prec \rho(z)$ [42] such that

$$f(z) = z\, exp\left(\int_0^z \frac{r(t)-1}{t} dt\right). \tag{3.0.2}$$

As ρ is observed to be univalent in $\mathcal{D}$, $r_k(\lceil) \subset \rho(\mathcal{D})$ and $q_k(0) = \rho(0)$ for $k = 1, 2, 3$, then it follows that for each k, we get $q_i \prec \rho$. Therefore, $f_k(z) \in \mathcal{S}_\rho^*$ where $f_k(z)$ is obtained using (4.0.2) [21].

Now after taking $r(z) = 1 + sinh^{-1}(z)$ we conclude the following (4.0.2) [40] follows:

$$f_o(z) = z\, exp\left(\int_0^z \frac{1 + sinh^{-1}(t)}{t} dt\right) = z + z^2 + \frac{1}{2}z^3 + \frac{1}{9}z^4 - \frac{1}{72}z^5 - \frac{1}{225}z^6 + \ldots, \tag{3.0.3}$$

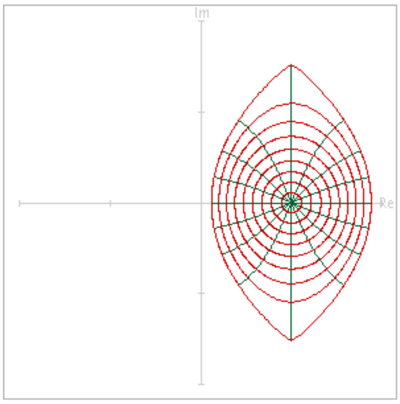


Figure 3.1: Petal shaped domain $\rho(\mathbb{D})$

the function mentioned above serve as extremal function for the class S^*_ρ helping in obtaining the sharp results.

We know that $sinh^{-1} = ln(\phi_c(z))$, where $\phi_c(z) = z + \sqrt{1+z^2}$. So, if we assume that $f \in \mathcal{S}^*_\rho$ satisfies $w = zf'(z)/f(z)$.Next, we note that $\mathcal{S}^*_\rho$ can also be expressed as $exp(w-1) \prec \phi_c(z)$. Here, $\phi_c(z)$ denotes the Crescent-shaped domain, bounded by two circular arcs with common end-points, $\Gamma_1 \subset \mathcal{T}(1,\sqrt{2})$, and $\Gamma_2 \subset \mathcal{T}(-1,\sqrt{2})$, as well as i and $-i$. Within the closed right half-plane lie Γ_1 and Γ_2. Starlike with respect to 1, $\phi_c(\mathcal{D}$ is a symmetric set with respect to the real axis [36]. Therefore, an exponential relation can be between the classes S^*_ρ and Δ^*. For a more comprehensive understanding of the research on crescent-shaped domains related to higher order starlike functions, one can refer to [5].

3.0.1 Certain properties around $\rho(z)$:

The following section outlines some observed geometric property of $1 + sinh^{-1}z$ [6,21]:

1. The function $\rho(z)$ is a convex univalent function, which can shown easily using the concept of subordination around the Carathéodory class.
2. The domain $\Omega_\rho = \rho(\mathcal{D})$, where $\rho'(0) > 0$ and $\phi_c(\overline{z}) = \overline{\phi_c(z)}$ is symmetric about the real axis. Further we obtain $\rho(\overline{z}) = \overline{\rho(z)}$.
3. The line $Re(w) = 1$ serves as the center of symmetry in the domain Ω_ρ.
4. Within $\rho(|z| \leq r)$, there is a maximal disk contained i.e., $\{w : |w-1| \leq sinh^{-1}(r)\}$.
5. $\rho(-r) \leq Re\rho(z) \leq \rho(r)$; $(|z| \leq r < 1)$.
6. $|Im\rho(z)| \leq 2$, where $(|z| \leq 1)$.
7. $\rho(-r) \leq |\rho(z)| \leq \rho(r)$, where $(|z| \leq r < 1)$.
8. $|arg(\rho(z))| \leq tan^{-1}(1/t)$, where $t = 4/\pi\sqrt{sinh^{-1}(1)(1 - sinh^{-1}(1))}$.

3.0.2 Inclusion Relations

Researchers are usually drawn to the use of special functions in geometric function theory because they have a wide range of applications in analytic univalent functions. Some researchers introduce operators (e.g., Carlson-Shaffer operator, Hohlov operator, Dziok-Srivastava operator [15, 41]) and get intriguing results by employing specific functions. Driven by the same, we derive several *inclusion relations* that correspond to the classes $M(\beta)$, $k-ST$, and S_p^*.

The expression

$$M(\beta) := \left\{ f \in A : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta, z \in \mathbb{D}, \beta > 1 \right\},$$

describes the class $M(\beta)$, which was initially examined by Uralegaddi et al [2]. Additionally, Kanas and Wiśniowska created the $k-ST$ class of k-starlike functions [4], which may be stated as follows:

$$k - ST := \left\{ f \in A : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, z \in \mathbb{D}, k \geq 0 \right\}.$$

Theorem 3.0.1. *The following relationships are satisfied by the class S_p^* [6, 21]:*

1. $S_p^* \subset S^* \cap C^*;\ \ 0 \leq \alpha \leq 1 - \sinh^{-1}(1);$
2. $S_p^* \subset M(\beta);\ \ \beta \geq 1 + \sinh^{-1}(1);$
3. $S_p^* \subset CS^*(\gamma);\ \ (2/\pi)\tan^{-1}(1/t) \leq \gamma \leq 1,\ \ t = \frac{4}{\pi}\sinh^{-1}(1)(1 - \sinh^{-1}(1));$
4. $k - ST \subset S_p^*;\ \ k \geq 1 + \sinh^{-1}(1).$

3.0.3 Sharp Coefficient Problems for the Class S_ρ^*

According to Kumar and Verma's 2022 [6] discussion of coefficient difficulties, there *exists* a Schwarz function $w(z) = \sum_{n=1}^{\infty} w_n z^n$ where $f \in \mathcal{S}_\rho^*$, such that

$$\frac{zf'(z)}{f(z)} = 1 + sinh^{-1}(w(z)). \tag{3.0.4}$$

Now let us assume

$$(p(z) + 1)w(z) = (p(z) - 1)$$

where

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}.$$

In (3.0.4) comparing the corresponding coefficients of the equation after putting the values of $p(z)$, $w(z)$ and $f(z)$, we get the following relation between a_n and p_n [21]:

$$a_2 = \frac{1}{2}p_1, \quad a_3 = \frac{1}{4}p_2, \quad a_4 = \frac{1}{144}\left(-p_1^3 - 6p_1p_2 + 24p_3\right) \tag{3.0.5}$$

$$a_5 = \frac{1}{1152}\left(5p_1^4 - 6p_1^2p_2 - 36p_2^2 - 48p_1p_3 + 144p_4\right) \tag{3.0.6}$$

$$a_6 = \frac{-54p_1^5 + 355p_1^3p_2 + 150p_1p_2^2 - 1680p_2p_3 - 1080p_1p_4 + 2880p_5}{28800} \tag{3.0.7}$$

$$a_7 = \frac{1}{2073600}\left(1031p_1^6 - 17220p_1^4p_2 + 26100p_1^2p_2^2 + 9000p_2^3 + 19200p_1^3p_3 + 33120p_1p_2p_3\right. \tag{3.0.8}$$

$$\left. -57600p_3^2 + 4320p_1^2p_4 - 108000p_2p_4 - 69120p_1p_5\right) \tag{3.0.9}$$

Theorem 3.0.2. *Let $f(z)$ be an analytic function $\in S_\rho^*$, then the sharpness of following bounds can been seen $|a2| \leq 1$, $|a3| \leq 1/2$, $|a4| \leq 1/3$, and $|a5| \leq 907/1632 \approx 0.55576$.*

3.0.4 Application

Petal-shaped domains have far-reaching applications across various fields, from statistical mechanics, probability theory, dynamical systems, quantum field theory,signal processing, image detection and processing. These domains help model, analyze, and understand complex systems and their behaviors in real-world scenarios.

Image edge detection

A key component in the creation of complex analysis is GFT. While many scholars have examined the geometrical characteristics of different sub-classes of analytic functions, very few have examined the use of GFT in image processing [29]. Therefore, this work aims to develop a novel approach for improving edge recognition in pictures by utilizing the coefficients found for the $\mathcal{SC}^{t,\rho}$ sub-class (CSKP model) [29].Five quality metrics—contrast, correlation, energy, homogeneity, and entropy have been applied in our method. We have displayed the metrics values for a number of photographs using these quality measures [29]:

1. A measurement of the minute differences that comprise an image is called *Contrast*.
2. *Correlation* examines the linearity of the picture.
3. The GLCM's squared element sum is measured by *Energy*.
4. The co-occurrence matrix of a homogeneous picture, or *Homogeneity*, will have both broad and narrow $k[i, j]$ values.
5. The unit of measurement for information content is *Entropy*. It measures the volatility of the intensity distribution.

Such a matrix, where the distance vector d contains no preferred pairs of gray levels, is used to represent an image [29] . When the magnitudes of all the $k[i,j]$ entries are the same, entropy is at its highest, and when the magnitudes of the $k[i,j]$ entries differ, it is at its lowest. One method for locating the picture edges needed to calculate the approximate absolute gradient magnitude at each location in a grayscale input image is edge detection [29]. The technique being employed makes it difficult to determine the proper absolute gradient magnitude for edges. The Sobel measures a two-dimensional spatial gradient applied to images.A statistically uncorrelated dataset is used to transmit a two-dimensional pixel array in order to minimize the amount of data needed to represent a digital image. This edge detector uses two 3×3 convolution masks, one for estimating gradients in the x-orientation and one for estimating gradients in the y-orientation [29]. The Sobel detector's great sensitivity to noise allows it to efficiently show noise as edges in photographs. As such, this operator is suggested in talks involving huge volumes of data that are detected during data transfer [29].

The image is convolued with each of the masks.At every pixel point, there are two numbers: $Q1$ and $Q2$, which stand for the row's and column's respective outputs from the mask. After applying equations (3.0.10) and (3.0.11) to those values, two matrices—the edge magnitude and orientation—are calculated:

$$Edge\ magnitude = \sqrt{Q_1^2 + Q_2^2} \tag{3.0.10}$$

$$Edge\ direction = tan^{-1}\frac{Q_1}{Q_2} \tag{3.0.11}$$

We have developed a novel edge detection improvement technique in this work by utilizing the ideas of convolution, kernels, and coefficient limits. To improve the outcomes, the parameter values were changed [29]. It is clear from a comparison of the other outcomes that the suggested strategy yields satisfactory results. The suggested method's limitation is that it won't yield a better image if we select an image with a lot of noise. We want to address this weakness in the future and create a modified edge detecting method.

Borel distribution

Numerous academics have thoroughly examined distributions including the binomial, Poisson, Pascal, logarithm, and hyper-geometric, as well as their applications to the class of univalent functions, from a variety of angles [30]. The application of the Borel distribution to the function class $\mathcal{RK}_{sinh}(\beta)$ outcomes is now covered [30].

Definition 3.0.3. [30] Let Δ be a unit disk and $0 \leq \theta \leq 1$. If a function $f \in \mathcal{A}$ meets the following subordination requirement, it is said to be in the class $\mathcal{RK}_{\sinh}(\theta)$ [36].

$$\left(\frac{f'(z)}{f'(z)}\right)^{\theta}\left(\frac{f''(z)}{f'(z)}\right)^{1-\theta} \leq 1 + \sinh^{-1} z, \quad z \in \Delta.$$

Note that

$$RK_{\sinh}(0) = K_{\sinh} = \left\{f \in \mathcal{A} : \frac{f''(z)}{f'(z)} \leq 1 + \sinh^{-1} z, z \in \Delta\right\}$$

and

$$RK_{\sinh}(1) = BT_s = \left\{ f \in \mathcal{A} : f'(z) \le 1 + \sinh^{-1} z, z \in \Delta \right\}.$$

Theorem 3.0.4. *[30] If f be an analytic function $\in \mathcal{RK}_{\sinh}(\theta)$. Then, for any $\nu \in \mathbb{C}$, we have*

$$|a_3 - \nu a_2| \le \frac{1}{\sqrt{3(2-\theta)}} \max\left\{1, \frac{3(2-\theta)\nu - 4(1-\theta)}{4}\right\}.$$

Theorem 3.0.5. *[30] If f be an analytic function $\in \mathcal{RK}_{\sinh}(\theta)$, then for any $\nu \in \mathbb{R}$, we have*

$$|a_3 - \nu a_2| \le \begin{cases} \sqrt{\frac{4(1-\theta)^3 - 3(2-\theta)^2}{12(2-\theta)}} & \text{if } \nu \le \frac{-4\theta}{3(2-\theta)}, \\ \frac{1}{\sqrt{3(2-\theta)}} & \text{if } -\frac{4\theta}{3(2-\theta)} < \nu \le \frac{4}{3}, \\ \sqrt{\frac{3(2-\theta)\nu - 4(1-\theta)}{12(2-\theta)}} & \text{if } \nu > \frac{4}{3}. \end{cases}$$

If $p(x)$ for a discrete random variable X has the formula $p(x = r) = (\upsilon r)^{r-1} e^{-\upsilon r}/r!, \quad r = 1, 2, 3, \ldots$, then X is said to follow a Borel distribution with parameter υ [30].

A power series whose coefficients reflect the probabilities of the Borel distribution was recently developed by Wanas and Khuttar [30] [36]. It looks like this:

$$M(\upsilon, z) = z + \sum_{n=2}^{\infty} (\upsilon(n-1))^{n-2} e^{-\upsilon(n-1)} z^n/(n-1)!$$

In this case, $0 \le \upsilon$. It is possible to demonstrate that the radius of convergence of the aforementioned series is infinite by applying the ratio test. We will now provide the linear operator $I_\upsilon : \mathcal{A} \to \mathcal{A}$

$$\begin{aligned} I_\upsilon(f)(z) &= M(\upsilon, z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\upsilon(n-1)^{n-2} e^{-\upsilon(n-1}}{(n-1)!} a_n z^n \\ &= z + \sum_{n=2}^{\infty} \alpha_n(\upsilon) a_n z^n \end{aligned}$$

$$= z + \alpha_2 a_2 z^2 + \alpha_3 a_3 z^3 + \cdots, \tag{3.0.12}$$

where $\alpha_n = \alpha_n(\upsilon)$.

The following is how we define the class RK^{υ}_{sinh}:

$$RK^{\upsilon}_{sinh}(\theta) = \{f \in \mathcal{A} : I\upsilon(f) \in \mathcal{RK}^{\upsilon}_{sinh}(\theta)\}$$

.

The Fekete-Szegö functional and coefficient bounds for the class RK^{υ}_{sinh} may be obtained from the equivalent estimates for the function of the class $\mathcal{RK}^{\upsilon}_{sinh}$ in the same manner as in Theorems 3.0.4 and 3.0.5 [30].

Theorem 3.0.6. *[30] Let $0 \leq \theta \leq 1$ and I_{vf} given by* (3.0.12)*. If $f \in \mathcal{RK}^{v}_{sinh}(\theta)$, then for any $\nu \in \mathbb{C}$, we have*

$$|a_3 - \nu a_2^2| \leq \frac{1}{3(2-\theta)\alpha_3} \max\left\{1, \frac{(\theta\alpha_3 - 4\alpha_2)^2 + 3(2-\theta)\nu\alpha_3}{4\alpha_2^2}\right\}.$$

Theorem 3.0.7. *[30] Let $0 \leq \theta \leq 1$ and L_{vf} given by* (3.0.12)*. For any $\nu \in \mathbb{R}$, we have*

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \frac{4\alpha_2^2(1-\alpha_3)-3(2-\theta)\alpha_3\nu}{12\alpha_2^2(2-\theta)\alpha_3}, & \nu \leq \frac{-4\theta\alpha_2^2}{3(2-\theta)}, \\ \frac{1}{3(2-\theta)\alpha_3}, & -\frac{4\theta\alpha_2^2}{3(2-\theta)} \leq \nu \leq \frac{4\alpha_2^2(2-\theta\alpha_3)}{3(2-\theta)\alpha_3}, \\ \frac{3(2-\theta)\alpha_3\nu-4\alpha_2^2(1-\alpha_3)}{12\alpha_2^2(2-\theta)\alpha_3}, & \nu > \frac{4(2-\theta\alpha_3)^2}{3(2-\theta)\alpha_3}. \end{cases}$$

The use of $(p,q)-$calculus, or more precisely $q-$calculus, has become more important in the theory of geometric function theory of complex analysis in recent years [30]. The class $\mathcal{RK}_{sinh}(\theta)$ may be altered by researchers using $q-$calculus, and all of this paper's findings can be expanded to the study of analytic or meromorphic functions [15, 30].

Chapter 4

Logarithmic Coefficient and Inverse Logarithmic Coefficient

The aim of this chapter to study about Logarithmic coefficients and Inverse Logarithmic coefficients. Additionally we'll also see the theorems on logarithmic and Inverse Logarithmic coefficients bounds for second order Hankel Dterminant.

Definition 4.0.1. (Logarithmic coefficients) [27] The logarithmic coefficients γ_n of the function $f \in S$ are defined with the help of given power series:

$$F_f(z) := log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n(f) z^n \quad z \in \mathbb{D} \setminus \{0\} \tag{4.0.1}$$

In the above equation $\gamma_n(f)$ are refered to as logarithmic coefficients of f. We mostly use γ_n instead of $\gamma_n(f)$.

Theorem 4.0.2. ***(Milin's conjecture)*** *[10] [11] According to Milin's conjecture , if $f \in \mathcal{S}$, then*

$$\sum_{m=1}^{n} \sum_{k=1}^{n} k|Y_k|^2 - \frac{1}{k} \leq 0.$$

where the equivalence is valid only in the case that f is the Koebe function's rotation. The Milin hypothesis, which supported the well-known Bieberbach conjecture, was demonstrated by De Branges. One of the reasons that logarithmic coefficients came into light was estimation of the sharp bound for the class $\mathcal{S}$. Since the Koebe function k plays the role of extremal function for most of the extremal problems in the class $\mathcal{S}$, it is expected that $|\gamma_n| \leq 1/n$ holds for functions in $\mathcal{S}$. But this is not true in general, even in order of magnitude.

Theorem 4.0.3. ***(Logarithmic Coefficients sharp bounds for class $\mathcal{S}$)*** *[27] For the class $\mathcal{S}$, sharp logarithmic coefficients for $n = 1$ and $n = 2$ are given below:*

$$|Y_1| \leq 1, \ |Y_2| \leq 1/2 + 1/e^2.$$

Until now it is known for only Y_1 and Y_2. The challenge of finding Sharp bounds of $Y_n, \ n \geq 3$, remains unsolved.

Theorem 4.0.4. ***(Lemma)*** *[27,37] By differentiating* (4.0.1) *and the equating coefficients we obtain Logarithmic coefficients:*

$$Y_1 = \frac{1}{2}a_2,$$

$$Y_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right),$$

$$Y_3 = \frac{1}{4}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right).$$

$$Y_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right)$$

$$Y_5 = \frac{1}{2}\left(a_6 - a_2a_5 - a_3a_4 + a_2a_3^2 + a_2^2a_4 - a_2^3a_3 + \frac{1}{5}a_2^5\right)$$

For $f \in \mathcal{S}$, it is easy to show that $|\gamma_1| \leq 1$.
Then for $f \in \mathcal{S}$:

$$H_{2,1}(F_f) = Y_1Y_3 - Y_2^2 = \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right).$$

Observing that under rotation, $H_{2,1}(F_f/2)$ is invariant, so at this point it is appropriate because for $f_\theta(z) := e^{-i\theta}f(e^{i\theta}z), \theta \in \mathbb{R}$ when $f \in \mathcal{S}$ [10,36], we obtain

$$H_{2,1}(F_{f_{\theta/2}}) = \frac{e^{4i\theta}}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right) = e^{4i\theta}H_{2,1}(F_{f/2}).$$

Theorem 4.0.5. ***(Fekete-Szegö inequality)*** *[17] The Fekete-Szegö inequality states that if*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

is a univalent analytic function on the unit disk and $\lambda \in [0, 1)$ then

$$|a_3 - \lambda a_2^2| \leq 1 + 2exp(-2\lambda/(1 - \lambda)).$$

Definition 4.0.6. Koebe 1/4 theorem [7,37] Let $f : \mathbb{D} \to \mathbb{C}$ be a univalent map. If $f(0) = 0$ and $|f'(0)| = 1$, then the image $f(\mathbb{D})$ contains the disk $\mathbb{D}(0, 1/4)$ [15] i.e.

$$\{w; |w| < 1/4\} \subset f(\mathbb{D}).$$

This constant 1/4 is sharp. This value is achieved by the Koebe function $k(z) = z/(1-z)^2$ which conformally maps $\mathbb{D}$ onto the slit plane $\mathbb{C} \setminus (-\infty, -1/4]$. In fact, this function serves as the extremal function in many other results in geometric function theory [14].
The 1/4-theorem put forward by Köebe allows us to define the inverse function F of f in a specific neighborhood of origin [15].

Definition 4.0.7. (Inverse Logarithmic coefficients) [37] Inverse function F of f in a specific neighborhood of origin as follows:

$$F(w) := f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n; \quad |w| < 1. \tag{4.0.2}$$

The logarithmic inverse coefficients $\Gamma_s,\ s \in \mathbb{N}$, of F are determined by the equation

$$log\left(\frac{F(z)}{z}\right) = 2\sum_{s=1}^{\infty} \Gamma_s w^s; \quad |w| < \frac{1}{4}.$$

Ponnusamy examined the logarithmic coefficients of the inverses of univalent functions that if $f \in \mathcal{S}$, then

$$|\Gamma_s(F)| \leq \frac{1}{2s}\binom{2s}{s}$$

and it was shown that equality for the above expression holds only either Köebe function or its rotations.

4.1 Estimation of the Hankel-determinant on Logarithmic coefficients

The estimation of the Hankel-determinant on logarithmic coefficients for starlike functions in the petal-shaped domain $\rho(\mathbb{D})$ will be the main part of this section [39]. Some significant lemmas that are crucial to the estimate of the theorems are provided in this section.

Lemma 4.1.1. *[23, 27, 32]: If $p \in \mathcal{P}$ of the form $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$, with $c_1 \geq 0$, then*

$$c_1 = 2\eta_1 \tag{4.1.1}$$

$$c_2 = 2\eta_1^2 + 2(1-\eta_1^2)\eta_2 \tag{4.1.2}$$

$$c_3 = 2\eta_1^3 + 4(1-\eta_1^2)\eta_1\eta_2 - 2(1-\eta_1^2)\eta_1\eta_2^2 + 2(1-\eta_1^2)(1-|\eta_2|^2)\eta_3 \tag{4.1.3}$$

for some $\eta_1 \in [0,1]$ and $\eta_2, \eta_3 \in \bar{\mathbb{D}} := \{z \in \mathbb{C} : |z| < 1\}$. For $\eta_1 \in \mathbb{D}$ and $\eta_2 \in \mathbb{T}$ then $z \in \mathbb{C}; |z| = 1$, there is a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (4.1.1), (4.1.2), *and* (4.1.3),

$$p(z) = \frac{1 + (\bar{\eta}_1\eta_2 + \eta_1)z + \eta_2 z^2}{1 + (\bar{\eta}_1\eta_2 - \eta_1)z - \eta_2 z^2} \tag{4.1.4}$$

where $z \in \mathbb{D}$.

For $\eta_1, \eta_2 \in \mathbb{D}$ *and* $\eta_3 \in \mathbb{T}$ $\exists$ *a unique* $p \in \mathcal{P}$ *with* c_1, c_2 *and* c_3 *as in* (4.1.1), (4.1.2), (4.1.3), *namely,*

$$\frac{1 + \bar{\eta}_1\eta_2 + \bar{\eta}_2\eta_3 + \eta_1)z + \bar{\eta}_1\eta_3 + \eta_1 + \eta_1\bar{\eta}_2\eta_3 + \eta_2)z^2 + \eta_3 z^3}{1 + (\bar{\eta}_1\eta_2 + \bar{\eta}_2\eta_3 - \eta_1)z + (\bar{\eta}_1\eta_3 - \eta_1\bar{\eta}_2\eta_3 - \eta_2)z^2 - \eta_3 z^3} \tag{4.1.5}$$

where $z \in \bar{\mathbb{D}}$.

Lemma 4.1.2. *[24]: If* A, B, C $\in \mathbb{R}$, *let us consider*

$$Y(A, B, C) := max\{|A + Bz + Cz^2| + 1 - |z|^2 : z \in \overline{\mathbb{D}}\}$$

1. If $AC \geq 0$, *then*

$$Y(A,B,C) := \begin{cases} |A| + |B| + |C|, & |B| \leq 2(1 - |C|), \\ 1 + |A| + \frac{B^2}{4(1-|C|)}, & |B| < 2(1 - |C|). \end{cases}$$

2. If $AC < 0$, *then*

$$Y(A,B,C) := \begin{cases} 1 - |A| + \frac{|B|}{4(1-|C|)}, & -4AC(C^{-2} - 1) \leq B^2 |B| < 2(1 - |C|), \\ 1 + |A| + \frac{B^2}{4(1+|C|)}, & B^2 < min\{4(1 + |C|^2), -4AC(C^{-2} - 1)\} \\ R(A, B, C), & else. \end{cases}$$

where

$$R(A,B,C) := \begin{cases} |A| + |B| - |C|, & |C|(|B| + 4|A|) \leq |AB|, \\ -|A| + |B| + |C|, & |AB| \leq |C|(|B| - 4|A|), \\ (|C| + |A|)\sqrt{1 - \frac{B^2}{4AC}}, & else. \end{cases}$$

Now we'll estimate the Hankel-determinant on logarithmic coefficients for starlike functions in the petal-shaped domain $\rho(\mathbb{D})$.

Theorem 4.1.3. *[3] If we assume that* $f \in \mathcal{S}_s^*$ *and* F_f *be given by (1.1), then the Hankel-determinant bound for* F_f *is proposed by the following inequality*

$$|H_{1,2}(F_f)| \leq \frac{1}{16}. \tag{4.1.6}$$

Above inequality is sharp.

Proof. Let us suppose that $f \in \mathcal{A}$ [36] satisfying ithe following equation,

$$\frac{2zf'(z)}{f(z)-f(-z)} = p(z), \tag{4.1.7}$$

where $p \in \mathcal{P}$ is provided by [27]. Since it is known that class $\mathcal{P}$ is invariant under rotation, we take $c_1 \in [0,2]$. We express the coefficients of $f(z)$, $a_i^{'}s$ $(i=1,2,3,4)$ in the form of $p_i^{'}s$ $(i=1,2,3)$ using the (4.1.7) equation in the following manner,

$$a_2 = \frac{1}{2}p_1,\ a_3 = \frac{1}{4}p_2,\ a_4 = \frac{1}{144}\left(-p_1^3 - 6p_1p_2 + 24p_3\right). \tag{4.1.8}$$

The following expression of $\eta_i^{'}s$, where $\eta_i \in \bar{\mathbb{D}}$ $(i=1,2,3,4)$ is obtained by using the Lemma (4.1.1).

$$\begin{aligned}
\mathcal{L} &:= \gamma_1\gamma_3 - \gamma_2^2 \\
&= \frac{1}{2}a_2\frac{1}{4}(a_4 - a_2a_3 + \frac{1}{3}a_2^3) - \frac{1}{2}(a_3 - \frac{1}{2}a_2^2)^2 \\
&= \frac{1}{576}(-p_1^4 + 3p_1^2p2 - 9p_2^2 + 6p_1p_3) \\
&= \frac{1}{144}(-\eta_1^4 - 9\eta_2^2 + 12\eta_1^2\eta_2^2 - 3\eta_1^4\eta_2^2 + 6\eta_1\eta_2 - 6\eta_1^3\eta_3 - 6\eta_1\eta_3|\eta_2^2| + 6\eta_1^3\eta_3|\eta_2|^2)
\end{aligned}$$

Above Expression leads to,

$$|\mathcal{L}| = \begin{cases} \frac{1}{16}, & \eta_1 = 0, \\ \\ \frac{1}{144}, & \eta_1 = 1, \end{cases}$$

For $\eta_1 \in (0,1)$ and inequality $|\eta_3| \le 1$, using Lemma (4.1.1) and the expression for $\mathcal{L}$, the following inequality is obtained.

$$|\mathcal{L}| = \frac{1}{24}\eta_1(1-\eta_1^2)\Psi(A,B,C) \tag{4.1.9}$$

where

$$A = \frac{-\eta_1^3}{6(1-\eta_1^2)},\quad B = 0,\quad C = \frac{\eta_1^2 - 3}{2\eta_1}. \tag{4.1.10}$$

We now examine the following cases in the context of Lemma 4.1.2 [27], based on the compositions of A, B, and C given in (4.1.10).

I. Suppose $\eta_1 \in X = (0,1)$. Clearly, we can observe that AC is greater than 0 and equality holds when η_1 approaches to 0.
Since $|B| - 2(1-|C|)$ is an increasing function on X, which is equal to $(4-\eta_1(2+\eta_1))/\eta_1$. Therefore, by using Lemma 1.1 to solve inequality (4.1.9), we will arrive at

$$|\mathcal{L}| \le \frac{1}{24}\eta_1(1-\eta_1^2)(1+|A| + \frac{B^2}{4(1-|C|)})$$

here, onwards

$$= \frac{1}{144}(6\eta_1 - 6\eta_1^3 + \eta_1^4)$$

So from above result we can see that inequality (4.1.6) holds.

Finally, the bound's sharpness is need to be shown. If we consider the function defined in (4.2.7) with $a_3 = 1/8$ and $a_2 = a_4 = 0$, then it is simply demonstrated by a straightforward computation that $|H_{2,1}(F_f)| = 1/16$. ∎

4.2 Estimation of the Hankel-determinant on Inverse Logarithmic coefficients

Now we'll do the estimation of the Hankel-determinant on coefficients of inverse logarithmic starlike functions in the petal-shaped domain $\rho(\mathbb{D})$ [39].

Theorem 4.2.1. *If we assume that $f \in \mathcal{S}_s^*$ and F_f be given by (1.1) [3], then the Hankel-determinant bound for F_f is proposed by the following inequality*

$$|H_{1,2}(F_f)| \leq \frac{1}{9}. \tag{4.2.1}$$

Above inequality is sharp [39].

Proof. Let us suppose that $f \in \mathcal{A}$ satisfying the following equation,

$$\frac{2zf'(z)}{f(z) - f(-z)} = p(z), \tag{4.2.2}$$

where $p \in \mathcal{P}$. Since it is known that class $\mathcal{P}$ is invariant under rotation, we take $c_1 \in [0, 2]$ [3]. We express the coefficients of $f(z)$, $a_i's$ $(i = 1, 2, 3, 4)$ in the form of $p_i's$ $(i = 1, 2, 3)$ using the (4.1.7) equation in the following manner,

$$a_2 = \frac{1}{2}p_1,\ a_3 = \frac{1}{4}p_2,\ a_4 = \frac{1}{144}\left(-p_1^3 - 6p_1p_2 + 24p_3\right). \tag{4.2.3}$$

Ponnusamy et al. [17] examined the logarithmic coefficients of the inverses of univalent functions.

$$\zeta_1 = \frac{-1}{2}a_2,\ \zeta_2 = \frac{-1}{2}a_3 + \frac{3}{4}a_2^2,\ \zeta_3 = \frac{-1}{2}a_4 + 2a_2a_3 - \frac{5}{3}a_2^3. \tag{4.2.4}$$

The following expressions is given in terms of $\eta_i's$ using Lemma 4.1.1, where $\eta_i \in \bar{\mathbb{D}}$ $(i = 1, 2, 3, 4)$.

here, onwards

$$
\begin{aligned}
\mathcal{L}: &= \gamma_1\gamma_3 - \gamma_2^2 \\
&= \frac{-1}{2}a_2\frac{-1}{2}(a_4 - 4a_2a_3 + \frac{10}{3}a_2^3) - \frac{-1}{2}(a_3 - \frac{3}{2}a_2^2)^2 \\
&= \frac{1}{576}(-p_1^4 + 3p_1^2p2 - 9p_2^2 + 6p_1p_3) \\
&= \frac{1}{2304}37p_1^4 - 48p_1^2p_2 - 36p_2^2 + 48p_1p_3 \\
&= \frac{1}{144}(6\zeta_1^2(-3+\zeta_2)\zeta_2 - 9\zeta_2^2 + \zeta_1^4(16 + 18\zeta_2 + 3\zeta_2^2) + 12\zeta_1\zeta_3 - 12\zeta_1^3\zeta_3 \\
&\quad +12\zeta_1(-1+\zeta_1^2)\zeta_3|\zeta_2|)^2
\end{aligned}
$$

Above expression leads to,

$$
|\mathcal{L}| = \begin{cases} \frac{11}{144}, & \zeta_1 = 0, \\ \frac{1}{9}, & \zeta_1 = 1. \end{cases}
$$

Using the above computation of $\mathcal{L}$ along with the help of Lemma 4.1.1 we achieve the subsequent inequality for $\zeta_1 \in (0,1)$ and $|\zeta_3| \leq 1$:

$$
|\mathcal{L}| = \frac{1}{24}\zeta_1(1-\zeta_1^2)\Psi(A,B,C) \tag{4.2.5}
$$

where

$$
A = \frac{-\zeta_1^3}{6(1-\zeta_1^2)},\ B = 0,\ C = \frac{\zeta_1^2 - 3}{2\zeta_1}. \tag{4.2.6}
$$

We now examine the following cases in the context of Lemma 4.1.2 [27], based on the compositions of A, B, and C given in (4.2.6).
Suppose $\zeta 1 \in X = (0,1)$. Clearly, we can observe that AC is less than 0. This case demands the following subcases:

(a) Simply for each $\zeta_1 \in X$

$$
T_1(\zeta_1) := |B| - 2(1 - |C|) = -2\left(1 - \frac{3}{4\|\zeta_1|}\right) < 0
$$

implies

$$
|B| < 2(1 - |C|)
$$

$$
T_2(\zeta_1) := -4AC\left(\frac{1}{C^2} - 1\right) - B^2 = \frac{4\zeta_1^2}{1 = \zeta^2}\left(\frac{16\zeta_1^2}{9} - 1\right) > 0
$$

implies

$$
= -4AC\left(\frac{1}{C^2} - 1\right) \leq B^2.
$$

Here, onwards We can infer from the above computations that there is only empty set that belongs to $T1(X^*) \cap T2(X^*)$. Consequently, for every $\zeta_1 \in X$, this case does not occur, as stated by Lemma 4.1.2.

(b) In view of $\zeta_1 \in X$, [27] the relation $4(1+|C|)^2$ and $-4AC\left(\frac{1}{C^2}-1\right)$ become

$$T_3(\zeta_1) := 4(1+|C|)^2 = 4\left(1+\frac{3}{4\zeta_1}\right)^2$$

$$T_4(\zeta_1) := -4AC\left(\frac{1}{C^2}-1\right) = T_2(\zeta_1) = \frac{4\zeta_1^2(16\zeta_1^2-9)}{9(1-\zeta_1^2}$$

$$T_3(x) = 4\left(1+\frac{3}{4x}\right)^2 = \frac{9+24x+16x^2}{4x^2}, \quad x \in (0,1)$$

$$T_3(1) = \frac{49}{16}$$

$$T_3^{'}(x) = \frac{24+32x}{4x^2} - \frac{9+24x+16x^2}{2x^3}$$

Therefore, $T_3^{'}(x)$ is non vanishing as, $T_3^{'}(1) = -21/2$ and $T_3(x) < 49/4$.

$$T_4(x) = \frac{4x^2(16x^2-9)}{9(1-x^2)} > 0$$

Thus inequality below holds for any $\zeta_1 \in X$,

$$B^2 = 0 < min\{4T_3(x), T_4(x)\} = 4T_3(x).$$

Due to Lemma 4.1.1 and value of $\mathcal{L}$ in terms of $\zeta_1, \zeta_2, \zeta_3$, and (4.2.5),

$$\mathcal{L} = \frac{1}{24}\zeta_1(1-\zeta_1^2)\left(1+|A|+\frac{B^2}{4(1-|C|)}\right) = \frac{1}{144}.$$

(c) The inequality does not hold for any $\zeta_1 \in X$, as $|C|(|B|+4|A|-|AB|) = 4(\zeta_1^2)/(1-\zeta_1^2) \geq 0$.

(d) The inequality does not hold for any $\zeta_1 \in X$, as $|AB|-|C|(|B|-4|A|)$ which is equal to $|AB|-|C|(|B|-4|A|) = 4(\zeta_1^2)/(1-\zeta_1^2 \geq 0$.

So from above result we can see that inequality (4.2.1) follows. Finally, the bound's sharpness is need to be shown. If we consider the function defined as:

$$f_o(z) = z\, exp\left(\int_0^z \frac{1+sinh^{-1}(t)}{t}dt\right) = z + z^2 + \frac{1}{2}z^3 + \frac{1}{9}z^4 - \frac{1}{72}z^5 - \frac{1}{225}z^6 + \ldots, \quad (4.2.7)$$

with $a_2 = a_4 = 0$ and $a_3 = 2/9$, then it is simply demonstrated by a straightforward computation that $|H_{2,1}(F_f)| = 1/9$. ∎

Chapter 5

Conclusion

This conclusion chapter summarizes the work carried out in this dissertation and suggests possible directions for future research. Utilizing the methodology of coefficient problems and leveraging known results from the Carathéodory class, this dissertation investigates the variation in bounds of the Hankel-determinant when its entries correspond to logarithmic and inverse logarithmic coefficients within the domain of petal-shaped $\rho(\mathbb{D})$. These findings are detailed in Chapter 4.

Bibliography

[1] Najla Alarifi. The third logarithmic coefficient for certain close-to-convex functions. *Journal of Mathematics*, 2022:1–5, 07 2022.

[2] Rosihan Ali, Maisarah Haji Mohd, See Lee, and V. Ravichandran. Radii of starlikeness, parabolic and strong starlikeness for janowski starlike functions with complex paramete. *Tamsui Oxf. J. Inf. Math. Sci.*, 27:253–267, 01 2011.

[3] Vasudevarao Allu and Amal Shaji. Moduli difference of inverse logarithmic coefficients of univalent functions. *arXiv preprint arXiv:2403.10031*, 2024.

[4] Aisha Amer and Maslina Darus. Coefficient estimates for starlike functions of order . *Acta Universitatis Apulensis. Mathematics - Informatics*, 26, 01 2011.

[5] N. Malathi andBK. Vijaya. Easychair preprint bi-starlike function of complex order based on double zeta functions associated with crescent shaped region bi-starlike function of complex order based on double zeta functions associated with crescent shaped region. 11 2021.

[6] K. Arora and S.S. Kumar. Starlike functions associated with a petal shaped domain. *Bulletin of the Korean Mathematical Society*, 59:993–1010, 07 2022.

[7] D.A. Brannan and W.E. Kirwan. On some classes of bounded univalent functions. *Journal of the London Mathematical Society*, 2(1):431–443, 1969.

[8] M. Buyankara and M. M. Çağlar. Hankel and toeplitz determinants for a subclass of analytic functions. *Matematychni Studii*, 60:132–137, Dec. 2023.

[9] M. Buyankara and M. Çağlar. Hankel and toeplitz determinants for a subclass of analytic functions. *Matematychni Studii*, 60:132–137, 12 2023.

[10] N. Tuneski D.K. Thomas and A. Vasudevarao. *Univalent Functions.* De Gruyter, Berlin, Boston, 2018.

[11] P.L. Duren. *Univalent functions.* 1983.

[12] C. Özel E. Ahmad and S. Koyuncu. Topology of hankel matrices and applications. *Journal of Geometry and Physics*, 199:105150, 02 2024.

[13] R.M. Goel and B. Mehrok. A coefficient inequality for certain classes of analytic functions. *Tamkang Journal of Mathematics*, 22, 06 1991.

[14] A.W. Goodman. *Univalent Functions*. Number v. 1 in Univalent Functions. Mariner Publishing Company, 1983.

[15] A.W. Goodman. *Univalent Functions*. Number v. 2 in Univalent Functions. Mariner Publishing Company, 1983.

[16] W. Hayman. On the second hankel determinant of mean univalent functions. *Proceedings of the London Mathematical Society*, 3(1):77–94, 1968.

[17] Y. Kim J. Choi and T. Sugawa. A general approach to the fekete-szegö problem. *Journal of the Mathematical Society of Japan*, 59, 07 2007.

[18] W. Kaplan. Close-to-convex schlicht functions. *Mich. Math. J.*, 1, 07 1952.

[19] W. Koepf and D. Schmersau. On the de branges theorem. *Complex Variables and Elliptic Equations*, 31:213–230, 04 1996.

[20] B. Kowalczyk and A. Lecko. The fekete-szegö inequality for close-to-convex functions with respect to a certain starlike function dependent on a real parameter. *Journal of Inequalities and Applications*, 2014:65, 02 2014.

[21] S. S. Kumar and N. Verma. Coefficient problems for starlike functions associated with a petal shaped domain, 08 2022.

[22] P. Kythe. *Handbook of Conformal Mappings and Applications*. 03 2019.

[23] R. Libera and E. Zlotktewtcz. Early coefficients of the inverse of a regular convex function. 85:225–230, 06 1982.

[24] R. Libera and E. Zlotktewtcz. Coefficient bounds for the inverse of a function with derivative in p. ii. *Proceedings of the American Mathematical Society*, 87, 02 1983.

[25] Sanford S. Miller and Petru T. Mocanu. Differential subordinations: Theory and applications. 2000.

[26] B. Khan M.M. Ibrahim, L. Ragoub, and A. Alahmade. Fekete-szegö and second hankel determinant for a subclass of holomorphic p-valent functions related to modified sigmoid. *International Journal of Analysis and Applications*, 22:90, 05 2024.

[27] M. Mundalia and S.S. Kumar. Coefficient problems for certain close-to-convex functions. *Bulletin of the Iranian Mathematical Society*, 49, 01 2023.

[28] N.E.K. and S. Baskara. Fekete-szegÖ inequality for sakaguchi type of functions in petal shaped domain. *Australian Journal of Mathematical Analysis and Applications*, 19:pp 1–8, 08 2022.

[29] N.E.K. and S. Baskara. Image edge detection enhancement using coefficients of sakaguchi type functions mapped onto petal shaped domain. *Heliyon*, 10:1–6, 05 2024.

[30] T. Panigrahi, G. Murugusundaramoorthy, and E. Pattnayak. Coefficient bounds for a family of analytic functions linked with a petal-shaped domain and applications to borel distribution. pages 33–50, 04 2023.

[31] C. Pommerenke. On the coefficients and hankel determinants of univalent functions. *London Mathematical Society*, pages 111–122, 1966.

[32] C. Pommerenke. On the hankel determinants of univalent functions. *Mathematika*, 14(1):108–112, 1967.

[33] C. Pommerenke. Univalent funtions. 2010.

[34] M.S. Robertson. On the theory of univalent functions. *Annals of Mathematics*, 37:374, 1936.

[35] J. Rosenthal. The bieberbach conjecture. 12 1995.

[36] V. Ravichandran S. Lee and S. Supramaniam. Bounds for the second hankel determinant of certain univalent functions. *Journal of Inequalities and Applications*, 2013, 03 2013.

[37] N. Sharma S. Ponnusamy and K. Wirths. Logarithmic coefficients of the inverse of univalent functions, 11 2018.

[38] K. Sakaguchi. On a certain univalent mapping. *Mathematical Society of Japan*, 11(1), 1959.

[39] Lei Shi, Muhammad Arif, Javed Iqbal, Khalil Ullah, and Syed Muhammad Ghufran. Sharp bounds of hankel determinant on logarithmic coefficients for functions starlike with exponential function. *Fractal and Fractional*, 6(11):645, 2022.

[40] Lei Shi, Muhammad Arif, Ayesha Rafiq, Muhammad Abbas, and Javed Iqbal. Sharp bounds of hankel determinant on logarithmic coefficients for functions of bounded turning associated with petal-shaped domain. *Mathematics*, 10:1939, 06 2022.

[41] Rajavadivelu Themangani, Saurabh Porwal, and Nanjundan Magesh. Inclusion relation between various subclasses of harmonic univalent functions associated with wright's generalized hypergeometric functions. *Abstract and Applied Analysis*, 2020:1–6, 11 2020.

[42] Zhi-Gang Wang, Muhammad Arif, Zhi-Hong Liu, Saira Zainab, Rabia Fayyaz, Muhammad Ihsan, and Meshal Shutaywi. Sharp bounds of hankel determinants for certain subclass of starlike functions. *Journal of Applied Analysis Computation*, 13, 01 2020.

[43] R. Wilson. Determinantal criteria for meromorphic functions. *Proceedings of The London Mathematical Society - Proc London Math SOC*, 4:357–374, 01 1954.

[44] K. Ye and L. Lim. Every matrix is a product of toeplitz matrices. *Foundations of Computational Mathematics*, 16, 07 2013.

KONGUNADU
College of Engineering and Technology
(Autonomous)

INSTITUTION'S INNOVATION COUNCIL
(Ministry of Education Initiative)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai, Accredited by NBA (CSE, ECE, EEE & MECH), Accredited by NAAC, Recognized by UGC with 2(f) & 12(B) and An ISO 9001:2015 Certified Institution.
Namakkal- Trichy Main Road, Tholurpatti (Post), Thottiam(Taluk), Trichy(Dt)-621 215,Tamilnadu, india.

Third International Conference on Applied Science Mathematics and Statistics (ICASMS-24)
Organized by
DEPARTMENT OF MATHEMATICS

CERTIFICATE OF PARTICIPATION

This is to Certify that Prof./Dr/Mr./Ms. Snehal of Delhi Technological University has Participated / Presented in the Third International Conference on "Applied Science Mathematics and Statistics (ICASMS-24)" held on 17.05.2024 in Hybrid Mode Organized by Department of Mathematics, Kongunadu College of Engineering and Technology (Autonomous), Trichy.

He / She has Presented a paper entitled: Estimates of logarithmic coefficient of second Hankel Determinant for S* (Rho).

Mr.J.Sophers	Dr.K.Balasubramanian	Dr.J.Yogapriya	Dr.R.Asokan
Organizing Secretary	HOD/S&H	Dean R&D	Principal

KONGUNADU

College of Engineering and Technology
(Autonomous)

INSTITUTION'S INNOVATION COUNCIL
(Ministry of Education Initiative)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai, Accredited by NBA (CSE, ECE, EEE & MECH), Accredited by NAAC, Recognized by UGC with 2(f) & 12(B) and An ISO 9001:2015 Certified Institution.
Namakkal- Trichy Main Road, Tholurpatti (Post), Thottiam(Taluk), Trichy(Dt)-621 215,Tamilnadu, india.

Third International Conference on Applied Science Mathematics and Statistics (ICASMS-24)
Organized by
DEPARTMENT OF MATHEMATICS

CERTIFICATE OF PARTICIPATION

This is to Certify that Prof./Dr/Mr./Ms. Arya Tripathi of Delhi Technological University has Participated / Presented in the Third International Conference on "Applied Science Mathematics and Statistics (ICASMS-24)" held on 17.05.2024 in Hybrid Mode Organized by Department of Mathematics, Kongunadu College of Engineering and Technology (Autonomous), Trichy.

He / She has Presented a paper entitled: Estimates of logarithmic coefficient of second Hankel Determinant for S* (Rho).

Mr.J.Sophers	Dr.K.Balasubramanian	Dr.J.Yogapriya	Dr.R.Asokan
Organizing Secretary	HOD/S&H	Dean R&D	Principal

PAPER NAME

ARYA SNEHAL (1).pdf

WORD COUNT **11663 Words**	CHARACTER COUNT **53118 Characters**
PAGE COUNT **46 Pages**	FILE SIZE **1018.1KB**
SUBMISSION DATE **Jun 4, 2024 8:44 PM GMT+5:30**	REPORT DATE **Jun 4, 2024 8:47 PM GMT+5:30**

10% Overall Similarity

The combined total of all matches, including overlapping sources, for each database.

- 8% Internet database
- 7% Publications database
- Crossref database
- Crossref Posted Content database
- 7% Submitted Works database

Excluded from Similarity Report

- Bibliographic material
- Quoted material
- Cited material
- Small Matches (Less then 8 words)
- Manually excluded text blocks

How much of this submission has been generated by AI?

0%

of qualifying text in this submission has been determined to be generated by AI.

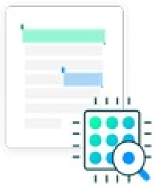
Caution: Percentage may not indicate academic misconduct. Review required.

It is essential to understand the limitations of AI detection before making decisions about a student's work. We encourage you to learn more about Turnitin's AI detection capabilities before using the tool.

Frequently Asked Questions

What does the percentage mean?

The percentage shown in the AI writing detection indicator and in the AI writing report is the amount of qualifying text within the submission that Turnitin's AI writing detection model determines was generated by AI.



Our testing has found that there is a higher incidence of false positives when the percentage is less than 20. In order to reduce the likelihood of misinterpretation, the AI indicator will display an asterisk for percentages less than 20 to call attention to the fact that the score is less reliable.

However, the final decision on whether any misconduct has occurred rests with the reviewer/instructor. They should use the percentage as a means to start a formative conversation with their student and/or use it to examine the submitted assignment in greater detail according to their school's policies.

How does Turnitin's indicator address false positives?

Our model only processes qualifying text in the form of long-form writing. Long-form writing means individual sentences contained in paragraphs that make up a longer piece of written work, such as an essay, a dissertation, or an article, etc. Qualifying text that has been determined to be AI-generated will be highlighted blue on the submission text.

Non-qualifying text, such as bullet points, annotated bibliographies, etc., will not be processed and can create disparity between the submission highlights and the percentage shown.

What does 'qualifying text' mean?

Sometimes false positives (incorrectly flagging human-written text as AI-generated), can include lists without a lot of structural variation, text that literally repeats itself, or text that has been paraphrased without developing new ideas. If our indicator shows a higher amount of AI writing in such text, we advise you to take that into consideration when looking at the percentage indicated.

In a longer document with a mix of authentic writing and AI generated text, it can be difficult to exactly determine where the AI writing begins and original writing ends, but our model should give you a reliable guide to start conversations with the submitting student.