A NOVEL TECHNIQUE FOR THE NUMERICAL APPROXIMATION OF THE SOLUTION OF SINGULARLY PERTURBED REACTION DIFFUSION EQUATION WITH DELAY AND ADVANCE

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Abstract

In this paper , we undertake numerical approaches to solve singularly perturbed reaction diffusion problems ,with Dirichlet boundary conditions. To analyze the layer behaviour of such problems , we will use standard finite difference scheme with uniform mesh and fitted mesh method with a piecewise uniform mesh introduced by Ivanovich Shishkin. A numerical example of reaction-diffusion type with delay as well as advance is solved to show the effect of standard finite difference method and fitted mesh finite difference method to show the convergence between the actual solution and the solution obtained by these numerical approaches. The boundedness, stability and convergence analysis for the numerical problem are discussed. Several graphs and tables are used to show the Error and Order of Convergence of the numerical methods.

Keywords:- Piecewise uniform mesh , Fitted mesh , Boundary layer , Singularly perturbed , Reaction-Diffusion

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Chapter 1

Introduction

1.1 Perturbation Theory

Differential equations are widely used in mathematical modeling to explain a wide range of physical phenomena in engineering and science. Differential equations are a useful tool for comprehending and forecasting how systems change over time and place by articulating the link between a variable's rate of change and the variable itself. Their application enables us to tackle and evaluate intricate issues related to the fundamental physical phenomena and offers valuable understanding of the actions and movements of the related systems. Their importance cuts across many academic fields, making them a vital resource for technology development and scientific research.

Perturbation theory in differential equations is used to approximate the solutions of equations that can't be solved exactly.For example ,

$$\epsilon u''(x) + Au'(x) + Bu(x) = 0$$

where, ϵ is a small parameter . If $\epsilon = 0$, then the differential equation reduces to simpler equation that might be easily solvable. The aim of perturbation theory is to find the solution of original equation by treating ϵ as small perturbation.

Perturbation theory is classified in two parts:-

1. Regularly Perturbed Differential Equation

2. Singular Perturbed Differential Equation

1.2 Regularly Perturbed Differential Equation

The perturbation problem in differential equations, D_ϵ refers to the situation where the derivative with the greatest order term is multiplied by a very small perturbation parameter ϵ. If the solution of D_ϵ as $\epsilon \to 0$ converges uniformly to the solution of the reduced problem D_0, which is accomplished by making

ϵ equal to zero in the perturbation problem, then the perturbation problem D_ϵ is known to be regularly disturbed.

Example:-

$$D_\epsilon = u''(x) - 2\epsilon u'(x) + u(x) = 1 \qquad , u(0) = 0, u(1) = 0 \tag{1.1}$$

Actual Solution of equation (1):-

$$u(x) = c_1 e^{(\epsilon+\sqrt{\epsilon^2+1})x} + c_2 e^{(\epsilon-\sqrt{\epsilon^2+1})x} - 1$$

where,

$$c_1 = \frac{1 - e^{\epsilon-\sqrt{\epsilon^2+1}}}{e^{\epsilon+\sqrt{\epsilon^2+1}} - e^{\epsilon-\sqrt{\epsilon^2+1}}} \quad ; \quad c_2 = 1 - c_1$$

When $\epsilon \to 0$,

$$D_0 = u''(x) - u(x) = 1 \tag{1.2}$$

Actual Solution of equation (2):-

$$u(x) = c_1 e^x + c_2 e^x - 1$$

where,

$$c_1 = \frac{1 - e^{-1}}{e - e^{-1}} \quad ; \quad c_2 = 1 - c_1$$

Hence , D_ϵ uniformly converges to D_0 when $\epsilon \to 0$.

1.3 Singularly Perturbed Differential Equation

The perturbation problem is known to be singularly perturbed if the solution of D_ϵ as $\epsilon \to 0$ does not converges uniformly to the solution of the reduced problem D_0 , which is achieved by setting ϵ equal to zero in the perturbation problem D_ϵ [21]. Such a breakdown of singular perturbation problems is limited to short time or restricted space intervals. The solution quickly transforms and separates into layers in these confined areas. These slender areas are commonly known as Strokes lines, transition points in quantum mechanics, boundary layers in fluid mechanics, edge layers in solid mechanics, skin layers in electrical applications, and shock layers in fluid and solid mechanics [8].

The singular perturbation parameter's drop to zero in the limit situation entirely alters the nature of the differential equations, which justifies the term "singular perturbation." For instance, the equations for the conservation of energy and momenta become nonlinear hyperbolic equations from nonlinear parabolic equations.

In engineering, biology, economics, and physics, singularly perturbed boundary value problems are widely used to mathematically characterize and model a wide range of real-world phenomena. These include fluid dynamics, thermodynamics, magnetohydrodynamics, rarefied gas dynamics, chemical reactor theory, elasticity, quantum mechanics, oceanography, plasticity, meteorology, and radiating flows. The Michelis-Menten theory for enzyme reactions is another common application of singularly perturbed boundary value problems. With a considerable amount of history, singular perturbation is currently a fairly developed mathematics topic [5]. These days, the topic is frequently covered in graduate programs in applied mathematics and various engineering domains.

[12]This paper deals with singularly perturbed differential difference equation which contains ϵ (where ϵ is a small positive value such that $0 < \epsilon <<< 1$) [31]as perturbed parameter multiplied with the highest order derivative term. When $\epsilon \to 0$, the solution of the equation form layers in a narrow region called boundary layers. Since, the classical approaches fails to give exact behaviour of solution for each value of perturbed parameter ϵ , we will use standard finite difference method to study the boundary layer behaviour.

Generally, a singularly perturbed differential difference equation is of the type ,

$$D_\epsilon = -\epsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x)$$

[24] with Dirichlet boundary conditions,

$$u(0) = 0, u(1) = 0$$

Here , if $a(x) = 0$ and $b(x) \neq 0$ then it becomes Reaction- diffusion problem and if $b(x) = 0$ and $a(x) \neq 0$ then it becomes Convection- diffusion problem.

In this paper,we consider a model problem of reaction diffusion type with delay as well as advance parameters.

1.4 Delay and Advanced Differential Equation

When the derivative of an unknown function at one point in time is stated in terms of the values of the function at earlier times, a mathematical differential equation known as a delay differential equation (DDE) arises. Systems with aftereffect or dead time, equations with delay arguments, time-delay systems, hereditary systems, and systems with delay arguments are some terms used in the literature to refer to DDEs.[3]

In many branches of the biological sciences, including population dynamics, epidemiology, immunology, physiology, and neural networks, mathematical modeling utilizing DDEs is employed for analysis and forecasting [2]. Considering that the current state of the model depends on its prior history, the temporal delays amplify the memory effects of these models. The length of some hidden processes, such as life cycle stages, the interval between virus infection and viral replication, the length of the infectious and immunological periods, and so on, may be connected to the delay.

Advanced differential equations (ADEs) find widespread application in situations where the evolution rate is dependent on both the present and the future. Advances in ADEs, which account for the influence of potential future actions that are currently available on the system, indicate the prospective memory of the future, whereas delays in DDEs represent the retrospective memory of the past [2]. Fields such as population dynamics, mechanical control engineering, and economic challenges are common places where such phenomena are believed to occur.

Compared to the delay differential equation, the differential difference equation is more broadly applicable. It is possible for a differential difference equation to concurrently have advance and delay terms. The differential equation $u'(x) + u(x - \delta) + u(x + \eta) = g(x)$ is a differential difference equation with both delay as well as advance.

1.5 Numerical Methods

1.5.1 Numerical Methods of Singularly Perturbed Problems

For issues that are usually not solvable using closed-form solutions, numerical approaches are used to provide an approximate solution. These approaches provide quantitative information about the problem and are intended to tackle a broad variety of issues. These approaches' quantitative nature means that the qualitative answers they produce are very different from those of asymptotic methods.

Researchers have created a number of numerical methods for handling solitary perturbation issues during the past few decades. These techniques fall into two main categories: computational methods and parameter uniform numerical methods. The standard finite difference, finite element, or finite volume methods—collectively, known as classical computational methods [11]—are found to be inadequate on uniform meshes and require an extraordinarily high number of mesh points in order to produce accurate numerical solutions when the perturbation parameter is set to a critically small value.

The presence of steep gradients in the boundary layer(s) of the analytical solution is the cause of this computational approach constraint. It is not possible for these methods to minimize the maximum point-wise error until the singular perturbation parameter and the mesh size are of the same order of magnitude [36]. Conversely, increasing the number of mesh points and associated processing overhead results from refining the mesh size to the order of the perturbation parameter. Therefore, the primary limitation of the computational approach is the reliance of the domain discretization on the perturbation parameter. It is therefore desirable to develop robust computational approaches that are independent of the perturbation parameter with respect to order of convergence, error, and discretization. We refer to these techniques as parameter uniform numerical techniques.The fitted mesh method and the fitted finite difference operator are the two basic categories into which parameter uniform approaches can be roughly classified.

1.5.2 Numerical Treatment for Singularly Perturbed Differential Difference Equations with Delay as well as Advance [17]

This thesis focuses on a particular class of differential equation problems where the highest order derivative is multiplied by a minor parameter called ϵ [15]. For differential difference equations perturbed singularly, very little literature has been reported. The numerical study of such problems has attracted more attention in recent years due to its applications in fields such as optimal control theory, neurobiology, the study of an optically bistable device, describing the so-called human pupil-light reflex, and various models for physiological processes or diseases.Prior research on singularly perturbed differential difference equations has mostly focused on the existence and uniqueness of the solutions to the problems, giving the problem of creating approximate solutions far less attention.

The study of boundary value problems (BVPs) for singularly perturbed differential difference equations individually was started in 1982 by Lange and Miura [16]. The authors used the asymptotic approach to find an approximate solution to these boundary value problems (BVPs) and published a number of research articles on the subject. The highest order derivative of a class of BVPs for linear second order differential difference equations, in which[35] shifts of both positive and negative types are multiplied by a tiny parameter, was examined by the authors. The words "positive shift" for "advance" and "negative shift" for "delay" were employed by the writers. These works concentrated on solving problems whose solutions display turning point behavior, resonance behavior, fast oscillations, and boundary layer and inner layer phenomena.

A study on asymptotic stability for a homogeneous singularly perturbed system of differential equations with unbounded delay was published in 1992 by Voulov et al. The authors of this study derive necessary requirements for the null solution of a homogeneous system of differential equations with unbounded delay to be equiasymptotically stable under singular perturbation [34].

The BVPs for singularly perturbed linear second order differential difference equations [23] were the subject of further investigation by Lange and Miura. The authors examined shifts of a fixed type in their earlier research on such BVPs. They also examined modified versions of classical singly perturbed ordinary differential equations. Matching asymptotic expansions is a technique that is extended to the analysis of BVPs for linear differential equations with minor shifts whose solutions show layer behavior. Analyses of analogous boundary value issues with oscillatory behavior in their solutions are presented in the companion paper.

In 2002, a numerical investigation of boundary value problems for second order differential difference equations with minor shifts and unique perturbations was started [1]. The authors of this paper examine the situation in which boundary layer behavior is displayed by the solutions of such BVPs. To get the numerical solution for these BVPs, a difference technique based on finite differences is developed. The stability and convergence of the given difference scheme are examined. The authors use a number of numerical experiments to show how shifts affect the behavior of the boundary layer in the solution.

Chapter 2

Singularly Perturbed Reaction Diffusion Problems with Delay and Advance

Boundary layer behavior is typically seen in the solution of the singularly perturbed differential equation. The singular perturbation parameter ϵ generally does not behave consistently well for all values of the traditional numerical methods, and in particular, the results are not satisfactory when the singular perturbation parameter ϵ is small.[30] There are primarily two methods based on the fitted mesh and the fitted operator to solve this issue.[15] First, a finite difference operator that reflects the differential operator's singularly perturbed nature is used in place of the standard finite difference operator. The numerical techniques using such difference operators on a uniform mesh are known as fitted operator methods, and these difference operators are generally referred to as fitted finite difference operators [30].In most of the cases, the fitted finite difference operator is used at all points of the mesh [15]. However, Farrell demonstrated that, in some circumstances, the mesh points in the boundary layer region might employ the fitted finite difference operator, while the remainder of the domain uses the normal finite difference operator.

However, some issues cannot be solved with an ϵ-uniform fitted mesh approach; instead, an ϵ-uniform finite difference operator on a uniform mesh can be used to solve the problem. The finite mesh approach consists of a standard finite difference operator as well as a unique kind of piecewise uniform mesh condensed in the boundary layer regions to reflect the singularly perturbed nature of the solution [15]. These kinds of meshes were introduced by Shishkin. The fitted mesh method is a numerical technique that uses a piecewise uniform mesh and a typical finite difference operator. Miller et al. published the first numerical findings obtained with a fitted mesh approach.

We establish two numerical schemes in this chapter to solve boundary value problems for a class of differential difference equations[32] that are singularly perturbed and have small delay and advance with layer behavior. These schemes are based on two different approaches: (i) the standard upwind finite difference scheme on a uniform mesh; (ii) the fitted mesh finite difference method, which consists of a standard finite upwind difference scheme on a piecewise-uniform mesh[13]. We apply Taylor's series expansion to the terms that contain advance or delay, given that both the advance and the delay are of order o(ϵ).

The coeffcient of the reaction term in the differential equation obtained after applying Taylor's series for the terms comprising both advance and delay is also of o(ϵ) because both the advance and delay are of small order of ϵ.

2.1 Problem Description

Here, we consider a boundary value problem of reaction diffusion type with small delay as well as advance,

$$\epsilon u''(x) + p(x)u(x-\delta) + s(x)u(x) + q(x)u(x+\eta) = g(x)$$

on $\xi = (0, 1)$ subject to the discrete boundary conditions,

$$u(0) = \rho(x) = 1$$

$$u(1) = \sigma(x) = 1 \tag{2.1}$$

here , $p(x)$, $q(x)$,$s(x)$, $\rho(x)$ and $\sigma(x)$[14] are smooth functions.

Here, $0 < \epsilon <<< 1$ is perturbation parameter with δ and η [29]as delay and advance parameter respectively. For the solution $u(x)$ to be smooth . It must be continuous on [0,1] and continuously differentiable on (0,1) [18]. The layer behavior or oscillatory behavior of the problem depends on the sign of $p(X) + q(X) + s(X)$. The problem shows layer behavior if $p(X) + q(X) + s(X) < 0$.

Now, we consider a simple model problem of reaction diffusion type with dirichlet condition[30]

$$D_\epsilon = -\epsilon u''(x) + p(x)u(x) = g(x) \tag{2.2}$$

where , both $g(x)$ and $p(x)$ are smooth functions and $p(x) \geq p > 0$.

The solution of a boundary value problem shows boundary layer behaviour if the condition $p(x) + q(x) + s(x) < 0$ is satisfied.

In both cases , (2.1) and (2.2) , to understand the behaviour of solution in boundary layers, we first use standard upwind finite difference operator with uniform mesh. Since, ϵ- Unifrom mesh doesn't work well in most of the cases in the boundary layer region, we will use Shishkin mesh introduced by a Russian mathematician Grigorii Ivanovich Shishkin in 1988 , which is a piecewise uniform mesh, to study boundary layer behaviour. To simplify the delay and advance parameter which are of $O(\epsilon)$, we will use taylor series expansion.

2.2 Analytical Results

The solution of the problem (1.1),(1.2) is suffciently differentiable and the delay as well as advance are very small,[30] therefore by using Taylor's series expansion approximate the terms containing delay and advance [17],

$$u(x-\delta) \approx u(x) - \delta u'(x)$$

and

$$u(x+\eta) \approx u(x) + \eta u'(x) \tag{2.3}$$

using equation (2.2) in equation (1.1),(1.2), we obtain [19],

$$\Rightarrow \epsilon u''(x) + p(x)(u(x) - \delta u'(x)) + q(x)(u(x) + \eta u'(x)) + s(x)u(x) = g(x)$$

$$\Rightarrow \epsilon u''(x) + (q(x)\eta - p(x)\delta) + (p(x) + q(x) + s(x))u(x) = g(x) \tag{2.4}$$

and

$$u(0) = \rho_0, \rho_0 = \rho(0)$$
$$u(1) = \sigma_1, \sigma_1 = \sigma(1) \tag{2.5}$$

The differential operator D_ϵ corresponding to the problem (2.4), (2.5) for any smooth function is defined as $\phi \in C^2(\bar{\xi})$ is defined as,

$$D_\epsilon \phi(x) = \epsilon u''(x) + (q(x)\eta - p(x)\delta) + (p(x) + q(x) + s(x))u(x)$$

Minimum Principle:- Assume $\phi \in C^2(\bar{\xi})$ is a smooth function satisfying $\phi(0) \geq 0$, $\phi(1) \geq 0$ and $D_\epsilon \phi(t) \leq 0$ for all $x \in \xi$. Then $\phi(t) \geq 0$ for all $t \in \bar{\xi}$.

Proof:- Let $t^* \in \bar{\xi}$ be such that $\phi(t^*) = min_{x\in\bar{\xi}}\phi(t)$ an.d assume that $\phi(t^*) < 0$.

Clearly $t^* \notin \{0, 1\}$, therefore $\phi'(t^*) = 0$ and $\phi''(t^*) \geq 0$. We have

$$D_\epsilon \phi(t^*) = \epsilon \phi^{''}(t^*) + (q(t^*)\eta - p(t^*)\delta)\phi^{'}(t^*) + (p(t^*) + q(t^*) + s(t^*))\phi(t^*) > 0$$

which contradicts the hypothesis. Therefore $\phi(t^*) \geq 0$ and thus $\phi(t) \geq 0 \forall t \in \bar{\xi}$ [16].

Lemma 1. *Let $u(x)$ be the solution of the problem (6), (7), then we have*

$$\|u\| \leq \theta^{-1}\|g\| + \max(|\rho_0|, |\sigma_1|).$$

Proof:- *Let us consider the barrier functions [34] $\phi^{\pm}$ defined by*

$$\phi^{\pm}(x) = \theta^{-1}\|g\| + \max(|\rho_0|, |\sigma_1|) \pm u(x)$$

Then we have

$$\begin{aligned}\phi^{\pm}(0) &= \theta^{-1}\|g\| + \max(|\rho_0|, |\sigma_1|) \pm u(0)\\ &= \theta^{-1}\|g\| + \max(|\rho_0|, |\sigma_1|) \pm \rho_0, \quad \textit{since} \quad \text{u}(0) = \rho_0\\ &\geq 0,\end{aligned}$$

$$\begin{aligned}\phi^{\pm}(1) &= \theta^{-1}\|g\| + \max\left(|\rho_0|, |\sigma_1|\right) \pm u(1)\\ &= \theta^{-1}\|g\| + \max\left(|\rho_0|, |\sigma_1|\right) \pm \sigma_1, \qquad \textit{since} \quad \mathrm{u}(1)=\sigma_1\\ &\geq 0\end{aligned} \tag{2.6}$$

and we have from [34]

$$\begin{aligned}D_\epsilon\phi^{\pm}(x) &= \epsilon\left(\phi^{\pm}\right)''(x) + (q(x)\eta - p(x)\delta)\left(\phi^{\pm}\right)'(x) + (p(x)+q(x)\\ &\quad + s(x))\phi^{\pm}(x)\\ &= (p(x)+q(x)+s(x))\left(\theta^{-1}\|g\| + \max\left(|\phi_0|, |\sigma_1|\right)\right) \pm D_\epsilon u(x)\end{aligned} \tag{2.7}$$

Since $(p(x)+q(x)+s(x)) \leq -\theta < 0$, so $(p(x)+q(x)+s(x))\theta^{-1} \leq -1$. Using this inequality and the Eq. (2.4) for $D_\epsilon u(x)$ in the above equation, we get

$$\begin{aligned}D_\epsilon\phi^{\pm}(x) &= -\|g\| + (p(x)+q(x)+s(x))\max\left(|\rho_0|, |\sigma_1|\right)\big] \pm g(x)\\ D_\epsilon\phi^{\pm}(x) &\leq 0 \quad \forall x\in\xi, \qquad \text{since} \quad (p(\mathrm{x})+q(\mathrm{x})+\mathrm{s}(\mathrm{x}))<0 \quad \text{and} \quad \|\mathrm{g}\| \geq \mathrm{g}(\mathrm{x})\end{aligned} \tag{2.8}$$

Therefore by the minimum principle, we get,

$$\phi^{\pm}(x) = \theta^{-1}\|g\| + \max\left(|\rho_0|, |\sigma_1|\right) \pm u(x) \geq 0 \quad \forall \quad \mathrm{x}\in\bar{\xi}$$

which gives the required bound of the solution u.

Result 1. *If u is the solution of the problem (2.4) , (2.5). Then,*

$$\|u^{(k)}\| \leq C\epsilon^{-2k} \quad \textit{where} \quad k = 1, 2, \ldots$$

Result 2. *Let $u = s_\epsilon + t_\epsilon, s_\epsilon = s_0 + \epsilon s_1$ and $t_\epsilon = t_l + t_\tau$ be the decomposition of the solution of the problem (2.4), (2.5) and assume that $(q(x)\eta - p(x)\delta) > 0$. For all $0 \leq k \leq 3$ and for sufficiently small ϵ, the functions s_0, s_1, t_l, t_r and their derivatives satisfy the following bounds,[34]*

$$\begin{aligned}\left\|s_0^{(k)}\right\|_{\bar{\xi}} &\leq C,\\ \left\|s_1^{(k)}\right\|_{\bar{\xi}} &\leq C\epsilon^{-k},\\ \left|t_l^{(k)}(x)\right| &\leq C\epsilon^{-k}\exp(-x\sqrt{\theta}/\epsilon), \qquad x\in\bar{\xi}\\ \left|t_r^{(k)}(x)\right| &\leq C\epsilon^{-k}\exp(-(1-x)\sqrt{\theta}/\epsilon), \quad x\in\bar{\xi}\end{aligned}$$

2.3 Standerd Finite Difference Scheme

In some cases, it is difficult to find the analytic solution of the problem. [7] So finite difference method is a numerical technique use to find the approximate solution of differential equation. In finite difference method, we replace the derivative term of differential equation with some approximated finite difference formulas. These approximated finite difference formulas will transform the differential equation into system of algebraic equation. This system of algebric equations can be written into $AU = B$ where A is tridiagonal matrix and U is the set of solutions of equation.

The error that arises when a differential operator is converted to a difference operator determines the discrepancy between the exact and numerical answers. This type of error is called a "truncation error" or a "discretization error." The term "truncation error" describes how a limited part of a Taylor series is used in the approximation.

In equation (2.3) and (2.4), for discrete approximation , we use a uniform mesh of size $h = \frac{1}{N}$ and replace u'' and u' by central and forward difference approximations. $x_i = (i-1)h$ denote the value of mesh points. Where $i = 1, 2 \ldots N+1$

$$D_1^N u_i = g\left(x_i\right) \tag{2.9}$$

$$\begin{aligned} u_0 &= \rho_0, \quad \rho_0 = \rho(0), \\ u_N &= \sigma_1, \quad \sigma_1 = \sigma(1), \end{aligned} \tag{2.10}$$

where the discrete operator D_1^N is defined as

$$\begin{aligned} D_1^N u_i &= \epsilon D_+ D_- u_i + \left(q\left(x_i\right)\eta - p\left(x_i\right)\delta\right) D_+ y_i + \left(p\left(x_i\right) + q\left(x_i\right) + s\left(x_i\right)\right) u_i \\ D_+ D_- u_i &= \left(u_{i-1} - 2u_i + u_{i+1}\right)/h^2, D_+ u_i = \left(u_{i+1} - u_i\right)/h \quad and \\ D_- y_i &= \left(y_i - y_{i-1}\right)/h \end{aligned}$$

,

on simplification,

$$D_1^N u_i = E_i u_{i-1} - F_i u_i + G_i u_{i+1} = H_i \tag{2.11}$$

where

$$\begin{aligned} E_i &= \epsilon/h^2, \\ F_i &= 2\epsilon/h^2 + \left(q_i\eta - p_i\delta\right)/h - \left(p_i + q_i + s_i\right), \\ G_i &= \epsilon/h^2 + \left(q_i\eta - p_i\delta\right)/h, \\ H_i &= g_i, \quad i = 1, 2, \ldots N+1. \end{aligned} \tag{2.12}$$

The system of equations given by equation (2.12) will form a tridiagonal system of $N+1$ equations (two equations are given by the boundary points) with $N+1$ unknowns $u_o, u_1, \ldots u_N$

2.3.1 Finite Difference Discretization

Let us consider a singularly perturbed reaction diffusion problem .

$$D_\epsilon = -\epsilon u''(x) + p(x)u(x) = g(x) \quad , for \quad x \in (0,1) \tag{2.13}$$

with boundary conditions

$$u(0) = 0 \quad , u(1) = 0$$

where , $p(x) > 0 \quad , \epsilon <<<< 1$. [25]

Assume that $p(x)$ and $g(x)$ lies in the interval [0,1] into n subintervals by an equidistant mesh $x_i = a + (i-1) * h$ for $i = 1, 2, ..., N+1$ and $h = 1/N$.

To approximate the solution at these equidistant points we used central difference formulas or approximations,

$$u''(x) = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} \quad , u'(x) = \frac{u_{i+1} - u_{i-1}}{2h}$$

For $i = 1, 2, ..., N$ by using these central difference approximations in equation (10),we get,

$$u_{i-1}(-\epsilon) + u_i(2\epsilon + h^2\alpha) + u_{i+1}(-\epsilon) = h^2 f(x)$$

and further ,we can express it in the form of $N + 1 * N + 1$ tridiagonal matrix form $AU = D$ where,

$$\text{A}=\begin{bmatrix} 1 & 0 & & & \\ \text{-}\epsilon & 2\epsilon + h^2p & \text{-}\epsilon & & \\ 0 & \text{-}\epsilon & 2\epsilon + h^2p & \text{-}\epsilon & \\ & \ddots & \ddots & \ddots & \\ & & \text{-}\epsilon & 2\epsilon + h^2p & \text{-}\epsilon \\ & & & 0 & 1 \end{bmatrix}$$

$$\text{U}=\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n+1} \end{bmatrix}$$

and

$$D=\begin{bmatrix} h^2g(x_1) \\ h^2g(x_2) \\ \vdots \\ h^2g(x_{n+1}) \end{bmatrix}$$

2.3.2 Upwind Finite Difference Scheme

[10]Upwind Finite Difference Scheme is generally used to avoid the unnecessary oscillations form the obtained solutions.
upwinding occurs when the one sided difference taken on the side away from the layer i.e.,

$$u'(x) = \frac{u_i - u_{i-1}}{h}$$

in this paper , upwind finite difference scheme will be used for further approximation of solution.
The difference between standard finite difference method using forward difference approximation with uniform mesh and standard finite difference method using upwind scheme can be seen by the graphs given below,

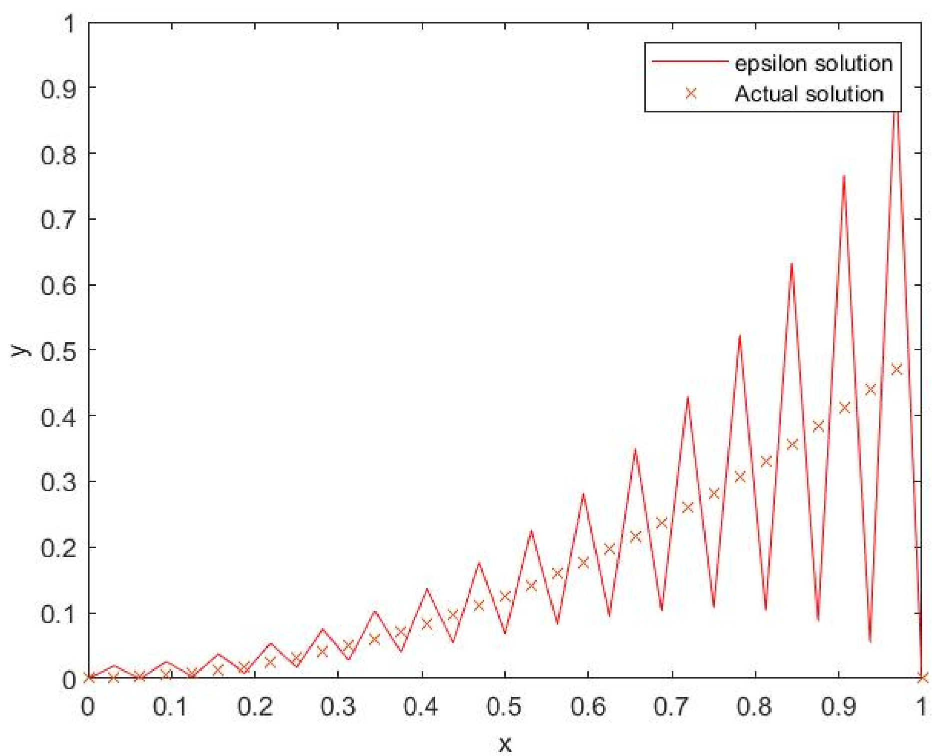


Figure 2.1: Finite Difference Method with Forward Difference

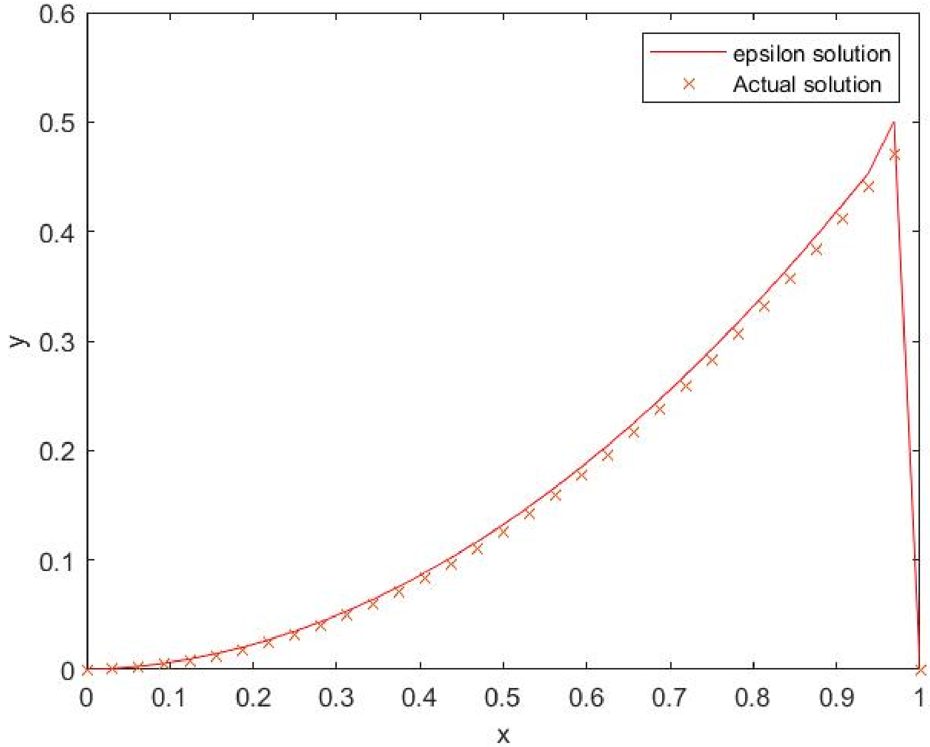


Figure 2.2: Finite Difference Method with upwind Scheme

2.3.3 Stability and Convergence Analysis

Theorem 1. *Let* $(q(x)\eta - p(x)\delta) \geq K > 0$ *and* $(p(x) + q(x) + s(x)) < -\theta < 0 \forall x \in [0, 1]$, *then the solution of the system of difference equation (13) together with the boundary conditions exists, is unique and satisfies,*

$$\|u\|_{h,\infty} \leq C^{-1}\|g\|_{h,\infty} + (\|\rho\|_{h,\infty} + \|\sigma\|_{h,\infty}) \tag{2.14}$$

where ,$C = M$ *or* $\|q\eta - p\delta\|_{h,\infty}$*,is a constant independent of* h *and* ε. $\|.\|_{h,\infty}$ *is the discrete* $l_\infty-$ *norm, given by*

$$\|x\|_{h,\infty} = \max_{0 \leq i \leq N} |x_i|$$

Proof:- *Let* v_i *be any mesh function satisfying*

$$D_1^N v_i = g_i$$

Combining this with Eq. (13) followed by a rearrangement of terms gives

$$F_i v_i = -g_i + E_i v_{i-1} + G_i v_{i+1}$$

Taking modulus on both the sides and using the non-negativity of the coefficients (E_i, F_i *and* G_i), *we obtain*

$$F_i |v_i| \leq |g_i| + E_i |v_{i-1}| + G_i |v_{i+1}|$$

Using Eq. (2.12), for E_i, F_i and G_i, the above inequality becomes,

$$\begin{aligned}\left(2\epsilon/h^2 + (q_i\eta - p_i\delta)/h - (p_i + q_i + s_i)\right)|v_i| \leq |g_i| + \left(\epsilon/h^2\right)|v_{i-1}| \\ + \left(\epsilon/h^2 + (q_i\eta - p_i\delta)/h\right)|v_{i+1}|\end{aligned} \tag{2.15}$$

$i = 1, 2, \ldots N+1$. A rearrangement in the above inequality (2.15) yields

$$\begin{aligned}&\epsilon\left(|v_{i-1}| - 2|v_i| + |v_{i+1}|\right)/h^2 + (q_i\eta - p_i\delta)\left(|v_{i+1}| - |v_i|\right)/h \\ &+ (p_i + q_i + s_i))|v_i| + |g_i| \geq 0.\end{aligned} \tag{2.16}$$

To replace the coefficient $(q_i\eta - p_i\delta)$ in above inequality(2.16), the sign of the expression $(|v_{i+1}| - |v_i|)$ to be noted.If $(|v_{i+1}| - |v_i|) \geq 0$, we use the inequality $0 < (q_i\eta - p_i\delta) \leq \|(q\eta - p\delta)\|_{h,\infty}$ and if $(|v_{i+1}| - |v_i|) < 0$, we use the inequality $(q_i\eta - p_i\delta) \geq K > 0$.Thus using the above results in above inequality (18), we get,

$$\epsilon\left(|v_{i-1}| - 2|v_i| + |v_{i+1}|\right)/h^2 + C\left(|v_{i+1}| - |v_i|\right)/h + (p_i + q_i + s_i)|v_i| + |g_i| \geq 0 \tag{2.17}$$

where ,$C = M$ or $\|(\beta\eta - \alpha\delta)\|_{\mathrm{h},\infty}$,is a positive constant which depends on the sign of the expression $(|z_{i+1}| - |z_i|)$. Now,on rearrangement of the terms in the inequality (19) gives,

$$\begin{aligned}&\epsilon\left(|v_{i+1}| - |v_i|\right)/h^2 - \epsilon\left(|v_i| - |v_{i-1}|\right)/h^2 + C\left(|v_{i+1}| - |v_i|\right)/h \\ &+ |f_i| + (p_i + q_i + s_i)|v_i| \geq 0.\end{aligned} \tag{2.18}$$

[27]To prove the existence and uniqueness of the solution of the system of linear difference equations (2.11), suppose $\langle u_i\rangle_{i=0}^N$ and $\langle v_i\rangle_{i=0}^N$ be two sets of solutions to the system of linear difference equations (2.11) satisfying the boundary conditions. Let $v_i = c_i - d_i$. This satisfies

$$D_1^N(v_i) = g_i$$

where

$$g_i = 0 \text{ and } \mathrm{v}_0 = 0 = {}_{\mathrm{N}}.$$

On taking summation (2.18) from 1 to $N+1$ and using $v_0 = 0 = v_N$, we obtain

$$-\left(\epsilon/h^2\right)|v_1| - \left(\epsilon/h^2\right)|v_{N-1}| - (C/h)|v_1| + \sum_{i=1}^{N-1}(p_i + q_i + s_i)|v_i| \geq 0. \tag{2.19}$$

Since $C > 0, (p_i + q_i + s_i) < 0$ and $|z_i| \geq 0, i = 1, 2, \ldots N+1$, therefore for inequality (2.19) to hold, we must have

$$v_i = 0, \quad i = 1, 2, \ldots N+1.$$

Thus, the uniqueness of the solution of the tridiagonal system is proved. The solution of the tridiagonal system of difference equatiuon exist and is unique(the existance is implied by uniqueness in case on linear equation). Now to establish the estimate, let

$$v_i = u_i - l_i$$

where u_i satisfies the difference equations (2.11) and the boundary conditions and

$$l_i = (1 - ih)\rho_0 + (ih)\sigma_1$$

Then $v_0 = 0 = v_N$ and v_i satisfy [28]

$$D_1^N(v_i) = g_i, \quad i = 1, 2, \dots N + 1$$

Now let

$$|v_n| = \|v\|_{h,\infty} \geq |v_i|, \quad i = 1, 2, \dots N + 1 \tag{2.20}$$

Then summing (2.17) from $i = n$ to $N + 1$ gives, [22]

$$\begin{aligned} &-\epsilon\left(|v_n| - |v_{n-1}|\right)/h^2 - \epsilon^2\left|v_{N-1}\right|/h^2 - C\left|v_n\right|/h \\ &+ \sum_{i=n}^{N-1}\left(p_i + q_i + s_i\right)|v_i| + \sum_{i=n}^{N-1}|g_i| \geq 0. \end{aligned} \tag{2.21}$$

From the above inequality (2.20), we have $(|v_n| - |v_{n-1}|) > 0$ and $(p_i + q_i + s_i) < 0$. After removing the first, second and fourth terms from these inequalities, we get by (2.21),

$$\begin{aligned} C\left|v_n\right| &\leq h\sum_{i=n}^{N-1}|g_i| \\ &\leq h\sum_{i=0}^{N}|g_i| \\ &\leq \|g\|_{h,\infty}, \end{aligned}$$

i.e., we have

$$\begin{aligned} |v_n| &\leq C^{-1}\|g\|_{h,\infty}. \\ \|u\|_{h,\infty} &= \max_{0 \leq i \leq N}|u_i| \end{aligned} \tag{2.22}$$

Using $u_i = v_i + l_i$, we obtain

$$\begin{aligned} \|u\|_{h,\infty} &\leq \|v\|_{h,\infty} + \|l\|_{h,\infty} \\ &= |v_n| + \|l\|_{h,\infty}, \quad \text{from Eq. (23).} \end{aligned} \tag{2.23}$$

Now, we shall find out the bound on $\|l\|_{h,\infty}$,

$$\begin{aligned}\|l\|_{h,\infty} &= \max_{i\in[0,N]} [|l_i \,|] \\ &\leq \max_{i\in[0,N]} [|(1-ih)\rho_0 + (ih)\sigma_1|] \\ &\leq \max_{i\in[0,N]} [(1-ih)\,|\rho_0| + (ih)\,|\,\sigma_1|]\,,\end{aligned}$$

i.e., we have

$$\begin{aligned}\|l\|_{h,\infty} &\leq |\rho_0| + |\sigma_1| \\ &\leq \|\rho\|_{h,\infty} + \|\sigma\|_{h,\infty}.\end{aligned} \tag{2.24}$$

Now using the inequalities (2.22) and (2.24) in the inequality (2.23), we obtain the required estimate (2.14)

$$\|u\|_{h,\infty} \leq C^{-1}\|g\|_{h,\infty} + (\|\rho\|_{h,\infty} + \|\sigma\|_{h,\infty})\,.$$

Thus , The theorem has demonstrated that the solution to the system of difference equations (2.11), regardless of the mesh size h and parameter ϵ, is uniformly limited. The scheme is therefore stable for all step sizes.[12]

2.4 Fitted Mesh Finite Difference method

2.4.1 Piecewise Uniform Shishkin Mesh

The uniform mesh $x_i = a + (i-1) * h$ could not determine the layer behaviour accurately. So to capture the sharp edges in the layer region. Russian mathematician G.I. Shishkin gave a piecewise uniform mesh called shishkin mesh.The width of Shishkin mesh can be adjust by the nature of solution.

The mesh spacing is always chosen in such a way that the layer region get maximum number of solution points and more the number of solution points in the layer region more final will be the region of interest can be studied. Shishkin mesh is generally used when the solution exhibits sharper edges and this strategy of using piecewise uniform mesh helps to get the important features in the layer region of the solution.

[30]In this part, The fitted mesh finite difference method is employed, comprising of a conventional upwind finite difference operator applied to a piecewise uniform mesh that condenses at the boundary points $x = 0$ and $x = 1.$, to discretize the boundary value problems (2.4), (2.5). In order to create the fitted piecewise-uniform mesh $\bar{\xi}^N$ on the interval [0,1], the interval is divided into three subintervals: $(0, \lambda)$, $(\lambda, 1-\lambda)$, and $(1-\lambda, 1)$.[6] A uniform mesh is created on each of these subintervals, i.e., $\frac{N}{4}+1$equal mesh points are created from the intervals $(0, \lambda)$ and $(1-\lambda, 1)$, and $\frac{N}{2}$ equal mesh points are created from the interval $(\lambda, 1-\lambda)$. One parameter, known as the transition parameter, determines the piecewise uniform mesh that is produced $\lambda = \min\left[\frac{1}{4}, (\frac{2}{\alpha})\epsilon logN\right]$. In order to ensure that there is at least one point in the boundary layer, we assume that $N = 2^r$ with $r \geq 3$.

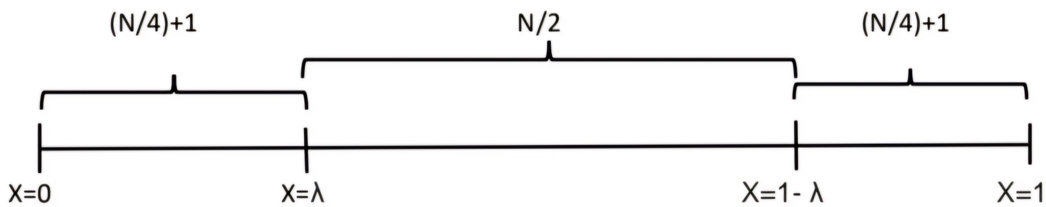


Figure 2.3: Interval Distribution of Shishkin Mesh for Reaction-Diffusion

The difference in mesh spacing between standard finite difference method with uniform mesh and fitted finite difference scheme under piecewise uniform mesh is shown by the figures 2.4 and 2.5 below,

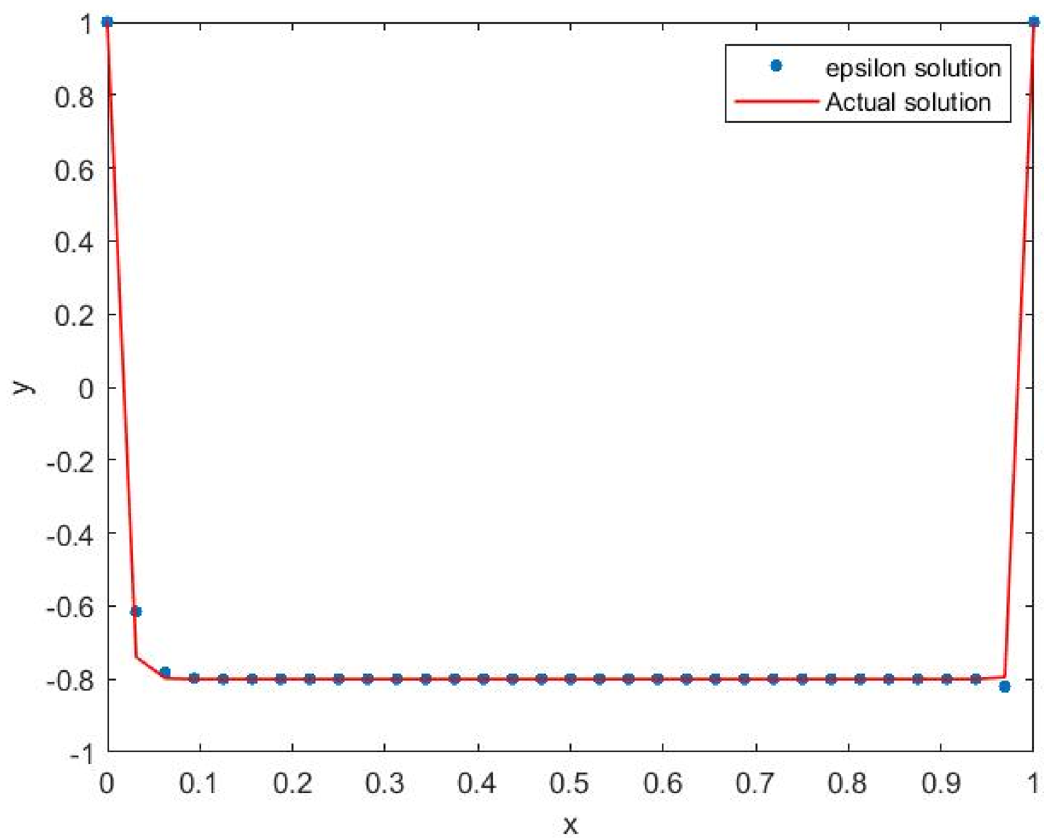


Figure 2.4: Mesh spacing under Standard Finite Difference Method

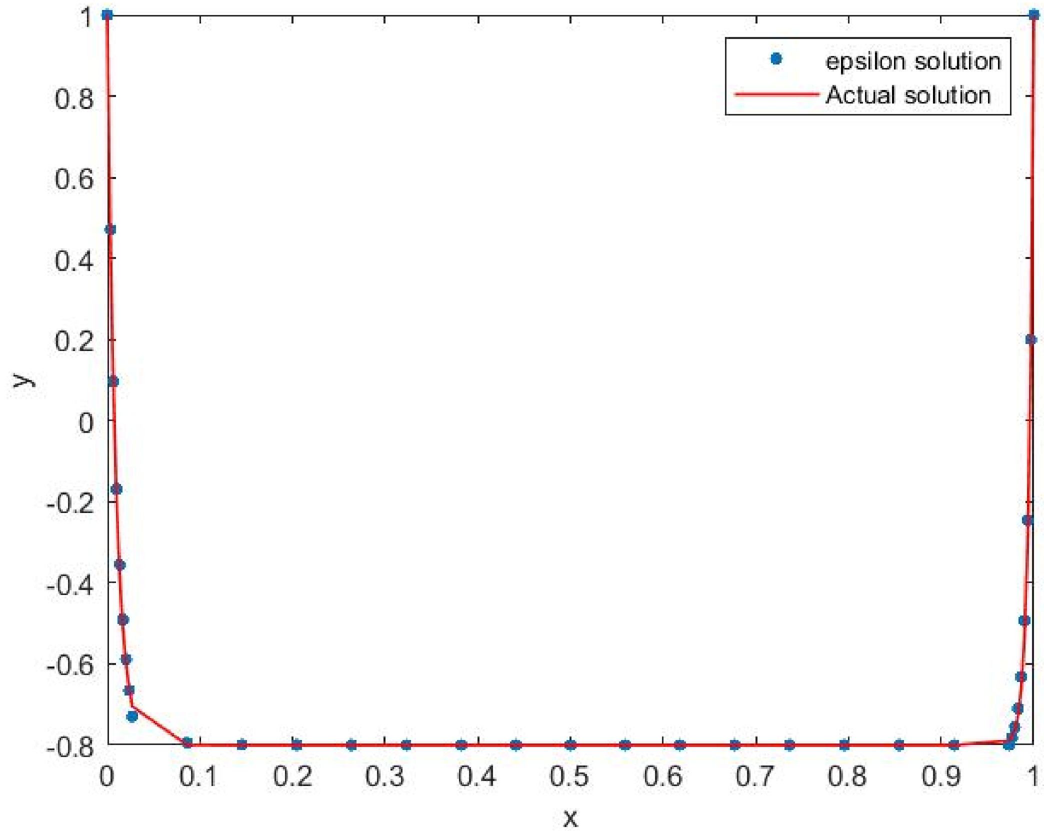


Figure 2.5: Mesh spacing under Shishkin Mesh

2.4.2 Discrete Minimum Principle

Suppose $\phi_0 \geq 0$ and $\phi_N \geq 0$. The $D_3^N \phi_i \leq 0$ for all $x_i \in \xi^N$ implies that $\phi_i \geq 0$ for all $x \in \bar{\xi}^N$. [34]
Proof:- Let k be such that $\phi_k = \min_{0 \leq i \leq N} \phi_i$ and assume $\phi_k < 0$. Then we have $\phi_k - \phi_{k-1} \leq 0, \phi_{k+1} - \phi_k \geq 0$ and

$$\begin{aligned} D_3^N \phi_k =& \epsilon D^+ D^- \phi_k + \left(q\left(x_k\right) \eta - p\left(x_k\right) \delta\right) D^+ \phi_k \\ &+ \left(p\left(x_k\right) + q\left(x_k\right) + s\left(x_k\right)\right) \phi_k \\ =& 2\epsilon \left(\frac{\left(\phi_{k+1} - \phi_k\right)}{h_{k+1}} - \frac{\left(\phi_k - \phi_{k-1}\right)}{h_k}\right) / \left(h_k + h_{k+1}\right) \\ &+ \left(q\left(x_k\right) \eta - p\left(x_k\right) \delta\right) \left(\phi_{k+1} - \phi_k\right) / h_{k+1} \\ &+ \left(p\left(x_k\right) + q\left(x_k\right) + s\left(x_k\right)\right) \phi_k \\ >& 0 \end{aligned}$$

[16] which is a contradiction to the fact that $\phi_k < 0$, Hence $\phi_k \geq 0$.Choosing k as a fixed arbitrary number. We have $\phi_i \geq 0$ for all $i, 0 \leq i \leq N$.

Result 3. *Suppose H_i be any mesh function such that $H_0 = H_N = 0$. Then for all $i, 0 \leq i \leq N$*

$$\left|H_i\right| \leq \theta^{-1} \max_{1 \leq j \leq N-1} \left|D_3^N H_j\right|.$$

2.4.3 Error Estimaste

Theorem 2. *The solution $U^N = \langle u_i \rangle_{i=0}^N$ of the discrete problem (2.9), (2.10) and the solution $u(x)$ of the problem (2.4), (2.5) satisfy the following ϵ-uniform error estimate [16]*

$$\left\|U^N - u\right\| \leq CN^{-1} \ln N,$$

where C is a constant independent of ϵ.

Proof:-*In the same way that we broke down the solution u of the continuous problem (2.4), (2.5), we do the same for the solution U^N of the discrete problem (2.09), (2.10).. Thus [34]*

$$U^N = S_c^N + T_c^N$$

where S_ϵ^N is the solution of the inhomogeneous problem

$$D_3^N S_\varepsilon^N (x_i) = g (x_i) \quad \textit{for all } \mathrm{x_i} \in \xi^{\mathrm{N}}, \quad \mathrm{S}_c^{\mathrm{N}}(0) = \mathrm{s}_\epsilon(0), \quad \mathrm{S}_\varepsilon^{\mathrm{N}}(1) = \mathrm{s}_\epsilon(1)$$

and T_ϵ^N is the solution of the homogeneous problem

$$D_3^N T_c^N (x_i) = 0 \quad \textit{for all } \mathrm{x_i} \in \xi^{\mathrm{N}}, \quad \mathrm{S}_c^{\mathrm{N}}(0) = \mathrm{s}_c(0), \quad \mathrm{S}_\epsilon^{\mathrm{N}}(1) = \mathrm{s}_c(1).$$

Then the error can be written in the form

$$U^N - u = \left(S_\epsilon^N - s_\epsilon\right) + \left(T_\epsilon^N - t_\epsilon\right). \tag{2.25}$$

Now, the error in the regular and singular components will be estimated separately [9].For estimating the error for the regular component from the differential and difference equations, we have

$$\begin{aligned} D_3^N \left(S_\epsilon^N - s_\epsilon\right)(x_i) &= g(x_i) - D_3^N s_\epsilon (x_i) = \left(D_k - D_3^N\right) s_\epsilon (x_i) \\ &= \epsilon \left(\frac{d^2}{dx^2} - D^+ D^-\right) s_\epsilon (x_i) + (q(x)\eta - p(x)\delta) \left(\frac{d}{dx} - D^+\right) s_\epsilon (x_i). \end{aligned} \tag{2.26}$$

Let $x_i \in \xi^N$. Then for any $\phi \in C^2(\xi)$,

$$\left|\left(D^+ - \frac{d}{dx}\right) \phi (x_i)\right| \leq (x_{i+1} - x_i) \left\|\phi^{(2)}\right\| /2$$

and for any $\phi \in C^3(\bar{\xi})$, [13]

$$\left|\left(D^+ D^- - \frac{d^2}{dx^2}\right) \phi (x_i)\right| \leq (x_{i+1} - x_{i-1}) \left\|\phi^{(3)}\right\| /3$$

Using above results in Eq. (2.26) and after a simplification, we get,

$$\left|D_\epsilon\left(S_\epsilon^N-s_\epsilon\right)(x_i)\right|\le C\left(x_{i+1}-x_{i-1}\right)\left(\epsilon\left\|s_\epsilon^{(3)}\right\|+\|q\eta-p\delta\|\left\|s_\epsilon^{(2)}\right\|\right),\quad x_i\in\xi^N.$$

[34]Since $x_{i+1}-x_{i-1}\le 2N^{-1},\eta,\delta$ *are of* $o\left(\epsilon^2\right)$ *and using Result(2) to estimate* $s_\epsilon^{(2)}$ *and* $s_c^{(3)}$*, we get,*

$$\left|D_\epsilon\left(S_\epsilon^N-s_\epsilon\right)(x_i)\right|\le CN^{-1},\quad x_i\in\xi^N. \tag{2.27}$$

An application of Result 3 ,the mesh function $\left(S_c^N-s_k\right)(x_i)$ *gives*

$$\left|\left(S_\epsilon^N-s_\epsilon\right)(x_i)\right|\le\theta^{-1}\max_{j\in[1,N-1]}\left|D_\epsilon\left(V_\epsilon^N-s_\epsilon\right)(x_j)\right| \tag{2.28}$$

Using inequality (2.27) in the above inequality (2.28), we get

$$\left|\left(S_c^N-s_c\right)(x_i)\right|\le CN^{-1},\quad x_i\in\xi^N. \tag{2.29}$$

The estimates of the singular component of the error $D_3^N\left(T_c^N-t_c\right)$ *rely on the transition parameter* λ*, and can be determined by either* $\lambda=1/4$ *or* $\lambda=C\epsilon\ln N$*, where* $C=1/\sqrt{\theta}$*.*

Case i) $C\epsilon\ln N\ge 1/4$*, i.e.,* ***when the mesh is uniform [16]***
When the smooth component of the error is estimated using classical method, it results in,

$$\left|D_3^N\left(T_\epsilon^N-t_\epsilon\right)(x_i)\right|\le C\left(x_{i+1}-x_{i-1}\right)\left(\epsilon\left\|t_\epsilon^{(3)}\right\|+\|(q\eta-p\delta)\|\left\|t_\epsilon^{(2)}\right\|\right),\quad x_i\in\xi^N$$

Using the bounds for the derivatives of t (Result 2), the fact that $(x_{i+1}-x_{i-1})=2N^{-1}$ *and the restriction on* $\eta,\delta\,(\eta,\delta$ *are of* $o(\epsilon))$*, we get, [34]*

$$\left|D_\epsilon^N\left(T_\varepsilon^N-t_\epsilon\right)(x_i)\right|\le C\epsilon^{-1}N^{-1},\quad x_i\in\xi^N.$$

Using $\varepsilon^{-1}\le 4C\ln N$ *in the above inequality, we get,*

$$\left|D_3^N\left(T_\epsilon^N-t_\epsilon\right)(x_i)\right|\le CN^{-1}\ln N,\quad x_i\in\xi^N. \tag{2.30}$$

Now by usinf Result 3 for the mesh function $\left(T_\epsilon^N-t_\epsilon\right)(x_i)$*, we get*

$$\left|\left(T_\epsilon^N-t_\epsilon\right)(x_i)\right|\le\theta^{-1}\max_{1\le j\le N-1}\left|D_3^N\left(T_\varepsilon^N-t_\epsilon\right)(x_j)\right|. \tag{2.31}$$

Combining (2.30) with (2.31), we get

$$\left|\left(T_\epsilon^N-t_\epsilon\right)(x_i)\right|\le CN^{-1}\ln N,\quad x_i\in\xi^N. \tag{2.32}$$

Case ii)*[34]* $C\epsilon \ln N < 1/4$, ***i.e., when the mesh is piecewise uniform with mesh spacing*** $4\lambda/N$ ***in the subintervals*** $[0,\lambda],[1-\lambda,1]$ ***and*** $2(1-\lambda)/N$ ***in the subinterval*** $[\lambda,1-\lambda]$

We will now estimate the error's singular component in each subinterval independently. In the boundary layer regions, that is, in the subintervals $[0,\lambda]$ *and* $[1-\lambda,1]$*, we first estimate the singular component. We apply comparable classical reasoning to xi lying in the open subintervals* $(0,\lambda)$ *and* $(1-\lambda,1)$*, as we did in the first example, and get,*

$$\left|D_3^N\left(T_\epsilon^N-t_\epsilon\right)(x_i)\right| \leq C\lambda N^{-1}\epsilon^{-1}$$

In the above inequality using $\lambda = C\epsilon\ln N$*, we get,*

$$\left|D_3^N\left(T_c^N-t_s\right)(x_i)\right| \leq CN^{-1}\ln N \tag{2.33}$$

On estimating the singular component of the error in the smooth region, i.e., for $x_i \in [\lambda, 1-\lambda]$*. We have [16]*

$$\begin{aligned}\left|D_3^N\left(T_\epsilon^N-t_c\right)(x_i)\right| \leq & \epsilon\left|\left(D^+D^-t_c-t_c''\right)(x_i)\right| \\ & +\|q\eta-p\delta\|\left|\left(D^+t_\epsilon-t_\epsilon'\right)(x_i)\right|\end{aligned} \tag{2.34}$$

But we have

$$\left|D^+D^-t_c(x_i)\right| \leq \max_{x_{i-1}\leq x_i\leq x_{i+1}}|t_c''| \;\; \textit{and} \;\; \left|\mathrm{D^+t_c(x_i)}\right| \leq \max_{\mathrm{x_{i-1}\leq x_i\leq x_{i+1}}}|\mathrm{t}_c'|$$

Using these results in the above inequality (2.34), we get,

$$\left|D_3^N\left(T_\epsilon^N-t_\epsilon\right)(x_i)\right| \leq 2\epsilon \max_{x_{i-1}\leq x_i\leq x_{i+1}}|t_\epsilon''| + 2\|q\eta-p\delta\| \max_{x_{i-1}\leq x_i\leq x_{i+1}}|t_\epsilon'|$$

Using the fact that η,δ *are of* $o(\epsilon)$ *and the bounds for* t_ϵ'' *and* t_e' *from Result 2, we get,[34]*

$$\begin{aligned}\left|D_\epsilon^N\left(T_\epsilon^N-t_\epsilon\right)(x_i)\right| &\leq C\exp\left(-x_{i-1}\sqrt{\theta}/\epsilon\right) \qquad \textit{if } \mathrm{x_i} \leq 1/2 \\ \left|D_\epsilon^N\left(T_\varepsilon^N-t_\epsilon\right)(x_i)\right| &\leq C\exp\left(-(1-x_i)\sqrt{\theta}/\epsilon\right) \quad \textit{if } \mathrm{x_i} \geq 1/2\end{aligned} \tag{2.35}$$

Case When $x_i \leq 1/2$*; Since* $x_i \in [\lambda, 1-\lambda]$*, either* $x_i > \lambda$ *or* $x_i = \lambda$*.*
a) *For* $x_i > \lambda$*, we have* $x_{i-1} \geq \lambda$ *since* $x_i \in [\lambda, 1-\lambda]$*.*
[4]Since $\exp(-x)$ *is a decreasing function for all*$x \in R^+$*, therefore from this fact we have*

$$\begin{aligned}\exp\left(-x_{i-1}\sqrt{\theta}/\varepsilon\right) &\leq \exp(-\tau\sqrt{\theta}/\epsilon) \\ &= \exp(-\ln N), \qquad \textit{since } \lambda = \epsilon\ln N/\sqrt{\theta} \\ &= N^{-1}\end{aligned} \tag{2.36}$$

b) *For $x_i = \lambda$, we have $x_{i-1} = x_i - h_i$. But $h_i = 4\tau/N$ for $i = 1, 2, \ldots N/4$. Thus*

$$\begin{aligned}\exp\left(-x_{i-1}\sqrt{\theta}/\epsilon\right) &= \exp(-(\lambda - 4\lambda/N)\sqrt{\theta}/\epsilon)\\ &= \exp(-\lambda\sqrt{\theta}/\epsilon)\exp\left(4N^{-1}\sqrt{\theta}/\epsilon\right)\\ &= \exp(-\ln N)\cdot\exp\left(4N^{-1}\ln N\right), \qquad \textit{since } \lambda = \epsilon\ln N/\sqrt{\theta}\\ &= N^{-1}(N)^{4/N}\end{aligned}$$

Since we have $N^{1/N} \leq C'\forall N \geq 1$, where C' is constant, using this inequality into the above equation, we get

$$\exp\left(-x_{i-1}\sqrt{\theta}/\epsilon\right) \leq C'N^{-1} \tag{2.37}$$

Using inequalities (2.36), (2.37) in the inequality (2.35) for $x_i \leq 1/2$, we have

$$\left|D_3^N\left(T_\epsilon^N - t_\epsilon\right)(x_i)\right| \leq CN^{-1}\ln N. \tag{2.38}$$

Similarly the same result can be obtained when $x_i \geq 1/2$ Now, combining the inequalities (2.38) with the inequality (2.33), we get,

$$\left|D_3^N\left(T_\epsilon^N - t_\epsilon\right)(x_i)\right| \leq CN^{-1}\ln N \quad \forall x_i \in \xi^N \tag{2.39}$$

Now an application of Result 3 for the mesh function, $(T_\epsilon - t_\epsilon)(x_i)$ gives,

$$\left|\left(t_\epsilon^N - t_\epsilon\right)(x_i)\right| \leq \theta^{-1}\max_{1\leq j\leq N-1}\left|D_3^N\left(T_\epsilon^N - t_\epsilon\right)(x_j)\right| \tag{2.40}$$

Using inequality (41) in the inequality (42), we get the estimate for the singular component of the error in whole domain,[13] i.e., for all $x_i \in \xi^N$

$$\left|\left(W_\varepsilon^N - t_\epsilon\right)(x_i)\right| \leq CN^{-1}\ln N \tag{2.41}$$

Hence the inequalities (2.29) and (2.40) gives the required error estimate.

Chapter 3

Numerical Computations

Consider a simple model problem of reaction diffusion type ,

$$\epsilon^2 u''(x) - u(x-\delta) + u(x) - 1.25u(x+\eta) = 1 \tag{3.1}$$

with boundary conditions[20],

$$u(x) = 1 \quad , -\delta \leq x \leq 0$$

and

$$u(x) = 1 \quad , 1 \leq x \leq 1+\eta$$

Using Taylor's series expansion, [17]

$$u(x-\delta) \approx u(x) - \delta u'(x)$$

and

$$u(x+\eta) \approx u(x) + \eta u'(x)$$

we get,

$$\epsilon^2 u''(x) + (\delta - 1.25\eta)u'(x) - 1.25u(x) = 1 \tag{3.2}$$

the actual solution of the problem is given by,

$$u(x) = c_1 e^{(s+t)x} + c_2 e^{(s-t)x} - 0.8 \tag{3.3}$$

where,

$$c_1 = \frac{1.8(1-e^{s-t})}{e^{s+t} - e^{s-t}} \quad , c_2 = 1.8 - c_1$$

$$s = \frac{-(\delta - 1.25\eta)}{2\epsilon^2} \quad , t = \frac{\sqrt{(\delta - 1.25\eta)^2 + 5\epsilon^2}}{2\epsilon^2} \tag{3.4}$$

Now , we show the Boundedness, Stability and Convergence for the model problem (3.1).

3.1 Boundedness

Here we consider mainly three cases, and by using Lemma 1,

Case 1:- When $\delta = 0$ and $\eta \neq 0$, then using this information in (3.4) ,we get,

$$s - t < 0$$

Case 2:- When $\delta \neq 0$ and $\eta = 0$, and since, the magnitude of "s" is less then the magnitude of "t " , then using this information in (3.4), we get,

$$s - t < 0$$

Case 3:- When $\delta = \eta$,

$$s - t < 0$$

So. we get that $s - t < 0$ for each of three cases, then the term $e^{s-t} \to 0$ as $\epsilon \to 0$. Therefore, (3.3) becomes,

$$u(x) \leq 1.8e^{[(s+t)(x-1)]} - 0.8 \leq 1.8$$

so, we get

$$u(x) \leq 1.8$$

Then , Lemma 1 proved and $u(x)$ is bounded.

3.2 Boundedness of Derivatives

Theorem 3. *[33]Let u be the solution of the problem (2.4), (2.5). Then for* $k = 1, 2, 3$

$$\left\| u^{(k)} \right\| \leq C\epsilon^{-k}.$$

Proof:- *For* $x \in \xi$*, construct a neighborhood* $N_x = (c, c + \epsilon)$*, where* c *is a positive constant chosen so that* $x \in N_x$ *and* $N_x \subset \Omega$*. [13] Then by the Mean Value Theorem, for some* $z \in N_x$*, we have*

$$u'(v) = \frac{y\,(c + \epsilon) - u(c)}{\epsilon}$$

which on simplification gives

$$|u'(v)| \leq 2\epsilon^{-1} \|u\|$$

Using Lemma 1 for the bound on u in the above inequality, we get [13]

$$|u'(v)| \leq 2\epsilon^{-1} \left(\|g\| \theta^{-1} + \max\left(|\rho_0|, |\sigma_1|\right) \right). \tag{3.5}$$

using equation (3.1),

$$\begin{aligned} |u'(v)| &\leq 2\epsilon^{-1}(1.8) \\ &\leq 3.6\epsilon^{-1} \end{aligned} \tag{3.6}$$

We have,

$$\begin{aligned} \textstyle\int_v^x u''(t)dt = u'(x) - u'(z) \\ i.e.,\ u'(x) = u'(v) + \textstyle\int_v^x u''(t)dt \end{aligned} \tag{3.7}$$

Using the differential equation (3.1) for substituting the value of $u''(t)$ in the above Eq. (3.7) and then taking modulus on both the sides , we get,

$$|u'(x)| \leq |u'(v)| + \epsilon^{-1}\int_v^x [1 - (\delta - 1.25\eta)u'(t) + 1.25u(t)]dt$$

Using the fact that the maximum norm of a function is always greater than or equal to the function value on the domain of consideration in the above inequality, we get [32]

$$\begin{aligned} |u'(x)| \leq |u'(v)| + \epsilon^{-1}|x - v| + \epsilon^{-1}(\delta - 1.25\eta)\left|\int_z^x u'(t)dt\right| \\ + 1.25\epsilon^{-1}\|u(t)\||x - v| \end{aligned}$$

Using Lemma 1 for the bound on u and inequality (3.5) in the above inequality, we get ,

$$\begin{aligned} |u'(x)| \leq 3.6\epsilon^{-1} + \epsilon^{-1}|x - v| + \epsilon^{-1}(\delta - 1.25\eta)\,|u(x) - u(v)| \\ + 1.25\epsilon^{-1} * 1.8 * |x - v| \end{aligned}$$

Using the inequality $0 < |x - v| \leq 1$ followed by a simplification we use,

$$|u'(x)| \leq C\epsilon^{-1}, \quad x \in \xi$$

which gives $\|u'\| \leq C\epsilon^{-1}$, where C is a constant and [13]

$$C = 3.6 + (\delta - 1.25\eta)\,|u(x) - u(v)| + 3.25|x - v|$$

Thus we have obtained the result for k=1 . Similarly , by differentiating equation (3.1) and using bounds on $u(x)$ and $u'(x)$ we can obtain the required bounds of the second and third derivatives of the solution u

3.3 Stability and Convergence

On comparing equation (3.2) with equation (2.4), we get,

$$q(x)\eta - p(x)\delta = -1.25\eta + \delta$$

$$p(x) + q(x) + s(x) = -1.25; \quad g(x) = 1$$

and $u(x)$ is given by equation (3.3).Now,using $\|.\|$ as the discrete l_∞- norm i.e. $\|x\|_{h,\infty} = \max_{0\leq i\leq N} |x_i|$

$$\begin{aligned} \|u\|_{h,\infty} &\leq C^{-1}\|g\|_{h,\infty} + (\|\rho\|_{h,\infty} + \|\sigma\|_{h,\infty}) \quad \text{where,} C = \|q\eta - p\delta\| \\ &\leq C^{-1}(1) + (1+1) \\ &\leq C^{-1}(1) + 2 \end{aligned} \tag{3.8}$$

Since , the solution exist uniquely with the boundary conditions and satisfies (3.5). Therefore , the solution of difference equation is uniformly bounded and independent of mesh size h as well as parameter ϵ. Hence, the scheme is Stable for all step sizes.[12] and the Convergence is given by the theorem below,

Theorem 4. *Under the conditions for Theorem 1, the error $e_i = u(x_i) - u_i$ between the solution $u(x)$ of the continuous problem (2.4), (2.5) and the solution u_i of the discretized problem (2.11) with boundary conditions, satisfies the estimate*

$$\|e\|_{h,\infty} \leq K^{-1}\|W\| \tag{3.9}$$

where

$$\begin{aligned} |W_i| \leq &\max_{x'\in[x_{i-1},xi+1]} \left[\frac{h}{2}|(q(x)\eta - p(x)\delta)|\,|u''(x)|\right] \\ &+ \max_{x\in[x_{i-1},xi+1]} \left[\frac{h^2}{6}\,|(q(x)\eta - p(x)\delta)\|u'''(x)|\right] \\ &+ \max_{x\in[x_{i-1},xi+1]} \left[\frac{h^2}{24}\left\{2\epsilon + h|(q(x)\eta - p(x)\delta)|\right\}\left|u^{iv}(x)\right|\right]. \end{aligned}$$

Proof:- *The truncation error W_i is defined as*

$$\begin{aligned} W_i =&\epsilon\left[(u_{i-1} - 2u_i + u_{i+1})/h^2 - u''(x_i)\right] \\ &+ (q(x)\eta - p(x)\delta)\left[(u_{i+1} - u_i)/h - u'(x_i)\right]. \end{aligned}$$

$$\begin{aligned} |W_i| \leq &\max_{x\in[x_{i-1},xi+1]} \left[\frac{h}{2}|(q(x)\eta - p(x)\delta)\||u''(x)\right|\Big] \\ &+ \max_{x\in[x_{i-1},xi+1]} \left[\frac{h^2}{6}\,|(q(x)\eta - p(x)\delta)\|u'''(x)|\right] \\ &+ \max_{x\in[x_{i-1},xi+1]} \left[\frac{h^2}{24}\left\{2\epsilon^2 + h|(q(x)\eta - p(x)\delta)|\right\}\left|u^{iv}(x)\right|\right]. \end{aligned}$$

now thw error e_i satisfy,

$$D_1^N e(x_i) = D_1^N u(x_i) - D_1^N u_i = W_i \quad i = 1, 2, \ldots N+1$$

and $e_0 = 0 = e_N$. Using an application of Theorem 1 for the mesh function e_i gives

$$\|e\|_{h,\infty} \leq C^{-1}\|W\|_{h,\infty}.$$

For fixed values of parameter ϵ,convergence of the difference scheme is established by estimate (3.6).

3.4 Error and Order of Convergence

Consider $u(x_i)$ is the exact solution and u_i is the numerical solution. Then the Error at each mesh point is given by,

$$e_i = |u(x_i) - u_i|$$

The maximum norm error is given by,

$$E_N = max\|u(x_i) - u_i\|$$

and the Order of Convergence is given by,

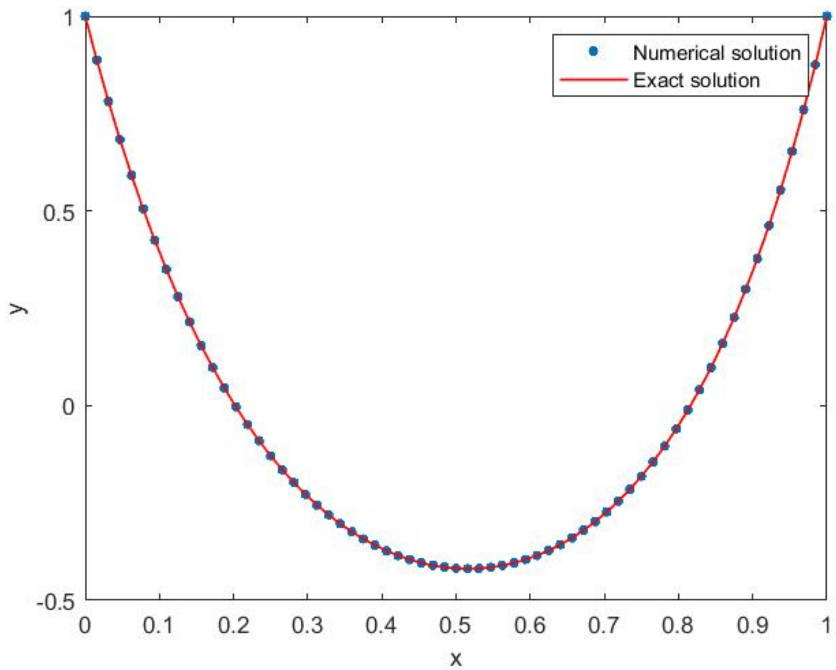
$$C_N = \frac{log(\frac{E_N}{E_N+1})}{log2}$$

Consider the numerical problem (3.1) with given boundary conditions and actual solution. First we will draw tables showing maximum norm error and order of convergence for different values of ϵ, N , δ and η.

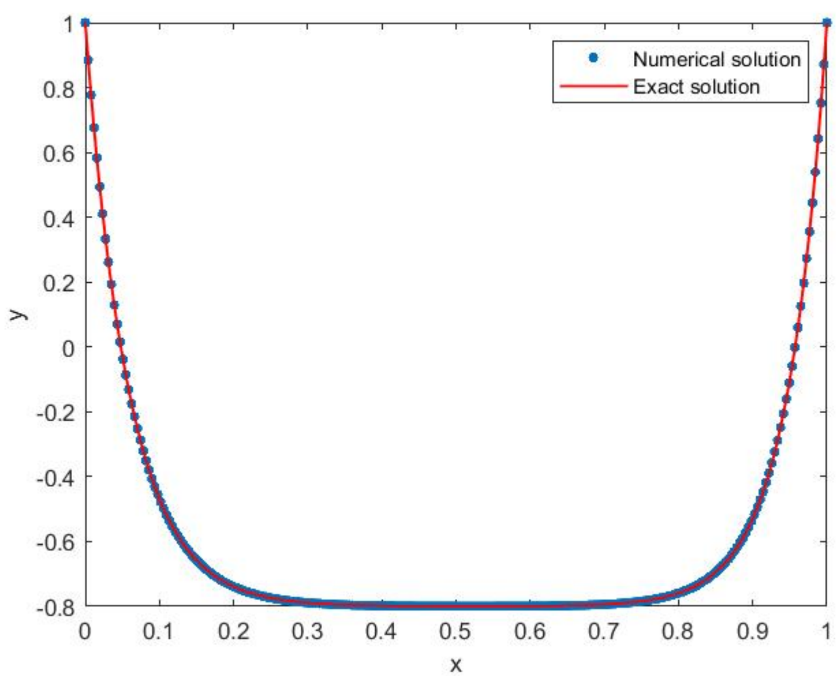
Further, we will draw graphs showing convergence between actual solution and numerical solution. in the table each box has two parts for the value of ϵ and N. The upper part of the box represents the Max. norm error (E_N) and the lower part of the box represents the Order of Convergence (C_N)

Table 3.1: The maximum norm error and Order of Convergence for $\delta = \eta = 0.5\epsilon$ under Standard finite difference method.

ϵ/N	32	64	128	256	512
2^{-1}	0.000192	0.0000481	0.000012	0.00000301	0.00000075
	1.94	2.00	1.99	2.00	2.05
2^{-2}	0.000684	0.000171	0.000042	0.0000107	0.0000026
	2.00	2.02	1.97	2.04	1.97
2^{-3}	0.0022	0.000563	0.000141	0.0000353	0.0000088
	1.96	1.99	1.99	2.00	2.00
2^{-4}	0.0087	0.0022	0.00056	0.000140	0.000035
	1.98	1.97	2.00	2.00	2.00
2^{-5}	0.0325	0.0087	0.0022	0.000560	0.00014
	1.90	1.98	1.97	2.00	2.00
2^{-6}	0.0781	0.0325	0.0087	0.0022	0.00056
	1.26	1.90	1.98	1.97	2.00



(a) Solution plot for $\epsilon = 2^{-2}$ and $N = 64$



(b) Solution plot for $\epsilon = 2^{-4}$ and $N = 256$

Figure 3.1: Solution Plots for Table 3.1

Table 3.2: The maximum norm error and Order of Convergence for $\delta = 0, \eta = 0.5\epsilon$ under Standard finite difference method.

N/ ϵ	**32**	**64**	**128**	**256**	**512**
2^{-1}	0.000136	0.0000341	0.0000085	0.0000021	0.00000053
	1.99	2.00	2.01	1.98	2.02
2^{-2}	0.000582	0.000146	0.000036	0.0000091	0.0000092
	1.99	2.01	1.98	2.04	1.94
2^{-3}	0.0022	0.00055	0.00014	0.000035	0.0000087
	2.00	1.97	2.00	2.01	2.05
2^{-4}	0.0086	0.0022	0.00055	0.000140	0.000035
	1.97	2.00	1.99	2.00	2.01
2^{-5}	0.0320	0.0086	0.0022	0.00055	0.00014
	1.89	1.97	2.00	1.97	2.00
2^{-6}	0.0933	0.0320	0.0086	0.0022	0.00055
	1.54	1.89	1.97	2.00	1.97

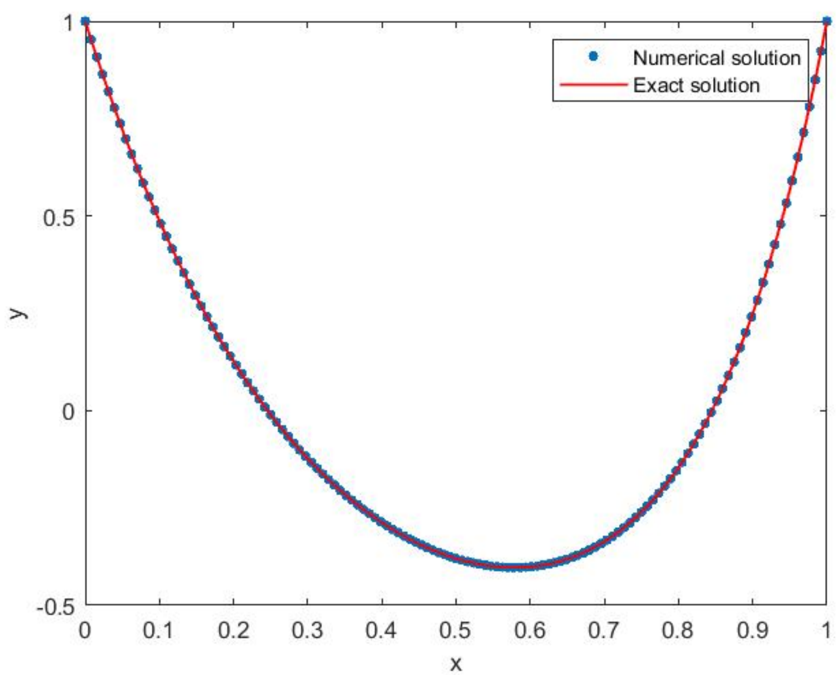


Figure 3.2: Solutin plot for $\epsilon = 2^{-2}$ and $N = 128$, For table 3.2

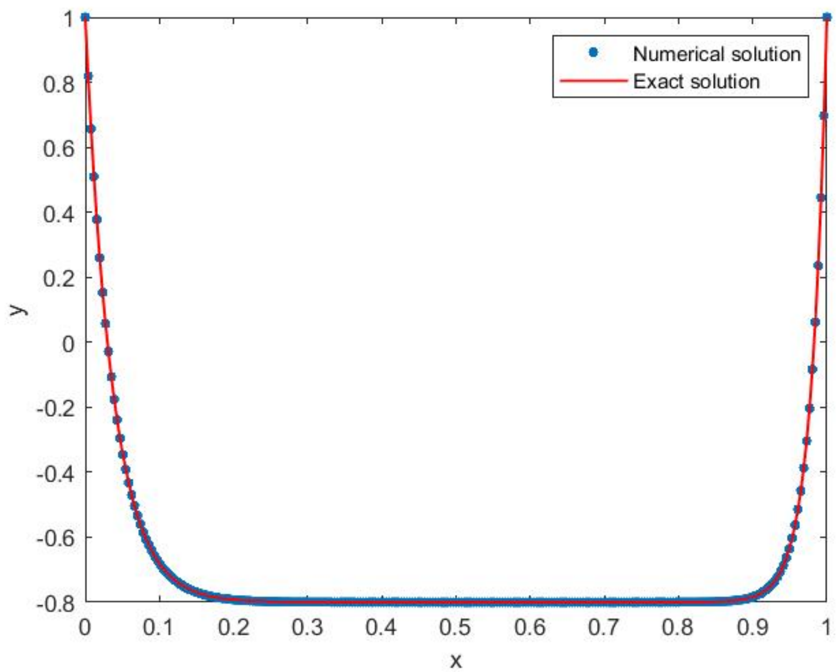
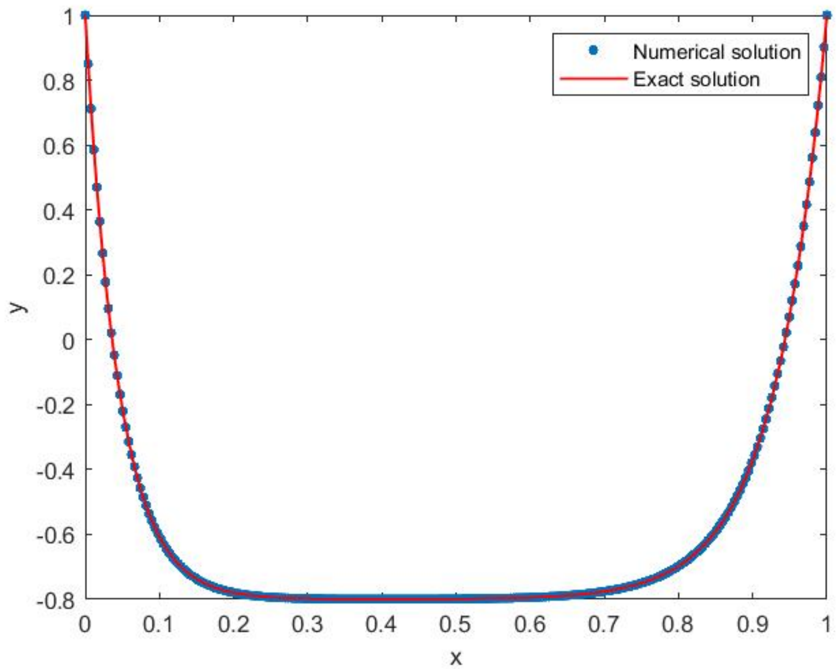


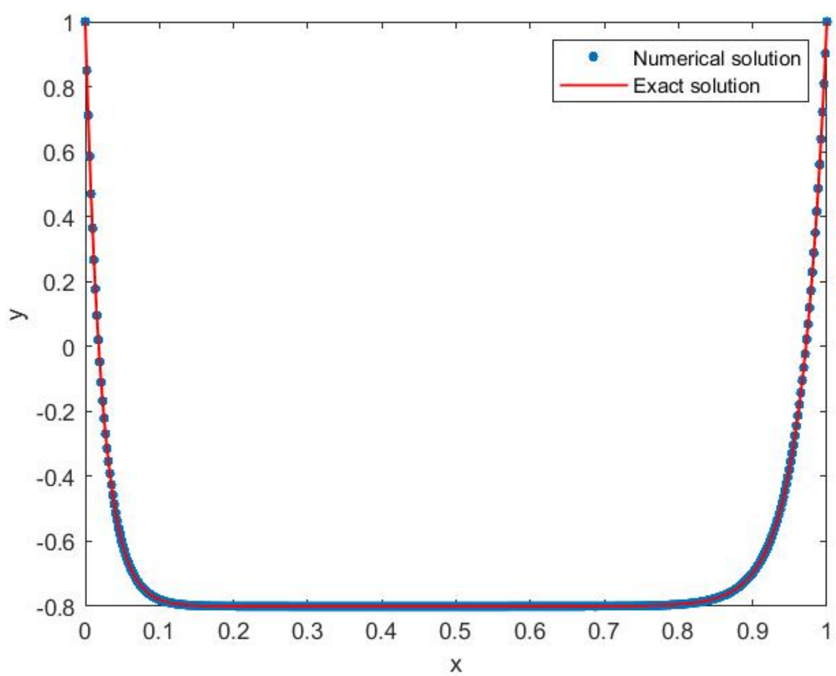
Figure 3.3: Solutin plot for $\epsilon = 2^{-5}$ and $N = 256$, For table 3.2

Table 3.3 : The maximum norm error and Order of Convergence for $\delta = 0.5\epsilon, \eta = 0$ under Standard finite difference method.

N/ ϵ	**32**	**64**	**128**	**256**	**512**
2^{-1}	0.00015	0.000039	0.0000098	0.0000024	0.00000061
	1.94	1.99	2.02	1.97	1.99
2^{-2}	0.00062	0.00015	0.000038	0.0000097	0.0000024
	2.04	1.98	1.96	2.01	2.00
2^{-3}	0.0023	0.00057	0.000143	0.000035	0.0000089
	2.01	1.99	2.03	1.97	2.01
2^{-4}	0.0089	0.0023	0.00057	0.000143	0.0000555
	1.95	2.01	1.99	2.03	1.97
2^{-5}	0.0329	0.0089	0.0023	0.00057	0.000143
	1.88	1.95	2.01	1.99	2.03
2^{-6}	0.0917	0.0329	0.00089	0.0023	0.00057
	1.47	1.88	1.95	2.01	1.99



(a) Solution plot for $\epsilon = 2^{-4}$ and $N = 256$



(b) Solution plot for $\epsilon = 2^{-5}$ and $N = 512$

Figure 3.4: Solution Plots for Table 3.3

Table 3.4 : The maximum norm error and Order of Convergence for $\delta = \eta = 0.5\epsilon$ under Fitted mesh finite difference method with Shishkin mesh.

N/ ϵ	**32**	**64**	**128**	**256**	**512**
2^{-1}	0.0021	0.0009828	0.0004797	0.0002369	0.0001177
	1.09	1.03	1.02	1.01	1.02
2^{-2}	0.0041	0.0019	0.0008937	0.0004366	0.0002157
	0.11	1.08	1.03	1.02	1.01
2^{-3}	0.0074	0.0032	0.0015	0.0007189	0.0003517
	1.20	1.09	1.06	1.03	1.04
2^{-4}	0.0147	0.0073	0.0032	0.0015	0.0007133
	1.01	1.18	1.09	1.07	1.06
2^{-5}	0.0223	0.0078	0.0040	0.0021	0.0011
	1.51	0.96	0.92	0.93	0.85
2^{-6}	0.0280	0.0115	0.0057	0.0029	0.0014
	1.28	1.01	0.97	1.05	1.17
2^{-7}	0.0295	0.0127	0.0069	0.0037	0.0018
	1.21	0.84	0.86	0.92	1.03
2^{-8}	0.0298	0.0131	0.0073	0.0040	0.0021
	1.18	0.84	0.86	0.92	1.07

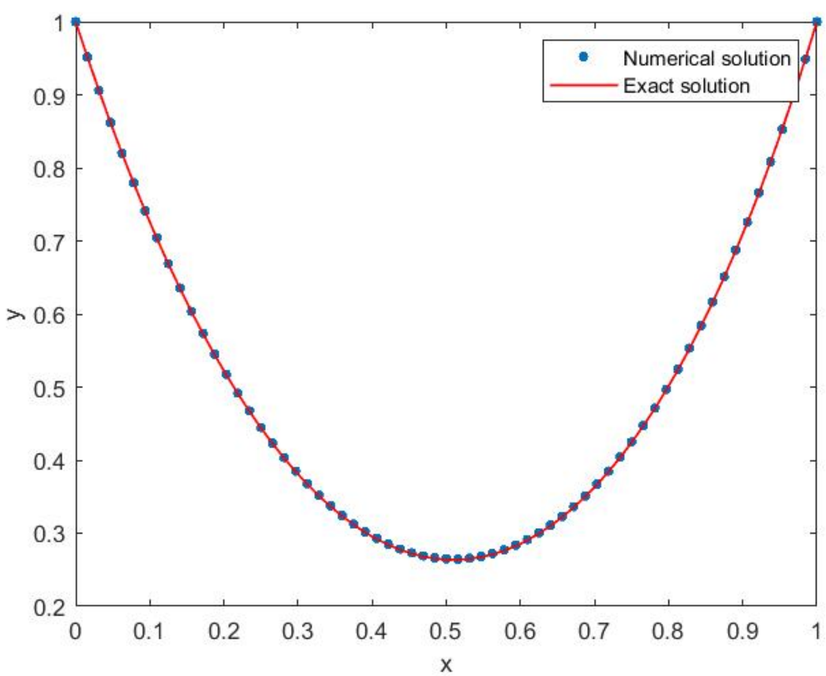


Figure 3.5: Solutin plot for $\epsilon = 2^{-1}$ and $N = 64$, For table 3.4

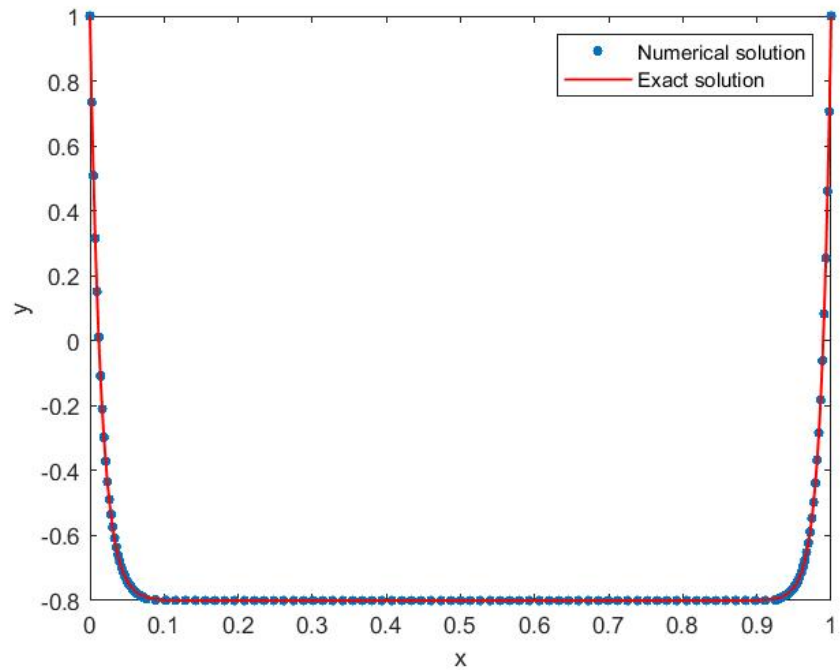
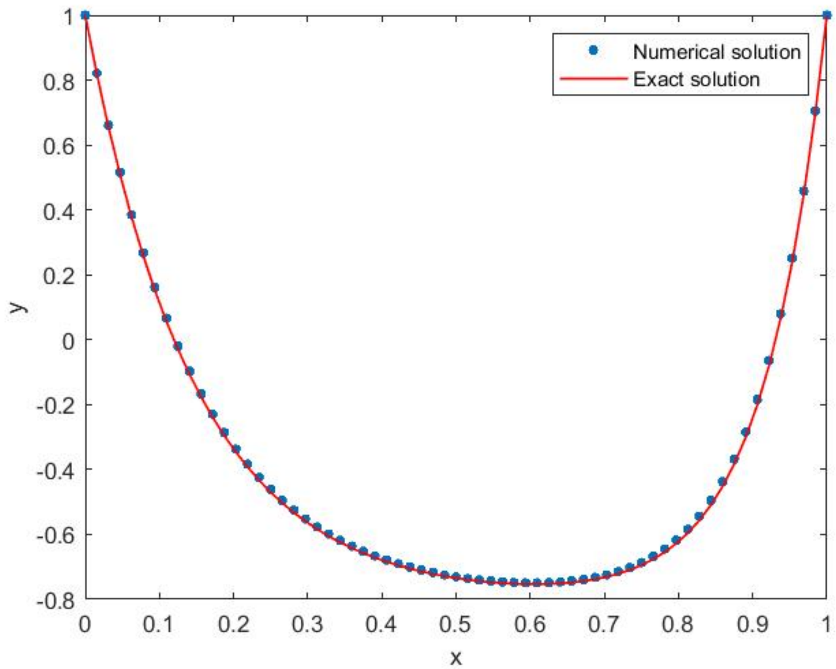


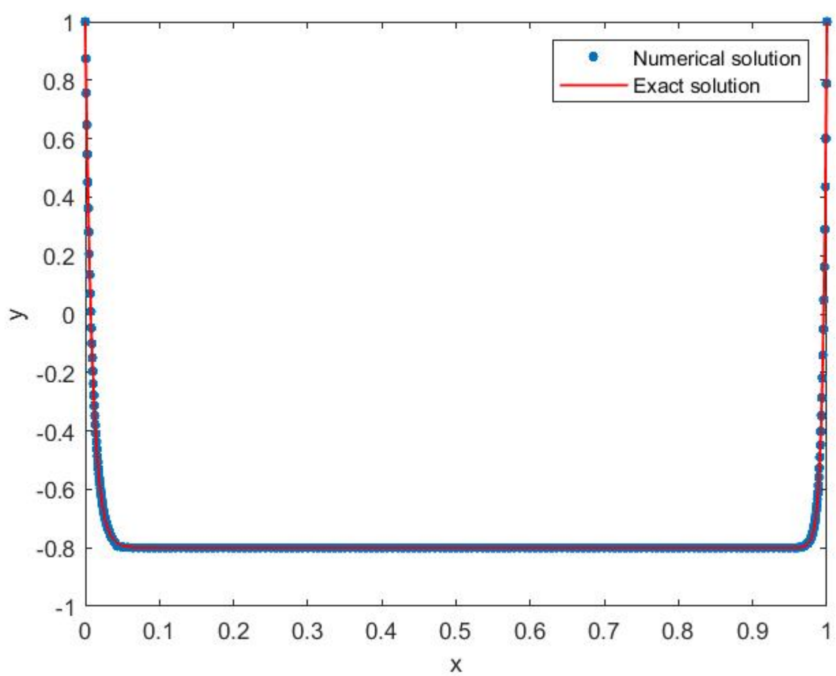
Figure 3.6: Solutin plot for $\epsilon = 2^{-6}$ and $N = 128$, For table 3.4

Table 3.5 : The maximum norm error and Order of Convergence for $\delta = 0, \eta = 0.5\epsilon$ under Fitted mesh finite difference method with Shishkin mesh. [30]

N/ ϵ	**32**	**64**	**128**	**256**	**512**
2^{-1}	0.0092	0.0046	0.0023	0.0012	0.0005762
	1.00	1.00	0.94	1.05	1.00
2^{-2}	0.0193	0.0097	0.0048	0.0024	0.0012
	0.99	1.01	1.00	1.00	1.00
2^{-3}	0.0327	0.0165	0.0083	0.0042	0.0021
	0.98	0.99	0.98	1.00	1.07
2^{-4}	0.0536	0.0312	0.0162	0.0082	0.0041
	0.78	0.94	0.98	1.00	1.03
2^{-5}	0.0527	0.0331	0.0196	0.0113	0.0064
	0.67	0.75	0.79	0.82	0.87
2^{-6}	0.0524	0.0330	0.0195	0.0113	0.0064
	0.66	0.78	0.78	0.82	0.87
2^{-7}	0.0523	0.0330	0.0195	0.0113	0.0064
	0.66	0.78	0.78	0.82	0.87
2^{-8}	0.0523	0.0330	0.0195	0.0121	0.0072
	0.66	0.78	0.68	0.74	0.87



(a) Solution plot for $\epsilon = 2^{-3}$ and $N = 64$



(b) Solution plot for $\epsilon = 2^{-7}$ and $N = 256$

Figure 3.7: Solution Plots for Table 3.5

Table 3.6 : The maximum norm error and Order of Convergence for $\delta = 0.5\epsilon, \eta = 0$ under Fitted mesh finite difference method with Shishkin mesh. [30]

N/ ϵ	**32**	**64**	**128**	**256**	**512**
2^{-1}	0.0073	0.0037	0.0018	0.0009259	0.0004633
	0.98	1.03	0.95	0.99	1.00
2^{-2}	0.0148	0.0075	0.0037	0.0019	0.0009394
	0.98	1.01	0.96	1.02	0.99
2^{-3}	0.0252	0.0127	0.0064	0.0032	0.0016
	0.99	0.99	1.00	1.00	1.00
2^{-4}	0.0420	0.0249	0.0125	0.0063	0.0031
	0.75	0.99	0.98	1.02	0.95
2^{-5}	0.0618	0.0258	0.0152	0.0087	0.0049
	1.26	0.76	0.80	0.82	0.86
2^{-6}	0.0763	0.0379	0.0180	0.0087	0.0049
	1.01	1.07	1.04	0.82	0.86
2^{-7}	0.0797	0.0420	0.0127	0.0109	0.0055
	0.92	0.95	0.99	0.98	0.97
2^{-8}	0.0805	0.0431	0.0229	0.0120	0.0063
	0.90	0.91	0.93	0.92	0.97

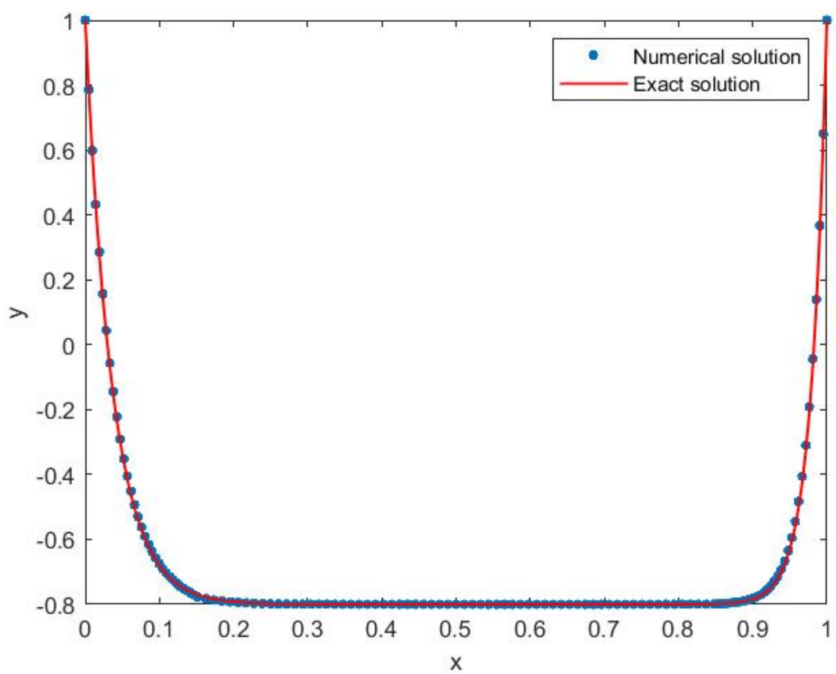


Figure 3.8: Solutin plot for $\epsilon = 2^{-5}$ and $N = 128$, For table 3.6

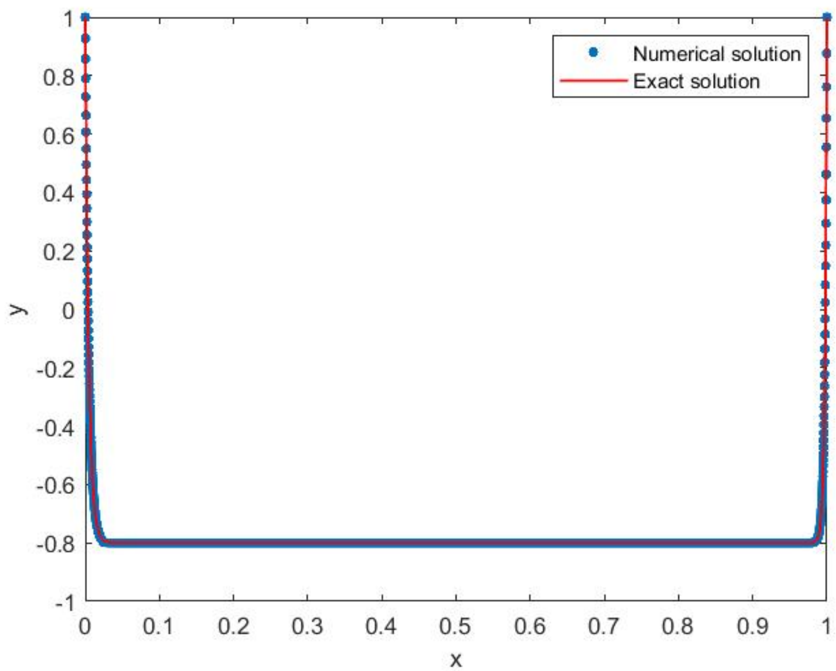


Figure 3.9: Solutin plot for $\epsilon = 2^{-8}$ and $N = 512$, For table 3.6

Chapter 4

Conclusion

We addressed a singularly perturbed differential difference equation in this paper. Two methods are offered to solve boundary-value problems for singularly perturbed differential-difference equations: a standard finite difference method with uniform mesh and an ϵ-piecewise uniform fitted mesh approach. Both methods account for delay and advance changes with layer behavior. These BVPs are found in the literature in several situations, such as the variational problem in control theory and the estimation of the expected time for the formation of nerve cell action potentials. We can summarize the working of the two numerical methods in the following manner.

1. **Standard finite Difference method with uniform mesh :-** This technique of standard finite difference method is based on a mesh spacing with equidistant mesh points throughout the interval. The mesh spacing is not biased towards the boundary layer region.The boundedness of the solution as well as the derivatives of the solution along with stability and convergence analysis is discussed in this paper [26].Several graphs and tables are used to show the error estimate and the order of convergence for the standard finite difference method.

2. **Fitted mesh finite difference method with piecewise uniform mesh :-** The method makes use of the conventional upwind finite difference operator and a specific type of mesh. Here, we investigate a piecewise uniform fitted mesh, which works well enough to construct the ρ-uniform method. The piecewise uniform mesh is intended to be primarily desirable due to its simplicity, while more complex meshes can be utilized. The established error estimate demonstrates the ρ-uniformity of the approach. We show how small changes impact the boundary layer solution by solving several numerical cases. Graphs illustrating the solution and numerical data reported in terms of maximum mistakes are supplied to demonstrate the approach's effectiveness.

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