Study on Some Generalised Approximation Operators

A Thesis
Submitted for the award of degree of
Doctor of Philosophy
in Mathematics
by

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First and foremost,

*To my **Lord**,*

for always being there for me,

*To my beloved **parents**,*

encourage me to follow my dreams,

*and my **brother**,*

who taught me to be courageous and robust.

Department of Applied Mathematics
Delhi Technological University, Delhi

Certificate

This is to certify that the research work embodied in the thesis entitled "**S**tudy on Some Generalised Approximation Operators" submitted by Neha (2K18/PHD/AM/08) is the result of her original research carried out in the Department of Applied Mathematics, Delhi Technological University, Delhi, for the award of **Doctor of Philosophy** under the supervision of **Prof. Naokant Deo**.

It is further certified that this work is original and has not been submitted in part or fully to any other university or institute for the award of any degree or diploma.

This is to certify that the above statement made by the candidate is correct to the best of our knowledge.

Date: 20 March 2024
Place: Delhi , India

Prof. Naokant Deo
(Supervisor)

Prof. Ramesh Srivastava
(Head of Department)

Declaration

I declare that the research work in this thesis entitled " **Study on Some Generalised Approximation Operators**" for the award of the degree of ***Doctor of Philosophy in Mathematics*** has been carried out by me under the supervision of ***Prof. Naokant Deo***, Department of Applied Mathematics, Delhi Technological University, Delhi, India, and has not been submitted by me earlier in part or full to any other university or institute for the award of any degree or diploma.

I declare that this thesis represents my ideas in my own words and where others' ideas or words have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission.

Neha
(2K18/PHD/AM/08)

Date: 20 March 2024

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Abstract

This thesis is mainly a study of convergence estimates of various approximation operators. Approximation theory is indeed an old topic in mathematical analysis that remains an appealing field of study with several applications. The findings presented here are related to the approximation of specific classes of linear positive operators. The introductory chapter is a collection of relevant definitions and literature of concepts that are used throughout this thesis.

- The second chapter is based on Durrmeyer-type modification of Apostol-Genocchi operators.
- The third chapter is devoted to a modification of the Lupaş-Kantrovich operator that preserves exponential function e^{-x}.
- The fourth chapter is based on the inverse Pólya-Eggenberger distribution.
- The chapter fifth is dedicated to estimating the difference between Mastroianni and Gupta operators.
- The chapter sixth is dedicated to Micchelli-type iterative combinations of generalized positive linear operators.
- The last chapter summarizes the thesis with a brief conclusion and also discusses the prospects of this thesis.

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$\mathbb{N}$	the set of natural numbers
$\mathbb{N} \cup \{0\}$	the set of natural number including zero
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers
$[a, b]$	a closed interval
(a, b)	an open interval
Λ	index set
e_n	denotes the n-th monomials with $e_n : [a, b] \to \mathbb{R}$, $e_n(x) = x^n$, $n \in \mathbb{N}_0$
$(x)_n$	the rising factorial $(x)_n := x\,(x+1)\,(x+2)\,...\,(x+n-1)\,,\ (x)_0 = 1$
$\Omega(f; \delta)$	the weighted modulus of continuity
$C[a, b]$	the set of all real-valued continuous functions defined on the compact interval $[a, b]$
$C^r[a, b]$	the set of all real-valued, r-times continuously differentiable function $(r \in \mathbb{N})$
$C[0, \infty)$	the set of all continuous functions defined on $[0, \infty)$
$C_B[0, \infty)$	the set of all continuous and bounded functions on $[0, \infty)$
$C_B^r[0, \infty)$	the set of all r-times continuously differentiable functions in $C_B[0, \infty)$ $(r \in \mathbb{N})$
$B_\rho[0, \infty)$	the set of all functions f defined on $[0, \infty)$ satisfying the condition : $\lvert f(x)\rvert \leq M\rho(x)$, M is a positive constant, and ρ is weight function.
$C_\rho[0, \infty)$	the subspace of all continuous function in $B_\rho[0, \infty)$

Chapter 1

Introduction

In mathematics, the main focus of the theory of approximation is on identifying the best ways to approximate functions with simpler ones and quantifying the errors that are introduced thereby. The foundation of approximation theory was laid on a result first given by Karl Weierstrass [147] in 1885, which states that for every continuous function f on a closed interval $[a, b]$ and any $\epsilon > 0$, there exists a polynomial p of degree n on $[a, b]$ such that

$$|f(x) - p(x)| < \epsilon, \ \forall x \in [a, b].$$

In other words, any continuous function on a closed and bounded interval can be uniformly approximated on that interval by polynomials to any degree of accuracy.

1.1 Preliminaries

In this section, we recall some definitions and properties regarding approximation operators discussed here that will be of interest to the whole thesis.

1.1.1 Linear Positive Operators

Definition 1.1.1 *Let X and Y be two linear spaces of real functions. Then the mapping $L : X \to Y$ is a linear operator if:*

$$L(\alpha f + \beta g; x) = \alpha L(f; x) + \beta L(g; x),$$

for all $f, g \in X$ and $\alpha, \beta \in \mathbb{R}$. If for all $f \in X$ and $f \geq 0$, it follows that $L(f; x) \geq 0$, then L is called a positive operator.

Next, we define the modulus of continuity, mainly used to measure quantitatively the uniform continuity of functions.

1.1.2 Usual and Higher Order Modulus of Continuity

Definition 1.1.2 *Let $f \in C[a,b]$ and $\delta \geq 0$, then*

$$\omega(f;\delta) = \sup\{|f(x+h) - f(x)| : x, x+h \in [a,b], 0 \leqslant h \leqslant \delta\}.$$

Where ω is known as the usual modulus of continuity or simply first order modulus of continuity which was introduced by H. Lebesgue in 1910.

Some of the error estimates in this thesis are given in terms of the modulus of continuity of higher order. Therefore we now give the definition of ω_r, $r \in \mathbb{N}$, as given in 1981 by L. L. Schumaker [135].

Definition 1.1.3 *Let $f \in C[a,b]$, then for $r \in \mathbb{N}$ and $\delta \geq 0$, the modulus of continuity of order r is defined as:*

$$\omega_r(f;\delta) = \sup\left\{\left|\Delta_h^r f(x)\right| : x, x+rh \in [a,b],\ 0 \leqslant h \leqslant \delta\right\},$$

where

$$\Delta_h^r f(x) = \sum_{i=0}^{r} (-1)^{r-i} \begin{pmatrix} r \\ i \end{pmatrix} f(x+ih),$$

denotes the forward difference with step size h. In particular, for $r = 1$, $\omega(f,\delta)$ is the usual modulus of continuity.

Proposition 1.1.4 *The modulus of continuity of order r verifies the following properties:*

1. *$\omega_r(f;.)$ is a positive, non decreasing and continuous function on $(0,\infty)$;*
2. *$\omega_r(f;0) = 0$;*
3. *$\omega_r(f;.)$ has sub-additive property;*
4. *$\omega_{r+1}(f;.) \leq 2\omega_r(f,.)$ for all $x \geq 0$;*
5. *$\omega_r(f;nx) \leq n^r \omega_r(f;x)$ for all $n \in \mathbb{N}$ and $x > 0$;*
6. *$\omega_r(f;kx) \leq (1+[k])^r \omega_r(f;x)$ for all $k > 0$ and $x > 0$, where $[\alpha]$ denotes the integer part of α;*

For $r = 1$, these properties are valid for the usual modulus of continuity $\omega(f;.)$.

1.1.3 Ditzian-Totik Modulus of Smoothness

We recall the definitions of the Ditzian-Totik first-order modulus of smoothness and the K-functional [54]. Let $\varphi(x) = \sqrt{x(1-x)}$ and $f \in C[0,1]$, then the first order modulus of smoothness is defined as:

$$\omega_\varphi(f;\delta) = \sup_{0\le h\le\delta}\left\{\left|f\left(x+\frac{h\varphi(x)}{2}\right)\right| - \left|f\left(x-\frac{h\varphi(x)}{2}\right)\right|, x \pm \frac{h\varphi(x)}{2} \in [0,1]\right\}. \tag{1.1}$$

Further, the corresponding Peetre's K-functional is given by

$$K_\varphi(f;\delta) = \inf_{g\in W_\varphi}\{\|f-g\| + \delta\,\|\varphi g'\|\}, \quad \delta > 0, \tag{1.2}$$

where

$$W_\varphi = \{g : \|\varphi g'\| < \infty,\ g \in AC[0,1]\},$$

and $AC[0,1]$ denotes the space of all absolutely continuous functions on every interval $[a,b] \subset (0,1)$ and $\|.\|$ is the uniform norm in $C[0,1]$. Moreover, from [[54], p. 11], there exists a constant $C > 0$ such that:

$$K_\varphi(f;\delta) \le C\omega_\varphi(f;\delta). \tag{1.3}$$

1.1.4 Weighted Spaces and Corresponding Modulus of Continuity

Let $B_\rho(I)$ be the space of all functions f defined on the interval $I \in \mathbb{R}$ for which there exists a constant $M > 0$ such that $|f(x)| \le M\rho(x)$, for every $x \in I$, where ρ is a positive continuous function called weight function. In 1974, A.D. Gadjiev [60; 61] introduced the weighted space $C_\rho(I)$, which is the set of all continuous functions f on the interval $I \in \mathbb{R}$ and $f \in B_\rho(I)$. This space is a Banach space, endowed with the norm

$$\|f\|_\rho = \sup_{x\in I}\frac{|f(x)|}{\rho(x)}.$$

For $I = [0,\infty)$, the subspace $C^*_\rho[0,\infty)$ is defined as follows:

$$C^*_\rho[0,\infty) := \{f \in C_\rho[0,\infty) : \lim_{x\to\infty}\frac{|f(x)|}{\rho(x)} = k < +\infty\}.$$

Many authors [15; 82] use the following weighted modulus of continuity $\Omega(f,\delta)$ for $f \in C_B(0,\infty)$:

$$\Omega(f;\delta) = \sup_{x\in[0,\infty),|h|<\delta}\frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)}.$$

Let us denote by $C^*[0,\infty)$, the Banach space of all real-valued continuous functions on $[0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and is finite endowed with the uniform norm. In [35], the following theorem is proved:

Theorem 1.1.5 *If the sequence $A_n : C^*[0,\infty) \to C^*[0,\infty)$ of positive linear operators satisfies the conditions*

$$\lim_{n\to\infty} A_n\left(e^{-kt};x\right) = e^{-kx},\ k = 0,1,2$$

uniformly in $[0,\infty)$*, then*

$$\lim_{n\to\infty} A_n(f;x) = f(x),$$

uniformly in $[0,\infty)$*, for every* $f \in C^*[0,\infty)$.

1.1.5 Modulus of Continuity for Exponential Functions

To find the rate of convergence of operators satisfying the conditions from the above theorem, we use the following modulus of continuity:

$$\omega^*(f;\delta) = \sup\left\{|f(x) - f(t)| :\ x,t \geqslant 0, \left|e^{-x} - e^{-t}\right| \leqslant \delta\right\}$$

defined for every $\delta \geq 0$ and every function $f \in C^*[0,\infty)$.

Proposition 1.1.6 *The modulus of continuity defined for exponential functions has the following properties:*

1. $\omega^*(f;\delta)$ *can be expressed in terms of usual modulus of continuity, by the relation*

$$\omega^*(f;\delta) = \omega(f^*;\delta),$$

where f^* is the continuous function on $[0,\infty)$ *given by:*

$$f^*(x) = \begin{cases} f(-\ln(x)), & x \in (0,\infty], \\ \lim\limits_{t\to\infty} f(t), & x = 0. \end{cases}$$

2. *For every $t,x \in [0,1]$ and $M > 0$, we have*

$$\omega^*(f;\delta) \leq \left(1 + e^M\right)\omega(f;\delta)\cdot$$

3. *The defined modulus of continuity ω^* possess the following property:*

$$|f(t) - f(x)| \leq \left(1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2}\right)\omega(f^*;\delta).$$

1.1.6 Lipschitz Spaces

Definition 1.1.7 *For non-negative real numbers a and b, the Lipschitz space [122] is defined as:*

$$Lip_M(\beta) = \left\{f \in C[0,1] : |f(t) - f(x)| \leq M\frac{|t-x|^\beta}{(ax^2 + bx + t)^{\beta/2}};\ x,t \in (0,1)\right\},$$

where $\beta \in (0,1]$ and M is a positive constant.

To discuss some direct estimates of operators in this thesis, we used Lipschitz-type maximal function of order β defined by Lenze [101] as follows:

$$\varpi_\beta(f;x) = \sup_{t\neq x, t\in[0,1]} \frac{|f(t)-f(x)|}{|t-x|^\beta}, \quad x\in[0,1], \tag{1.4}$$

where $\beta \in (0,1]$.

The definitions we provided above are for functions with a single variable. These definitions are slightly different for a function with two independent variables.

1.1.7 Total and Partial Modulus of Continuity

To establish the degree of approximation of bivariate operators in the space of continuous functions on the compact set $I^2 = [a,b]\times[a,b]$, the total modulus of continuity for the function $f \in C(I^2)$ is defined by:

$$\omega_{total}(f;\delta_1,\delta_2) = \sup\{|f(t_1,t_2)-f(x,y)| : (t_1,t_2),\ (x,y)\in I^2,\ |t_1-x|\leq\delta_1,\ |t_2-y|\leq\delta_2\}.$$

Further, the partial moduli of continuity with respect to the independent variables x and y is given as:

$$\omega^{(1)}(f;\delta) = \sup\{|f(x_1,y)-f(x_2,y)| : y\in I,\ |x_1-x_2|\leq\delta\},$$

and

$$\omega^{(2)}(f;\delta) = \sup\{|f(x,y_1)-f(x,y_2)| : x\in I,\ |y_1-y_2|\leq\delta\}.$$

Both total and partial modulus of continuity for bivariate functions satisfy the properties of the usual modulus of continuity and can be studied more in [22].

1.2 Historical Background and Literature Review

A simple yet powerful tool for deciding whether a given sequence of linear positive operators on $C[0,1]$ or $C[0,2\pi]$ is an approximation process or not are the Korovkin theorems. These theorems are abstract results in approximation which gives conditions for uniform approximation of continuous functions on a compact metric space. The Korovkin theorem [[21] pp.218] elegantly says that, if $(L_n)_{n\geq1}$ is an arbitrary sequence of linear positive operators on the space $C[a,b]$, and if

$$\lim_{n\to\infty} L_n(e_i;x) \to e_i \ \text{ uniformly on } [a,b],$$

for the test functions $e_i(x) = x^i$, $i = 0, 1, 2$ then

$$\lim_{n\to\infty} L_n(f; x) \to f \text{ uniformly on } [a, b],$$

for each $f \in C[a, b]$.

The above theorem, known as Korovkin's first theorem, was proposed by P. P. Korovkin [100] in 1953. Korovkin's second theorem has a similar statement, but the space $C[0, 1]$ is replaced by the space $C[0, 2\pi]$, i.e., the space of all 2π periodic real-valued functions on $\mathbb{R}$. The test functions e_i in this case belong to the set $\{1, \cos(x), \sin(x)\}$ for $i = 0, 1, 2$ respectively. H. Bohmann [34] in 1952 had proved a result similar to Korovkin's first theorem but concerning sequences of linear positive operators on $C[0, 1]$ of the form

$$L(f; x) = \sum_{i\in I} f(a_i)\,\phi_i, \quad f \in C[0, 1],$$

where $(a_i)_{i\in\Lambda}$ is a finite set of numbers in $[0, 1]$ and $\phi_i \in C[0, 1]$, $i \in \Lambda$. Therefore, Korovkin's first theorem is also known as Bohman-Korovkin Theorem. An immediate analogue of Korovkin's theorem does not hold if the domain of definition of the function f becomes unbounded and hence requires the function to have some finite limit at infinity. For continuous and unbounded functions on $[0, \infty)$, A. D Gadžiev [60] in 1974 introduced a weighted space $C_\rho[0, \infty)$ defined as the set of all continuous functions f on the interval $[0, \infty)$ for which there exists a positive constant M such that $|f(x)| \leq M\rho(x)$, for every $x \in [0, \infty)$. Here ρ is a positive continuous function called the weight function. The space $C_\rho[0, \infty)$ is a Banach space equipped with the norm

$$||f||_\rho = \sup_{x\in[0,\infty)} \frac{|f(x)|}{\rho(x)}.$$

The Korovkin theorem by Gadžiev is given as: Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous, strictly increasing and unbounded function. Set $\rho(x) = 1 + \varphi^2(x)$. If the sequence of linear positive operators $L_n : C_\rho[0, \infty) \to C_\rho[0, \infty)$ verifies

$$\lim_{n\to\infty} \left\| L_n(\varphi^i; x) - \varphi^i(x) \right\|_\rho = 0, \ \ i = 0, 1, 2.$$

Then,

$$\lim_{n\to\infty} ||L_n(f; x) - f(x)||_\rho = 0,$$

for every $f \in C_\rho[0, \infty)$ for which $\lim\limits_{n\to\infty} \frac{f(x)}{\rho(x)}$ exists and is finite.

With the application of Korovkin theorems to study the uniform convergence of linear positive operators, advancement in approximation theory began with the development of new linear positive operators, the first and most important of which are the Bernstein polynomials. In 1912, S. N. Bernstein [32] gave an elegant proof of the famous Weierstrass approximation theorem by defining a sequence of polynomials called Bernstein operators on the closed interval $[0,1]$ (extended on $[a,b]$ by simple manipulations). These operators are defined as:

Let f be a bounded function on $[0,1]$. The Bernstein operator of degree n with respect to f is defined as:

$$B_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right); \quad x \in [0,1], \tag{1.5}$$

where

$$b_{n,k}(x) = \left(\begin{array}{c} n \\ k \end{array}\right) x^k (1-x)^{n-k}; k = 0,1,2,\cdots n,$$

and $\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}$ represents the binomial coefficient. It should be noted that $b_{n,k}(x) \in P_n, \ k = 0,1,2,\cdots,n,$ where P_n denotes the space of all polynomials of degree at most n, are the so-called Bernstein polynomials.

Proposition 1.2.1 *Some important properties of the Bernstein polynomials are listed as follows:*

1. *The Bernstein polynomials of degree n form a basis for P_n ;*

2. *The Bernstein polynomials satisfy symmetry property $b_{n,k}(x) = b_{n,n-k}(1-x), \ k = 0,1,2,\cdots,n$;*

3. *The Bernstein polynomials are all positive over $[0,1]$, that is $b_{n,k}(x) \geq 0, \ \forall x \in [0,1]$;*

4. *Another important property is that the Bernstein polynomials form a partition of unity:*

$$\sum_{k=0}^{n} b_{n,k}(x) = 1;$$

5. *The recursive formula for the Bernstein polynomials is as follows:*

$$b_{n,k}(x) = (1-x)\, b_{n-1,k}(x) + x b_{n-1,k-1}(x).$$

Since the Bernstein operators were only suitable for approximating functions on a compact interval, O. Szász in 1950 [143], and G. Mirakiyan in 1941 presented a generalization of these operators for a continuous function f on the interval $[0, \infty)$, which later came to be known as Szász-Mirakiyan operators. These operators are defined as:

$$S_n(f; x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, \infty), \tag{1.6}$$

where

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}.$$

In 1957, V. A. Baskakov [30] introduced another sequence of linear positive operators on the interval $[0, \infty)$ called Baskakov operators which are defined as:

$$V_n(f; x) = \sum_{k=0}^{\infty} v_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, \infty), \tag{1.7}$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}.$$

To approximate integrable functions on the compact interval $[a, b]$, Kantorovich [92] was the first to define the integral variant of Bernstein operators by replacing the weight function with the average mean of the weight function in the vicinity of the point $\frac{k}{n}$ as:

$$\hat{B}_n(f; x) = (n+1)\sum_{k=0}^{n} b_{n,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\,dt,$$

where $b_{n,k}(x)$ is defined in (1.5).

Similarly, Szász-Kantorovich operators on the unbounded interval $[0, \infty)$ for given basis function $s_{n,k}(x)$ in (1.6) are defined as:

$$\hat{S}_n(f; x) = n\sum_{k=0}^{\infty} s_{n,k}(x) \int_{k/n}^{(k+1)/n} f(t)\,dt. \tag{1.8}$$

For Baskakov operators, the integral variant on the semi-real axis is:

$$\hat{V}_n(f; x) = (n-1)\sum_{k=0}^{\infty} v_{n,k}(x) \int_{k/(n-1)}^{(k+1)/(n-1)} f(t)\,dt, \tag{1.9}$$

where $v_{n,k}(x)$ is defined in similar manner as in (1.7). To estimate functions on an unbounded interval, Kantorovich forms of various approximation operators have been defined from time to time. For further reference, one can visit the articles [6; 14; 17; 45; 66; 118].

In 1967, J. L. Durrmeyer [55] gave a more generalised integral modification of Bernstein operators by replacing the values of $f(k/n)$ by an integral over the weight function on the interval $[0, 1]$. These so-called Bernstein-Durrmeyer operators were first studied by Derrienic [48] and are defined as:

$$\tilde{B}(f;x) = (n+1)\sum_{k=0}^{n} b_{n,k}(x)\int_{0}^{1} b_{n,k}(t)\, f(t)\, dt. \tag{1.10}$$

In the year 1985, Mazhar and Totik [115] introduced the Szász-Durrmeyer operators as follows:

$$\tilde{S}_n(f;x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_{0}^{\infty} s_{n,k}(t)\, f(t)\, dt. \tag{1.11}$$

In the same year Sahai and Prasad [133] also established the Baskakov-Durrmeyer operators defined as follows:

$$\tilde{V}_n(f;x) = (n-1)\sum_{k=0}^{\infty} v_{n,k}(x)\int_{0}^{\infty} v_{n,k}(t)\, f(t)\, dt. \tag{1.12}$$

where $b_{n,k}(x)$, $s_{n,k}(x)$ and $v_{n,k}(x)$ are same as in (1.5), (1.6) and (1.7) respectively. Durrmeyer-type variants of a number of linear positive operators were constructed in subsequent years. One can refer to the articles [9; 11; 42; 62; 96].

As approximation theory continues to advance, researchers have been motivated to explore the development of innovative approximation operators that exhibit faster convergence rates and are applicable across a diverse range of functions and spaces. In 1976, May [112] introduced operators of the following form:

$$W_\lambda(f;x) = \int_{-\infty}^{\infty} S(\lambda, x, t)\, f(t)\, dt,$$

and termed it as exponential operators provided they satisfy two conditions, first is the homogenous partial differential equation

$$\frac{\partial}{\partial x} S(\lambda, x, t) = \frac{\lambda(t-x)}{q(x)} S(\lambda, x, t), \tag{1.13}$$

where $S(\lambda, x, t) \geq 0$ is the kernel of these operators and q is a polynomial of at most degree n which is analytic and positive for $x \in (a, b)$ for some a, b such that $-\infty \leq a \leq b \leq +\infty$, while second is the normalization condition

$$W_\lambda(1;x) = \int_{-\infty}^{\infty} S(\lambda, x, t) dt = 1. \tag{1.14}$$

These operators satisfying the above conditions are, for example, the Bernstein operators, Szás- Mirakiyan operators, Post-Widder operators, Gauss-Weierstrass operators and Baskakov operators. These well-known operators are thus referred to as exponential operators. Some approximation properties were also studied for polynomials of degree at most 2.

Another approximation operator examined in this thesis is the beta-type operators. A vital tool among the researchers to study linear positive operators is Euler's beta function, which for $\alpha,\beta > 0$ is defined as follows:

$$B(\alpha,\beta) = \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}} dt = \frac{\lceil\alpha + \lceil\beta}{\lceil(\alpha+\beta)}. \tag{1.15}$$

The transform (1.15) reproduces the Durrmeyer type modification of the linear positive operators defined by (2.18) for all real-valued continuous and bounded functions f on $[0, \infty)$. The derived operators have been studied extensively by researchers over the past few decades; for instance, see [1; 43; 46; 52].

1.3 Improvement in the Order of Approximation

The central idea in approximation theory is to estimate the rate of convergence of the sequence of operators using various convergence methods. These methods aim to improve the rate of convergence of operators, thereby reducing the error induced during the approximation process. Let $E_n(f)$ be the error function for the best uniform approximation of function f by trigonometric or algebraic polynomials T_n of degree n, then

$$E_n(f) = \inf_{T_n} \sup_x |f(x) - T_n(x)|.$$

We expect that the smoother the function f is, the faster $E_n(f)$ converges to zero.
If f is r times continuously differentiable on some compact interval, then

$$E_n(f) \leqslant C_r \|f^r\| n^{-r}, \quad n = 1, 2, \cdots$$

For instance, we say that $E_n(f)$ tends to zero at least as fast as $1/n$ whenever f is differentiable i.e. degree of approximation of f is $1/n$ and $1\big/n^2$ when it is twice differentiable, and so on. Estimates of this type of estimating error have a rich history. The first results of this type were given by S. N. Bernstein [33] and later Favard [58] found the best constant C_r. Jackson [87] then refined the above given estimate of $E_n(f)$ by using subtler measures of the smoothness of a function f such as modulus of continuity $\omega(f,\delta)$ for $f \in C(I)$ as: If f is r times continuous differentiable, then

$$E_n(f) \leqslant C_r n^{-r} \omega\left(f^r, n^{-1}\right), \quad n = 1, 2, \cdots .$$

The continuity of f ensures that $\omega(f,\delta) \to 0$ as $\delta \to 0$. Although the linear positive operators are conceptually simpler and easy to construct and study, they lack the rapidity of convergence for sufficiently smooth functions. In the same context, a well-known theorem of Korovkin states that the optimal rate of convergence for any sequence of linear operators is at most $O(n^{-2})$. Thus if we want to have a better order of approximation for smoother functions, we must slacken the positivity condition. Several investigations indicate that even when a sequence or class of linear positive operators is saturated with a certain order of approximation, some carefully chosen linear combinations of its members give a better order of approximation of smoother functions.

Our interest in this thesis is to improve the order of approximation of classical and existing operators. For instance, many of the standard operators in approximation theory preserve the test functions e_0 and e_1 for all $n \in \mathbb{N}$, i.e.,

$$L_n(e_0; x) = e_0 \quad L_n(e_1; x) = e_1.$$

An important approach to improving the order of approximation was given by J. P. King in his pioneer work [98]. He presented a non-trivial sequence of positive linear operators defined on $C[0,1]$ that preserved the test functions e_0 and e_2. Let $\{r_n(x)\}$ be a sequence of continuous functions defined on $[0,1]$ such that $r_n(x) \in [0,1]$. Then the operators $V_{n,r_n} : C[0,1] \to C[0,1]$ are defined as:

$$V_{n,r_n}(f; x) = \sum_{k=0}^{n} (r_n(x))^k (1 - r_n(x))^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1],$$

where

$$r_n(x) = \begin{cases} x^2, & n = 1 \\ \frac{-1}{2(n-1)} + \sqrt{\left(\frac{n}{n-1}\right)x^2 + \frac{1}{4(n-1)^2}}, & n = 2, 3, \cdots \end{cases}$$

The operators V_{n,r_n} interpolate f at the endpoints 0 and 1 and are not polynomial operators. King also proved that the order of approximation of operators V_{n,r_n} is at least as good as the order of approximation of Bernstein operators for $x \in [0, \frac{1}{3}]$. Inspired by his work, other modifications of well-known operators were constructed as well to fix certain functions and to study their approximation and shape-preserving properties. In [36] Cárdenas-Morales et al. presented a family of sequences of linear Bernstein-type operators $B_{n,\alpha}$, $n > 1$, depending on a real parameter $\alpha \geq 0$, and fixing the polynomial function

$e_2+\alpha e_1$. Among other things, the authors prove that if f is convex and increasing on $[0,1]$, then $f(x) \leq B_{n,\alpha}(f;x) < B_n(f;x)$ for every $x \in [0,1]$. In their research paper, Duman and Özarsalan [56] proposed an improved version of the classical Szász-Mirakiyan operators that provide a more accurate error estimation. Similarly, Ozsarac and Acar [125] introduced a new modification of the Baskakov operators that preserve the functions $e^{\mu t}$ and $e^{2\mu t}$, where $\mu > 0$.

In this thesis, we have used King's approach as well as the sequential approach to present a better modification of various operators, thereby reducing the error and improving the rate of approximation of the considered operators.

1.4 Chapter-Wise Overview of the Thesis

The thesis consists of seven chapters, whose contents are described below:

The literature and historical context of some key approximation operators are covered in **Chapter 1**. Along with a brief summary of the chapters this thesis is divided into, we also discuss some preliminary instruments that will be employed subsequently to derive our main results.

Chapter 2 is dedicated to certain operators based on a class of orthogonal polynomials called Apostol-Genocchi polynomials. It is majorly divided into two sections. The **first section** considers a Durrmeyer-type modification of Apostol-Genocchi operators based on Jain operators for $f \in C[0,\infty)$.

We established a Durrmeyer-type modification of Apostol-Genocchi operators based on Jain operators:

$$\mathcal{H}_n(f;x) = \sum_{k=0}^{\infty} \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi)\, d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^{\infty} p_{n,k}^{(\beta)}(\xi) f(\xi)\, d\xi \tag{1.16}$$

$$= \sum_{k=0}^{\infty} \frac{\left\langle p_{n,k}^{(\beta)}(\xi), f(\xi) \right\rangle}{\left\langle p_{n,k}^{(\beta)}(\xi), 1 \right\rangle} b_{n,k}^{(\alpha)}(x),$$

where

$$b_{n,k}^{(\alpha)}(x) = e^{-nx} \left(\frac{1+e\lambda}{2} \right)^{\alpha} \frac{G_k^{(\alpha)}(nx;\lambda)}{k!}, \tag{1.17}$$

and

$$p_{n,k}^{(\beta)}(x) = \frac{nx(nx+k\beta)^{k-1} e^{-(nx+k\beta)}}{k!}.$$

Here $G_k^{(\alpha)}(nx;\lambda)$ is the generalized Apostol-Genocchi polynomials of order α, and $p_{n,k}^{(\beta)}(x)$ is the Jain basis. First, we estimate the convergence rate of Jain-Durrmeyer operators associated with the Apostol-Genocchi operators. We establish approximation estimates, such as a global approximation theorem along with some convergence estimates in terms of the usual modulus of continuity, and study the approximation behaviour of these operators (2.10) including $K-$functionality. We also estimate the convergence rate of the proposed operators for functions in Lipschitz-type space. Moreover, graphical interpretation to determine the absolute error for some particular values of the parameters is performed using Mathematica software.

The **second section** of this chapter is dedicated to a Durrmeyer-type modification of Apostol-Genocchi operators based on the beta operator. In this section, For $f \in C[0,\infty)$, Prakash [129] et al. considered a sequence of linear positive operators using Apostol-Genocchi operators polynomials. We presented a Durrmeyer form of Apostol-Genocchi operators based on the beta function as:

$$V_n^{(\alpha)}(f;x) = \sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x)\frac{1}{\beta(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}f(t)dt, \tag{1.18}$$

where

$$v_{n,k}^{(\alpha)}(x) = e^{-nx}\left(\frac{1+e\lambda}{2}\right)^{\alpha}\frac{G_k^{(\alpha)}(nx;\lambda)}{k!}, \tag{1.19}$$

and $\beta(k+1,n)$ is Beta function given as:

$$\beta(\alpha,\beta) = \int_0^{\infty}\frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}dt = \frac{\lceil\alpha + \lceil\beta}{\lceil(\alpha+\beta)}, \quad \alpha,\beta > 0.$$

Here $G_k^{(\alpha)}(nx;\lambda)$ is the generalized Apostol-Genocchi polynomials of order α, and $\beta(k+1,n)$ is the Beta basis. We propose the Beta function associated with the Apostal-Genocchi polynomials to study the approximation properties of these Durrmeyer operators and obtain the rate of convergence. Furthermore, we give a direct approximation theorem using first and second-order modulus of continuity, local approximation results for Lipschitz class functions and direct theorem for the usual modulus of continuity.

Chapter 3 deals with a modification of the Lupaş-Kantrovich operator that preserve exponential function e^{-x}. We considered these modified operators in the following way:

$$\tilde{\mathcal{K}}_m(f;x) = (m+1)\sum_{k=0}^{\infty}2^{-m\lambda_m(x)}\frac{(m\lambda_m(x))_k}{2^k k!}\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}}f(u)du, \tag{1.20}$$

where

$$\lambda_m(x) = \frac{x + \log\left((1+m)\left(1 - e^{\frac{-1}{m+1}}\right)\right)}{m \log\left(2 - e^{-\frac{1}{m+1}}\right)}. \tag{1.21}$$

Here $\lambda_m(x)$, is calculated under the assumption that these operators preserve exponential function e^{-x}. The moments and central moments of the proposed operators are evaluated with the help of moment-generating functions. We estimate the convergence rate of the operators in terms of both the usual and exponential modulus of continuity. Our analysis also includes a global estimate and quantitative Voronovskaya results. Some approximation results associated with the rate of convergence and order of approximation are also provided, along with some numerical examples and graphical representations.

Chapter 4 is related to the study of operators associated with inverse Pólya-Eggenberger distribution. It is divided into two sections and **the first section** focuses mainly on introducing Bézier variant of Baskakov operators associated with inverse Pólya-Eggenberger distribution, a concept originated by Stancu and then thoroughly researched by Deo and Dhamija[45]. For $\theta \geq 1$, Bézier variant is defined as:

$$\hat{L}^{(\beta)}_{n,\theta}(\varphi; x) = \sum_{r=1}^{\infty} w^{\theta,\beta}_{n,r}(x) \int_0^{\infty} \chi^{\rho}_{n,r}(t)\, \varphi(t) dt + w^{\theta,\beta}_{n,0}(x) \varphi(0), \tag{1.22}$$

where

$$\chi^{\rho}_{n,r}(t) = \begin{cases} \frac{n\rho}{\Gamma(r\rho)} e^{-n\rho t} (n\rho t)^{r\rho - 1}, & c = 0 \\ \\ \frac{\Gamma\left(\frac{n\rho}{c} + r\rho\right)}{\Gamma(r\rho)\Gamma\left(\frac{n\rho}{c}\right)} \frac{c^{r\rho} t^{r\rho-1}}{(1+ct)^{\frac{n\rho}{c} + r\rho}}, & c \in \mathbb{N}, \end{cases} \tag{1.23}$$

$$\int_0^{\infty} \chi^{\rho}_{n,r}(t) t^j dt = \begin{cases} \frac{\Gamma(r\rho + j)}{\Gamma(r\rho)} \frac{1}{\prod_{i=1}^{j} (n\rho - ic)}, & j \neq 0 \\ 1, & j = 0, \end{cases} \tag{1.24}$$

and

$$w^{(\beta)}_{n,r}(x) = \begin{pmatrix} n + r - 1 \\ r \end{pmatrix} \frac{\prod\limits_{i=0}^{r-1} (x + i\beta) \prod\limits_{j=0}^{n-1} (1 + j\beta)}{\prod\limits_{k=0}^{n+r-1} (x + 1 + k\beta)}$$

with $w^{\theta,\beta}_{n,r}(x) = \left(J^{\beta}_{n,r}(x)\right)^{\theta} - \left(J^{\beta}_{n,r+1}(x)\right)^{\theta}$ and $J^{\beta}_{n,r}(x,c) = \sum\limits_{j=r}^{\infty} w^{(\beta)}_{n,j}(x)$. It is obvious that the operators $\hat{L}^{(\beta)}_{n,\theta}(.; x)$ are the linear positive operators.

Special cases:

- For $c = 0$ and $\theta = 0$, the operators (4.5) reduce to Baskakov-Szász type operators based on inverse Pólya-Eggenberger-distribution [95].
- For $c = \beta = 0$ and $\theta = \rho = 1$, the operators (4.5) include Baskakov-Szász operators (see [2; 80]).
- For $c = \beta = 0$, $\theta = 1$ and $\rho \to \infty$, the operators (4.5) reduce to Baskakov operators [30].
- For $c = 0$, $\theta = 1$, $\beta > 0$ and $\rho \to \infty$, the operators (4.5) include Stancu operators [142].

We estimate the approximation behaviour of proposed operators in terms of first and second-order modulus of smoothness. Additionally, the degree of approximation is also established for the functions of the derivative of bounded variation.

The **second section** of this chapter is dedicated to α-Pólya-Baskakov operator [128] based on inverse Pólya-Eggenberger distribution as follows:

$$\overset{\rightharpoonup}{Q}_r^{(\alpha,e)}(h;x) = \sum_{s=0}^{\infty} q_{r,s}^{(\alpha,e)}(x) h\left(\frac{s}{r}\right), \tag{1.25}$$

where α being a non-negative parameter, which may depend only on the natural number r, with $\alpha \to 0$ when $r \to \infty$, $r \geq 1$, $x \in [0,\infty)$, and

$$\begin{aligned} q_{r,s}^{(\alpha,e)}(x) = \alpha &\begin{pmatrix} r+s-1 \\ s \end{pmatrix} \frac{1^{[r,-e]} x^{[s,-e]}}{(1+x)^{[r+s,-e]}} \\ &- (1-\alpha)\begin{pmatrix} r+s-3 \\ s-2 \end{pmatrix} \frac{1^{[r-1,-e]} x^{[s-1,-e]}}{(1+x)^{[r+s-2,-e]}} \\ &+ (1-\alpha)\begin{pmatrix} r+s-1 \\ s \end{pmatrix} \frac{1^{[r-1,-e]} x^{[s,-e]}}{(1+x)^{[r+s-1,-e]}}. \end{aligned}$$

This section explores the approximation properties of a non-negative real parametric generalization of the Baskakov operators based on inverse Pólya-Eggenberger (I-P-E) distribution. As a result of this study, we can obtain some approximation results, including the Voronovskaya type asymptotic formula, error estimate in terms of modulus of continuity and the sense of k-functional, and weighted approximation.

Chapter 5 investigates the difference between Mastroianni operators with Gupta operators in terms of modulus of continuity of first order. We considered here

Srivastava-Gupta [68] operators

$$\mathcal{G}_{n;c}(f;x) = \sum_{i=0}^{\infty} v_{n,i}(x,c)\mathcal{H}_{n,i}(f), \tag{1.26}$$

The Mastroianni operators [113; 114] are mentioned below:

$$\mathcal{M}_{n,c}(f;x) = \sum_{i=0}^{\infty} v_{n,i}(x,c)\mathcal{F}_{n,i}(f), \tag{1.27}$$

where

$$v_{n,i}(x,c) = \frac{(-x)^i}{i!}\tau_{n,c}^{(i)}(x), \mathcal{F}_{n,i}(f) = f\left(\frac{i}{n}\right),$$

with individual cases, which are mentioned below:

- If $\tau_{n,0}(x) = exp(-nx)$ then $v_{n,i}(x,0) = exp(-nx)\frac{(nx)^i}{i!}$, and the operators (5.1) reduce to Szász operators.
- If $c \in \mathbb{N}$ and $\tau_{n,c}(x) = \frac{1}{(1+cx)^{n/c}}$, then we have $v_{n,i}(x,c) = \frac{(n/c)_i}{i!}\frac{(cx)^i}{(1+cx)^{\frac{n}{c}+i}}$, and the operators (5.1) reduce to Baskakov operators.
- If $\tau_{n,-1}(x) = (1-x)^n$ then $v_{n,i}(x,-1) = \binom{n}{i}x^i(1-x)^{n-i}$, and the operators (5.1) reduce to Bernstein polynomials,

$$\mathcal{H}_{n,i}(f) = (n+c)\int_0^{\infty} v_{n+2c,i-1}(t,c)f(t)dt, \quad 1 \le i < \infty, \quad \mathcal{H}_{n,0}(f) = f(0).$$

We study the approximation properties of Gupta operators and the approximation of the difference of operators and find an estimate for the difference of Mastroianni operators with Gupta operators in terms of modulus of continuity of first order. We give the rate of convergence with the help of the moduli of continuity and Peetre's K-functional and in the last section, the weighted approximation of functions is studied.

Chapter 6 is dedicated to Micchelli-type iterative combinations of generalized positive linear operators $T_{n,k} : C_2(\mathbb{R}_+) \to C(\mathbb{R}_+)$, which is defined as:

$$T_{n,k}(f;x) = \sum_{r=1}^{k} (-1)^{r+1}\binom{k}{r} L_{n,c}^{r}(f;x). \tag{1.28}$$

The generalised form of linear positive operators $L_{n,c} : C_2(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined as:

$$L_{n,c}(f;x) = \sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.29}$$

where

$$q_{n.k}(x) = \frac{(-x)^k\,\phi_{n,c}^{(k)}(x)}{k!},$$

and

$$\phi_{n,c}(x) = \begin{cases} (1+cx)^{-n/c} \ ; \ c = -1, \ x \in [0,1] \\ (1+cx)^{-n/c} \ ; \ c > 0, \ \ x \in [0,\infty). \end{cases}$$

This is a generalised form of iterative combinations of positive linear operators with well-known Bernstein and Baskakov operators as its particular case. We have estimated the r^{th} moment of the iterative operator and found a recurrence relation between the central moments and their derivatives. We deduce the Voronovskaya type asymptotic formula and the relation between the error of continuous function and its norm with restrictions on its higher derivatives.

At the end of the thesis, we serve to summarize the research conducted, highlighting the key findings and implications of the study. This summary provides readers with a clear understanding of the research and its significance. Additionally, the author offers their thoughts on the future direction of the research, outlining potential areas for further investigation and providing insight into how the research could be expanded or improved. Overall, the thesis serves as a comprehensive conclusion, bringing together the various elements of the research and providing valuable insights for future research in the field.

We now move on to our second chapter, which explores some important operators based on a class of orthogonal polynomials called Apostol-Genocchi polynomials. These operators are particularly significant and we will explore them in depth throughout this chapter.

Chapter 2

On Durrmeyer Variant of Operators Involving Apostal-Genocchi Polynomials

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In the late 19th century, P. L. Chebyshev initiated the research on orthogonal polynomials, which was further developed by A. A. Markov and T. J. Stieltjes. This chapter focuses on certain operators based on a class of orthogonal polynomials known as Apostol-Genocchi polynomials. The first section constructs the Durrmeyer variant of these certain operators using the Jain operators with real parameters α, β, and λ, whereas the second section deals with the Durrmeyer variant associated with the Beta basis function. We establish approximation estimates such as a global approximation theorem along with some convergence estimates in terms of the usual modulus of continuity and examine the approximation behaviour of proposed operators including $K-$ functional. Furthermore, we estimate the rate of convergence of the proposed operators for function in Lipschitz-type space and local approximation results by using the modulus of continuity. Employing Mathematica software, we show the approximation and the absolute error graphically by varying the values of given parameters.

2.1 Integral Modification of Apostol-Genocchi Operators

2.1.1 Introduction

The classical Bernoulli polynomials $B_n(x)$, Euler polynomials $E_n(x)$, and Genocchi polynomials $G_n(x)$, together with their familiar generalizations $B^{(\alpha)}(x)$, $E^{(\alpha)}(x)$ and $G^{(\alpha)}(x)$ of (real or complex) order α, are usually by means of the following generating functions (see, [4; 110; 121; 134; 138; 141] for details):

$$\left(\frac{z}{e^z-1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!} \quad (|z|<2\pi;\ 1^{\alpha}:=1)\,, \tag{2.1}$$

$$\left(\frac{2}{e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{z^n}{n!} \quad (|z|<\pi\ 1^{\alpha}:=1)\,, \tag{2.2}$$

and

$$\left(\frac{2z}{e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{z^n}{n!} \quad (|z|<\pi\ 1^{\alpha}:=1)\,. \tag{2.3}$$

Obviously

$$B_n(x) := B_n^{(1)}(x), \quad E_n(x) := E_n^{(1)}(x), \quad and \quad G_n(x) := G_n^{(1)}(x), \quad (n\in\mathbb{N}_0)\,, \tag{2.4}$$

where $\mathbb{N}_0 := \mathbb{N}\cup\{0\}\,(\mathbb{N} := \{1,2,3,\cdots\})$.

From Equations (2.1), (2.2) and (2.3), it is easy to find the classical Bernoulli numbers $B_n(x)$, Euler numbers $E_n(x)$ and Genocchi numbers $G_n(x)$, which are defined as

$$B_n := B_n(0) = B_n^{(1)}(0), \quad E_n := E_n(0) = E_n^{(1)}(0), \quad and \quad G_n := G_n(0) = G_n^{(1)}(0),$$

respectively.

Analogues of the classical Bernoulli polynomials and numbers were first investigated by Apostol [23] and later on Srivastav [137]. An analogous extension of the generalized Euler polynomials as the Apostol-Euler polynomials studied by Luo [107].

Moreover, Luo [104; 105; 106; 107] introduced and investigated the Apostol-Genocchi polynomials of (real or complex) order α, which are defined as follows:

Definition 2.1.1 *[107] The Apostol-Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ of order α in variable x are defined by means of the generating function:*

$$\left(\frac{2z}{\lambda e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} \quad \Big(|z|<\big|\log(-\lambda)\big|;\ 1^{\alpha}:=1\Big). \tag{2.5}$$

with, of course,

$$G_n^{(\alpha)}(x) = G_n^{(\alpha)}(x;1) \quad \textit{and} \quad G_n^{(\alpha)}(\lambda) := G_n^{(\alpha)}(0;\lambda),$$

and

$$G_n(x;\lambda) := G_n^{(1)}(x;\lambda) \quad \textit{and} \quad G_n(\lambda) := G_n^{(1)}(\lambda),$$

where $G_n(\lambda)$, $G_n^{(\alpha)}(\lambda)$ *and* $G_n(x;\lambda)$ *denote the so-called Apostol-Genocchi numbers, Apostol-Genocchi numbers of order* α *and Apostol-Genocchi polynomials, respectively.*

For our convenience, we consider the operators in the following form:

For $f \in C[0,\infty)$, the operator is defined as:

$$\mathcal{A}_n^{\alpha,\lambda}(f;x) = \sum_{k=0}^{\infty} b_{n.k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) = e^{-nx}\left(\frac{1+e\lambda}{2}\right)^{\alpha} \sum_{k=0}^{\infty} \frac{G_k^{(\alpha)}(nx;\lambda)}{k!} f\left(\frac{k}{n}\right), \tag{2.6}$$

where $G_k^{(\alpha)}(x;\lambda)$ is generalized Apostol-Genocchi polynomials, which have the generating function of the form

$$\left(\frac{2t}{1+\lambda e^t}\right)^{\alpha} e^{xt} = \sum_{k=0}^{\infty} G_k^{(\alpha)}(x;\lambda)\frac{t^k}{k!}, \qquad (|t| < \pi). \tag{2.7}$$

The Apostol-Genocchi polynomials and their properties are studied by many researchers for the detail here we refer (cf. [24; 91; 105; 106; 124; 129; 140]).

In [108], the following explicit formula for the Apostol-Genocchi polynomials $G_k^{(\alpha)}(x;\lambda)$ is given:

$$\begin{aligned} G_k^{(\alpha)}(x;\lambda) &= 2^{\alpha}\alpha!\binom{k}{\alpha}\sum_{i=0}^{k-\alpha}\frac{\lambda^i}{(1+\lambda)^{\alpha+i}}\binom{k-\alpha}{i}\binom{\alpha+i-1}{i} \\ &\times \sum_{j=0}^{i}(-1)^j\binom{i}{j} j^i (x+j)^{k-i-\alpha}{}_2F_1[\alpha+i-k, i; i+1; j/(x+j)], \end{aligned} \tag{2.8}$$

where $k,\alpha \in \mathbb{N}\cup\{0\}$, $x \in \mathbb{R}$, $\lambda \in \mathbb{R}\setminus\{-1\}$ and ${}_2F_1[a,b;c;z]$ denotes the Gaussian hypergeometric function defined by

$${}_2F_1[a,b;c;z] \;=\; \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} = \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c}\frac{z^2}{2!} + \cdots,$$

where $(\alpha)_0 = 1$, $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}$, $(n \geq 1)$ and $0 \leqslant \alpha < 1$ (see [4], pp. 37).

Over the most recent two decades, an amazing number of papers showed up contemplating Genocchi numbers, their combinatorial relations, Genocchi polynomials, and their speculations alongside their different extensions and integral representations, which provides a new direction in the field of positive linear operators. To the readers, we suggest the following articles [5; 44; 64; 85].

Jain [88] introduced a new class of linear operators as:

$$J_n^{(\beta)}(f;x) = \sum_{k=0}^{\infty} p_{n,k}^{(\beta)}(x) f\left(\frac{k}{n}\right), \quad x \geq 0, \tag{2.9}$$

where $0 \leq \beta < 1$ and

$$p_{n,k}^{(\beta)}(x) = \frac{nx(nx+k\beta)^{k-1} e^{-(nx+k\beta)}}{k!}.$$

For $\beta = 0$, these operators reduce to Szász-Mirakyan operators. Several researchers studied Jain operators and their integral variant in [19; 20; 52; 57; 132; 145].

Durrmeyer variants of various operators are studied by several researchers [1; 43; 46; 52; 146] but in the year 2015, Gupta and Greubel [75] introduced the Durrmeyer variant of Jain operators (2.9). Motivated from [75], we now consider a Durrmeyer type modification of Apostol-Genocchi operators based on Jain operators. For $f \in C[0,\infty)$ the operators are defined as:

$$\begin{aligned} \mathcal{H}_n(f;x) &= \sum_{k=0}^{\infty} \left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi)\, d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^{\infty} p_{n,k}^{(\beta)}(\xi) f(\xi)\, d\xi \\ &= \sum_{k=0}^{\infty} \frac{\left\langle p_{n,k}^{(\beta)}(\xi), f(\xi) \right\rangle}{\left\langle p_{n,k}^{(\beta)}(\xi), 1 \right\rangle} b_{n,k}^{(\alpha)}(x), \end{aligned} \tag{2.10}$$

where $\langle f, g \rangle = \int_0^{\infty} f(\xi) g(\xi)\, d\xi$.

Some interesting results are studied by several mathematicians which have given a new direction in the field of positive linear operators (cf. [5; 44; 64; 85]).

2.1.2 Auxiliary Properties

Lemma 2.1.2 *[75] For* $0 \leq \beta < 1$, *we have*

$$\frac{\left\langle p_{n,k}^{(\beta)}(\xi), \xi^r \right\rangle}{\left\langle p_{n,k}^{(\beta)}(\xi), 1 \right\rangle} = \mathcal{P}_r(k;\beta), \quad r = 0, 1, 2, \cdots,$$

where $\mathcal{P}_r(k;\beta)$ is a polynomial of order r in variable k and $\langle f,g\rangle = \int_0^\infty f(\xi)g(\xi)d\xi$. In particular

$$\mathcal{P}_0(k;\beta) = 1;$$
$$\mathcal{P}_1(k;\beta) = \frac{1}{n}\left[(1-\beta)k + \frac{1}{1-\beta}\right];$$
$$\mathcal{P}_2(k;\beta) = \frac{1}{n^2}\left[(1-\beta)^2k^2 + 3k + \frac{2!}{1-\beta}\right];$$
$$\mathcal{P}_3(k;\beta) = \frac{1}{n^3}\left[(1-\beta)^3k^3 + 6(1-\beta)k^2 + \frac{(11-8\beta)k}{1-\beta} + \frac{3!}{1-\beta}\right];$$

$$\mathcal{P}_4(k;\beta) = \frac{1}{n^4}\left[(1-\beta)^4k^4 + 10(1-\beta)^2k^3 + 5(7-4\beta)k^2 + \frac{10(5-3\beta)k}{1-\beta} + \frac{4!}{1-\beta}\right];$$
$$\mathcal{P}_5(k;\beta) = \frac{1}{n^5}\left[(1-\beta)^5k^5 + 15(1-\beta)^3k^4 + 5(1-\beta)(17-8\beta)k^3 + \frac{15\left(15-20\beta+6\beta^2\right)k^2}{1-\beta} + \frac{(274-144\beta)k}{1-\beta} + \frac{5!}{1-\beta}\right].$$

Lemma 2.1.3 *[129] For $\mathcal{A}_n^{\alpha,\lambda}(t^m;x)$, $m = 0,1,2,3$ and 4, we have*

$$\mathcal{A}_n^{\alpha,\lambda}(1;x) = 1;$$
$$\mathcal{A}_n^{\alpha,\lambda}(\xi;x) = x + \frac{\alpha}{n(1+e\lambda)};$$
$$\mathcal{A}_n^{\alpha,\lambda}(\xi^2;x) = x^2 + \frac{(1+2\alpha+e\lambda)}{n(1+e\lambda)}x + \frac{\alpha^2 - 2\alpha e\lambda - \alpha e^2\lambda^2}{n^2(1+e\lambda)^2};$$
$$\mathcal{A}_n^{\alpha,\lambda}(\xi^3;x) = x^3 + \frac{(3+3\alpha+3e\lambda)}{n(1+e\lambda)}x^2 + \frac{\left(3\alpha^2+3\alpha+e^2\lambda^2-3\alpha e^2\lambda^2-3\alpha e\lambda+2e\lambda+1\right)}{n^2(1+e\lambda)^2}x + \frac{\left(\alpha^3-6\alpha^2e\lambda-3\alpha^2e^2\lambda^2-5\alpha e\lambda-4\alpha e^2\lambda^2-\alpha e^3\lambda^3\right)}{n^3(1+e\lambda)^3};$$
$$\mathcal{A}_n^{\alpha,\lambda}(\xi^4;x) = x^4 + \frac{(3+2\alpha+3e\lambda)}{n(1+e\lambda)}x^3 + \frac{\left(-6\alpha^2-25e^2\lambda^2-50e\lambda+6\alpha e^2\lambda^2-12\alpha-25\right)}{n^2(1+e\lambda)^2}x^2 + \frac{x}{n^3(1+e\lambda)^3}\left[2\alpha^3+7e^3\lambda^3-5\alpha e^3\lambda^3+21e^2\lambda^2+3\alpha^2-6\alpha^2e^2\lambda^2+3\alpha e^2\lambda^2 -9\alpha^2e\lambda+21e\lambda+20\alpha+24\alpha e\lambda\right].$$

Lemma 2.1.4 *The operators* (2.10), *for* $\mathcal{H}_n\left(\xi^i;x\right)$, *the moments up to second order are given by:*

$$\begin{aligned}
\mathcal{H}_n(1;x) &= 1;\\
\mathcal{H}_n(\xi;x) &= (1-\beta)\,x+\frac{1}{n}\left\{\frac{\alpha(1-\beta)}{1+e\lambda}+\frac{1}{1-\beta}\right\};\\
\mathcal{H}_n\left(\xi^2;x\right) &= (1-\beta)^2x^2+\left\{\frac{(1+2\alpha+e\lambda)(1-\beta)^2}{1+e\lambda}+3\right\}\frac{x}{n}\\
&\quad+\frac{1}{n^2}\left\{\frac{(1-\beta)^2\left(\alpha^2-2\alpha e\lambda-\alpha e^2\lambda^2\right)}{(1+e\lambda)^2}+\frac{3\alpha}{1+e\lambda}+\frac{2!}{1-\beta}\right\}.
\end{aligned}$$

Proof: By using Lemma 2.1.2 and Lemma 2.1.3, we get

$$\begin{aligned}
\mathcal{H}_n(1;x) &= \sum_{k=0}^{\infty}\mathcal{P}_0(k,\beta)\,b_{n,k}^{(\alpha)}(x)\\
&= 1,\\
\mathcal{H}_n(\xi;x) &= \sum_{k=0}^{\infty}\mathcal{P}_1(k,\alpha)\,b_{n,k}^{(\alpha)}(x)\\
&= \sum_{k=0}^{\infty}\frac{1}{n}\left[(1-\beta)\,k+\frac{1}{1-\beta}\right]b_{n,k}^{(\alpha)}(x)\\
&= (1-\beta)\,\mathcal{A}_n^{(\alpha)}(\xi;x)+\frac{1}{n(1-\beta)}\mathcal{A}_n^{(\alpha)}(1;x)\\
&= (1-\beta)\,x+\frac{1}{n}\left\{\frac{\alpha(1-\beta)}{1+e\lambda}+\frac{1}{1-\beta}\right\},\\
\mathcal{H}_n\left(\xi^2;x\right) &= \sum_{k=0}^{\infty}\mathcal{P}_2(k,\alpha)\,b_{n,k}^{(\alpha)}(x)\\
&= \sum_{k=0}^{\infty}\frac{1}{n^2}\left[(1-\beta)^2k^2+3k+\frac{2!}{1-\beta}\right]b_{n,k}^{(\alpha)}(x)\\
&= (1-\beta)^2\,\mathcal{A}_n^{(\alpha)}\left(\xi^2;x\right)+\frac{3}{n}\mathcal{A}_n^{(\alpha)}(\xi;x)+\frac{2!}{n^2(1-\beta)}\mathcal{A}_n^{(\alpha)}(1;x)\\
&= (1-\beta)^2\left\{x^2+\frac{(1+2\alpha+e\lambda)}{n(1+e\lambda)}x+\frac{\alpha^2-2\alpha e\lambda-\alpha e^2\lambda^2}{n^2(1+e\lambda)^2}\right\}\\
&\quad+\frac{3}{n}\left\{x+\frac{\alpha}{n(1+e\lambda)}\right\}+\frac{2!}{n^2(1-\beta)}\\
&= (1-\beta)^2x^2+\left\{\frac{(1+2\alpha+e\lambda)(1-\beta)^2}{1+e\lambda}+3\right\}\frac{x}{n}\\
&\quad+\frac{1}{n^2}\left\{\frac{(1-\beta)^2\left(\alpha^2-2\alpha e\lambda-\alpha e^2\lambda^2\right)}{(1+e\lambda)^2}+\frac{3\alpha}{1+e\lambda}+\frac{2!}{1-\beta}\right\}.
\end{aligned}$$

Lemma 2.1.5 *By direct computation, we have*

$$\mathcal{H}_n\left((\xi-x)\,;x\right)=\frac{1}{n}\left\{\frac{\alpha(1-\beta)}{1+e\lambda}+\frac{1}{1-\beta}\right\}-\beta x,$$

$$\mathcal{H}_n\left((\xi-x)^2\,;x\right)=\beta^2x^2+\left\{\frac{(1+2\alpha+e\lambda)(1-\beta)^2}{1+e\lambda}-\frac{2\alpha(1-\beta)}{1+e\lambda}-\frac{2}{1-\beta}+3\right\}\frac{x}{n}$$
$$+\left\{\frac{\left(\alpha^2-2\alpha e\lambda-\alpha e^2\lambda^2\right)(1-\beta)^2}{(1+e\lambda)^2}+\frac{3\alpha}{1+e\lambda}+\frac{2}{1-\beta}\right\}\frac{1}{n^2}.$$

Lemma 2.1.6 *For the operators $\mathcal{H}_n$, we have $|\mathcal{H}_n(f;x)|\leq\|f\|$, where $f\in C[0,\infty)$ and $\|f\|=\sup\limits_{x\in[0,\infty)}|f(x)|$.*

Proof: From operators (2.10) and using Lemma 2.1.4, we get

$$|\mathcal{H}_n(f;x)|\leq\left|\sum_{k=0}^{\infty}\left(\int_0^{\infty}p_{n,k}^{(\beta)}(\xi)\,d\xi\right)^{-1}b_{n,k}^{(\alpha)}(x)\int_0^{\infty}p_{n,k}^{(\beta)}(\xi)f(\xi)\,d\xi\right|$$
$$\leq\sum_{k=0}^{\infty}\left(\int_0^{\infty}p_{n,k}^{(\beta)}(\xi)\,d\xi\right)^{-1}b_{n,k}^{(\alpha)}(x)\int_0^{\infty}p_{n,k}^{(\beta)}(\xi)\,|f(\xi)|\,d\xi\leq\|f\|.$$

2.1.3 Main Results

Let $f\in C_B^2[0,\infty)$ be denoted the space of all functions $f\in C_B[0,\infty)$ such that f',f'' define in $C[0,\infty)$. Let $\|f\|$ be denoted the usual supremum norm of a bounded function f. Then Peetre's $K-$functional

$$K(f;\delta)=\inf_{g\in C_B^2[0,\infty)}\left\{\|f-g\|+\delta\,\|g''\|\right\},\tag{2.11}$$

and for $\delta>0$ the modulus of continuity of second-order

$$\omega_2(f;\delta)=\sup_{0<h\leq\delta,x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|\,.\tag{2.12}$$

Also from ([49], p. 177, Theorem 2.4), there exists a constant $C>0$ such that

$$K(f;\delta)\leq C\omega_2\left(f;\sqrt{\delta}\right).\tag{2.13}$$

Now we get the following approximation results.

The Bohman-Korovkin-Popoviciu theorem [99] is a powerful mathematical tool used to prove uniform convergence. In this context, it has been applied to the Apostol-Genocchi-Jain-Durremyer operators (2.10) to establish their uniform convergence.

Theorem 2.1.7 *Let us $f \in C[0,\infty) \cap \mho$ and this function also belongs to the class*

$$\mho := \left\{ f : x \in [0,\infty), \frac{f(x)}{1+x^2} \textit{is convergent as } \; x \to \infty \right\},$$

Then, the uniformly on each compact subset of $[0,\infty)$, *where* $C[0,\infty)$ *is the space of all real-valued continuous functions on* $[0,\infty)$, *i.e.,*

$$\lim_{n\to\infty} V_n^{(\alpha)}(f;x) = f(x),$$

where $\alpha(n)$ be such that $\alpha \to 0$ as $n \to \infty$.

Proof: *As $\alpha \to 0$ as $n \to \infty$, from Lemma 2.1.4, we have*

$$\lim_{n\to\infty} \mathcal{H}_n\left(\xi^i; x\right) = x^i, \quad i = 0,1,2,$$

uniformly on each compact subset of the non-negative half-line real axis. Hence, we get the desired result by applying the well-known Korovkin-type theorem [21] regarding the convergence of a sequence of positive linear operators.

Theorem 2.1.8 *If $f \in C_B[0,\infty)$ then for $x \in [0,\infty)$, we have*

$$|\mathcal{H}_n(f;x) - f(x)| \le 2\omega\left(f; \sqrt{\mathcal{H}_n\left((\xi - x)^2; x\right)}\right),$$

where ω is the modulus of continuity of f [49] defined as:

$$\omega(f;x) := \sup_{\substack{x,y\in[0,\infty) \\ |x-y|\le\delta}} |f(x) - f(y)|.$$

Proof: *Applying the well-known property of $\omega(f;x)$, Lemma 2.1.4, and from operators* (2.10), *we have*

$$\begin{aligned}
|\mathcal{H}_n(f;x) - f(x)| &= \left| \sum_{k=0}^{\infty} \left(\int_0^\infty p_{n,k}^{(\beta)}(\xi)\, d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^\infty p_{n,k}^{(\beta)}(\xi) \left(f(\xi) - f(x)\right) d\xi \right| \\
&\le \sum_{k=0}^{\infty} \left(\int_0^\infty p_{n,k}^{(\beta)}(\xi)\, d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^\infty p_{n,k}^{(\beta)}(\xi) \left|\left(f(\xi) - f(x)\right)\right| d\xi \\
&\le \sum_{k=0}^{\infty} \left(\int_0^\infty p_{n,k}^{(\beta)}(\xi)\, d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^\infty \omega(f;\delta)\, p_{n,k}^{(\beta)}(\xi) \left(1 + \frac{1}{\delta}|\xi - x|\right) d\xi \\
&= \left[1 + \frac{1}{\delta} \sum_{k=0}^{\infty} \left(\int_0^\infty p_{n,k}^{(\beta)}(\xi)\, d\xi \right)^{-1} b_{n,k}^{(\alpha)}(x) \int_0^\infty p_{n,k}^{(\beta)}(\xi) |\xi - x|\, d\xi \right] \omega(f;\delta).
\end{aligned}$$

For the integration, the following result holds by using Cauchy-Schwarz inequality

$$|\mathcal{H}_n(f;x) - f(x)| \le \left[1 + \frac{1}{\delta}\sum_{k=0}^{\infty}\left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi)\,d\xi\right)^{-1} b_{n,k}^{(\alpha)}(x)\left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi)d\xi\right)^{1/2} \right.$$
$$\left.\times\left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi)(\xi - x)^2\,d\xi\right)^{1/2} d\xi\right]\omega(f;\delta).$$

Now using the last inequality for infinite sum and we have

$$\begin{aligned}
&|\mathcal{H}_n(f;x) - f(x)| \\
&\le \left[1 + \frac{1}{\delta}\left\{\sum_{k=0}^{\infty}\left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi)\,d\xi\right)^{-1} b_{n,k}^{(\alpha)}(x)\int_0^{\infty} p_{n,k}^{(\beta)}(\xi)d\xi\right\}^{1/2}\right. \\
&\quad\left.\times\left\{\sum_{k=0}^{\infty}\left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi)\,d\xi\right)^{-1} b_{n,k}^{(\alpha)}(x)\int_0^{\infty} p_{n,k}^{(\beta)}(\xi)(\xi - x)^2\,d\xi\right\}^{1/2}\right]\omega(f;\delta) \\
&= \left[1 + \frac{1}{\delta}\{\mathcal{H}_n(1;x)\}^{1/2}\left\{\mathcal{H}_n\left((\xi - x)^2;x\right)\right\}^{1/2}\right]\omega(f;\delta).
\end{aligned}$$

By taking

$$\delta = \left\{\mathcal{H}_n\left((\xi - x)^2;x\right)\right\}^{1/2}.$$

We get the required result.

Now, for $0 < \varrho \le 1$ and let us present approximation in terms of Lipschitz constant defined as:

$$Lip_K^{\varrho} = \left\{f \in C_B[0,\infty) : |f(\eta_1) - f(\eta_2)| \le K\frac{|\eta_1 - \eta_2|^{\varrho}}{(\eta_1 + \eta_2)^{\varrho/2}}\right\},\quad \eta_1, \eta_2 \in [0,\infty),$$

where $K > 0$ is a constant.

Theorem 2.1.9 *Suppose that $f \in Lip_K^{\varrho}$, then*

$$|\mathcal{H}_n(f;x) - f(x)| \le K\left\{\frac{1}{x}\mathcal{H}_n\left((\xi - x)^2;x\right)\right\}^{\varrho/2}.$$

Proof: *Since $f \in Lip_K^{\varrho}$ and $0 < \varrho \le 1$, we have*

$$\begin{aligned}
|\mathcal{H}_n(f;x) - f(x)| &= |\mathcal{H}_n(f(\xi) - f(x);x)| \\
&\le \mathcal{H}_n(|f(\xi) - f(x)|;x) \\
&\le K\mathcal{H}_n\left(\frac{|\xi - x|^{\varrho}}{(\xi + x)^{\varrho/2}};x\right). \qquad (2.14)
\end{aligned}$$

From (2.14), *it becomes*

$$\begin{aligned}
\mathcal{H}_n\left(\frac{|\xi - x|^{\varrho}}{(\xi + x)^{\varrho/2}};x\right) &= \sum_{k=0}^{\infty}\left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi)\,d\xi\right)^{-1} b_{n,k}^{(\alpha)}(x)\int_0^{\infty} p_{n,k}^{(\beta)}(\xi)\frac{|\xi - x|^{\varrho}}{(\xi + x)^{\varrho/2}}d\xi \\
&\le \left\{\sum_{k=0}^{\infty}\left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi)\,d\xi\right)^{-1} b_{n,k}^{(\alpha)}(x)\int_0^{\infty} p_{n,k}^{(\beta)}(\xi)\,d\xi\right\}^{(2-\varrho)/2} \\
&\quad\times\left\{\sum_{k=0}^{\infty}\left(\int_0^{\infty} p_{n,k}^{(\beta)}(\xi)\,d\xi\right)^{-1} b_{n,k}^{(\alpha)}(x)\int_0^{\infty} p_{n,k}^{(\beta)}(\xi)\frac{(\xi - x)^2}{(\xi + x)}d\xi\right\}^{\varrho/2},
\end{aligned}$$

where we use the Hölder inequality by taking $p = \frac{2}{\varrho}$ *and* $q = \frac{2}{2-\varrho}$. *Now using the fact that* $\frac{1}{\xi+x} \leq \frac{1}{x}$, *we have*

$$\mathcal{H}_n\left(\frac{|\xi - x|^{\varrho}}{(\xi + x)^{\varrho/2}}; x\right) = \{\mathcal{H}_n(1; x)\}^{(2-\varrho)/2}\left\{\frac{1}{x}\mathcal{H}_n\left((\xi - x)^2; x\right)\right\}^{\varrho/2}.$$

This proves the required results.

Theorem 2.1.10 *For every* $f \in C_B[0, \infty)$, *we have*

$$|\mathcal{H}_n(f; x) - f(x)| \leq M\omega_2\left(f, \frac{\delta_n(x, \alpha)}{\sqrt{2}}\right) + \omega(f, \tau_\omega),$$

where $M > 0$ *is a constant.*

Proof: *For* $f \in C_\beta[0, \infty)$ *and by the definition of the operators* $\widetilde{\mathcal{H}}_n$, *we have*

$$\widetilde{\mathcal{H}}_n(f; x) := \mathcal{H}_n(f; x) - f\left((1 - \beta)x + \frac{1}{n}\left\{\frac{\alpha(1 - \beta)}{1 + e\lambda} + \frac{1}{1 - \beta}\right\}\right) + f(x). \tag{2.15}$$

Using Lemma 2.1.4, we obtain

$$\widetilde{\mathcal{H}}_n(e_0; x) = 1, \quad \textit{and} \quad \widetilde{\mathcal{H}}_n(e_1; x) = x,$$

i.e., $\widetilde{\mathcal{H}}_n$ *preserve constants and linear functions. Therefore*

$$\widetilde{\mathcal{H}}_n((e_1 - x); x) = 0. \tag{2.16}$$

Let $f \in C_B^2[0, \infty)$ *and using Taylor's expansion*

$$g(\xi) = g(x) + (\xi - x)g'(x) + \int_x^\xi (\xi - u)g''(u)du, \quad \xi, x \in [0, \infty). \tag{2.17}$$

Applying $\widetilde{\mathcal{H}}_n$ *to above expansion and using* (2.17), *we have*

$$\widetilde{\mathcal{H}}_n(g; x) - g(\xi) = g'(x)\widetilde{\mathcal{H}}_n(\xi - x; x) + \widetilde{\mathcal{H}}_n\left(\int_x^\xi (\xi - u)g''(u)du; x\right),$$

and from (2.16), *we have*

$$\begin{aligned}
&\left|\widetilde{\mathcal{H}}_n(g; x) - g(\xi)\right| \\
&\leq \widetilde{\mathcal{H}}_n\left(\left|\int_x^\xi (\xi - u)g''(u)du\right|; x\right) \\
&\leq \mathcal{H}_n\left(\left|\int_x^\xi (\xi - u)g''(u)du\right|; x\right) \\
&- \left|\int_x^{(1-\beta)x+\frac{1}{n}\left\{\frac{\alpha(1-\beta)}{1+e\lambda}+\frac{1}{1-\beta}\right\}}\left((1 - \beta)x + \frac{1}{n}\left\{\frac{\alpha(1 - \beta)}{1 + e\lambda} + \frac{1}{1 - \beta}\right\} - u\right)g''(u)du\right|.
\end{aligned}$$

Now we consider

$$\left|\int_{\xi}^{x}(\xi-u)\,g''(u))du\right|\leq(\xi-x)^{2}\,\|g''\|\,,$$

then we get

$$\left|\widetilde{\mathcal{H}}_n\,(g;x)-g(x)\right|\leq\left(\mathcal{H}_n\left((\xi-x)^2\,;x\right)+\left(\mathcal{H}_n\,(\xi-x;x)^2\right)\right)\|g''\|\,.$$

From Lemma 2.1.4 and using (2.15)*, for the operators* $\widetilde{\mathcal{H}}_n^{(\alpha)}$*, we have*

$$\begin{aligned}\left|\widetilde{\mathcal{H}}_n\,(f;x)-f(x)\right| &\leq\left|\widetilde{\mathcal{H}}_n\,(f-g;x)\right|+\left|\widetilde{\mathcal{H}}_n\,(g;x)-g(x)\right|+|g(x)-f(x)|\\ &\quad+\left|f\left((1-\beta)\,x+\frac{1}{n}\left\{\frac{\alpha(1-\beta)}{1+e\lambda}+\frac{1}{1-\beta}\right\}\right)-f(x)\right|\\ &\leq 4\,\|f-g\|+\left(\mathcal{H}_n\left((\xi-x)^2\,;x\right)+\left(\mathcal{H}_n\,(\xi-x;x)^2\right)\right)\|g''\|\\ &\quad+\omega\,(f,\mathcal{H}_n\,(\xi-x;x))\,.\end{aligned}$$

Taking infimum on the right hand side over $g\in W_\infty^2$ *and from* (2.13)*, we have*

$$\begin{aligned}\left|\widetilde{\mathcal{H}}_n\,(f;x)-f(x)\right| &\leq 4K_2\left(f,\frac{\tau_n^4\,(x,\alpha)}{4}\right)+\omega\,(f,\tau_\omega)\\ &\leq M\omega_2\left(f,\frac{\delta_n(x,\alpha)}{2}\right)+\omega\,(f,\tau_\omega)\,,\end{aligned}$$

where

$$\tau_n^2(x,\alpha)=\mathcal{H}_n\left((\xi-x)^2\,;x\right)+\left(\mathcal{H}_n\,(\xi-x;x)^2\right)\ \textit{and}\ \tau_\omega=\mathcal{H}_n\,(\xi-x;x)\,.$$

2.1.4 Numerical Results

Let $f(x)=x^3-2x^2+x-2$, $\alpha=2$, $\beta=0.01$, $\lambda=4$ and $n\in\{10,20,30\}$. The convergence of the defined operators $\widetilde{H}_n$ towards the function $f(x)$ and the absolute error $E_n(x)=\left|\widetilde{H}_n(f;x)-f(x)\right|$ of the operators are shown in Fig(a) and Fig(b), respectively. The absolute error of the operators is also computed in Table 1 for some values in $[1,3]$.

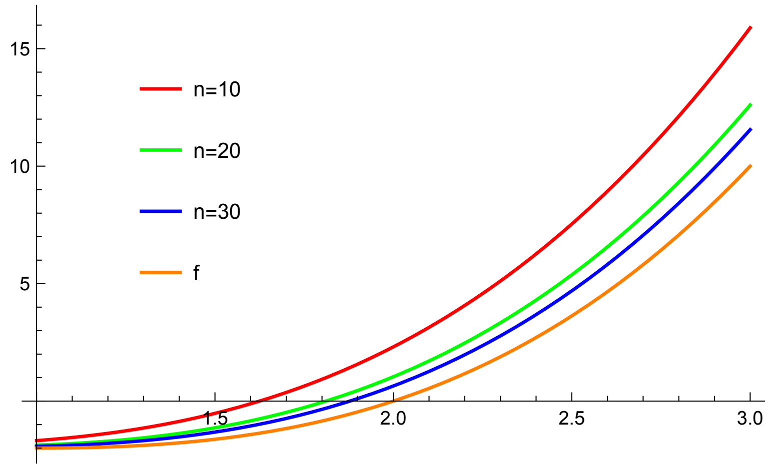

Figure 2.1: Considering $n = [10, 20, 30]$, the convergence of operators $\widetilde{H}_n$ towards the function $f(x) = x^3 - 2x^2 + x - 2$ with parameters $\alpha = 2,\ \beta = 0.01$ and $\lambda = 4$.

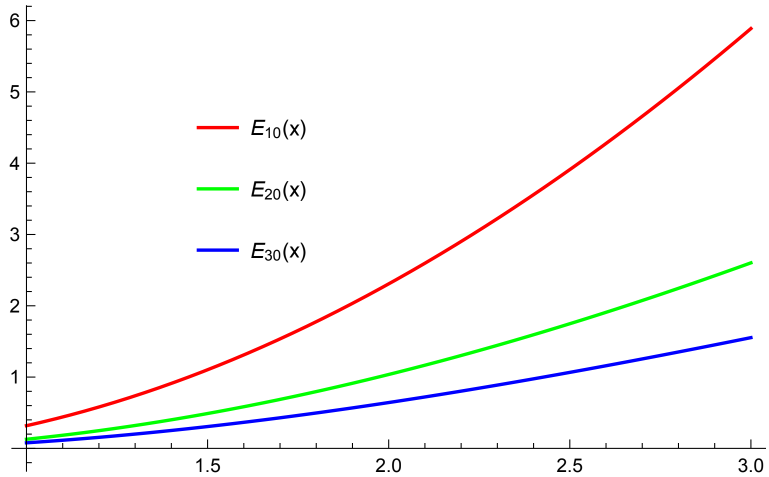

Figure 2.2: Graphical representation of absolute error $E_n(x) = \left|\widetilde{H}_n(f;x) - f(x)\right|$ to the function $f(x) = x^3 - 2x^2 + x - 2$ with parameters $\alpha = 2,\ \beta = 0.01,\ \lambda = 4$ and $n = \{10, 20, 30\}$.

2.2 Integral Modification of Beta type Apostol-Genocchi Operators

For $f \in C[0,\infty)$, Prakash et al. [129] considered the following operators:

$$\mathcal{A}_n^{\alpha,\beta}(f;x) = \sum_{k=0}^{\infty} v_{n.k}^{(\alpha)}(x)\, f\left(\frac{k}{n}\right) = e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha} \sum_{k=0}^{\infty} \frac{\psi_k^{\alpha}(nx;\beta)}{k!} f\left(\frac{k}{n}\right), \tag{2.18}$$

where $\psi_k^{\alpha}(x;\beta)$ is generalized Apostol-Genocchi polynomials, which have the generating function of the form:

$$\left(\frac{2t}{1+\beta e^t}\right)^{\alpha} e^{xt} = \sum_{k=0}^{\infty} \psi_k^{\alpha}(x;\beta)\frac{t^k}{k!}, \qquad (|t| < \pi). \tag{2.19}$$

Table 2.1: Estimation of absolute error $E_n(x)$ for some values of $x \in [1, 3]$.

x	$n = 10$	$n = 20$	$n = 30$
1	0.31827	0.12793	0.07812
1.2	0.58023	0.24848	0.15482
1.4	0.91155	0.40104	0.25106
1.6	1.31081	0.58416	0.36541
1.8	1.77657	0.79643	0.49645
2.0	2.30742	1.03641	0.64276
2.2	2.90192	1.30269	0.80290
2.4	3.55866	1.59383	0.97545
2.6	4.27620	1.90842	1.15890
2.8	5.05312	2.24502	1.35209
3.0	5.88800	2.60221	1.55333

The Apostol-Genocchi polynomials and their properties have been studied by many researchers over the past few decades; for instance, see (cf. [24; 91; 105; 106; 124; 140]).

In [108], the following explicit formula for the Apostol-Genocchi polynomials $\psi_k^{\alpha}(x;\beta)$ is given:

$$\psi_k^{\alpha}(x;\beta) = 2^{\alpha}\alpha!\binom{k}{\alpha}\sum_{i=0}^{k-\alpha}\frac{\beta^i}{(1+\beta)^{\alpha+i}}\binom{k-\alpha}{i}\binom{\alpha+i-1}{i}$$
$$\times\sum_{j=0}^{i}(-1)^j\binom{i}{j}j^i(x+j)^{k-i-\alpha}{}_2F_1[\alpha+i-k, i; i+1; j/(x+j)], \tag{2.20}$$

where $k, \alpha \in \mathbb{N}\cup\{0\}$, $\beta \in \mathbb{C}\setminus\{-1\}$ and ${}_2F_1[a, b; c; z]$ denotes the Gaussian hypergeometric function defined by

$${}_2F_1[a, b; c; z] = \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\frac{z^n}{n!} = \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c}\frac{z^2}{2!} + \cdots,$$

where $(\alpha)_0 = 1$, $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}$, $(n \geq 1)$ and $0 \leqslant \alpha < 1$.

Durrmeyer variants of various operators are studied by several researchers [1; 43; 46; 52; 146]. In this paper, we consider the Durrmeyer type modification of the linear positive operators defined by (2.18) for all real-valued continuous and bounded functions f on $[0, \infty)$ as follows:

$$V_n^{(\alpha)}(f;x) = \sum_{k=0}^{\infty}v_{n,k}^{(\alpha)}(x)\frac{1}{\beta(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}f(t)dt, \tag{2.21}$$

where $\beta(k+1,n)$ is beta function.

We proceed by obtaining the moments and giving a direct approximation theorem using first and second-order modulus of continuity, local approximation results for Lipschitz class functions and direct theorem for the usual modulus of continuity.

2.2.1 Basic Properties

Lemma 2.2.1 *The moments of $V_n^{(\alpha)}(f;x)$ are given as follows:*

$$\begin{aligned}
V_n^{(\alpha)}(1;x) &= 1; \\
V_n^{(\alpha)}(t;x) &= \frac{1}{(n-1)}\left[nx+\alpha+1+e\beta\right] \\
V_n^{(\alpha)}(t^2;x) &= n^2x^2+\left[\frac{(1+2\alpha+e\beta)}{(1+e\beta)}+1\right]nx \\
&\quad +\left[\frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{(1+e\beta)^2}+\frac{\alpha}{(1+e\beta)}+2\right]\frac{1}{(n-1)(n-2)}.
\end{aligned}$$

Lemma 2.2.2 *For $\delta_n(x)=V_n^{(\alpha)}[(t-x)^n;x]$ and $n\in\{1,2\}$ we have*

$$\begin{aligned}
\delta_1(x) &= V_n^{(\alpha)}\left[(t-x);x\right] = \frac{1}{(n-1)}\left[x+\alpha+1+e\beta\right]; \\
\delta_2(x) &= V_n^{(\alpha)}(t-x);x = \left[\frac{n^2}{(n-1)(n-2)}+1-\frac{2x}{n-1}\right]x^2 \\
&+ \left[\frac{n}{(n-2)}\left(\frac{(1+2\alpha+e\beta)}{(1+e\beta)}+1\right)-2(\alpha+1+e\beta)\right]\frac{x}{(n-1)} \\
&+ \left[\frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{(1+e\beta)^2}+\frac{\alpha}{(1+e\beta)}+2\right]\frac{1}{(n-1)(n-2)}.
\end{aligned}$$

Lemma 2.2.3 *For $\mathcal{A}_n^{\alpha,\beta}(t^m;x)$, $m=0,1,2,3$ and 4, we have*

$$\begin{aligned}
\mathcal{A}_n^{\alpha,\beta}(1;x) &= 1;\\
\mathcal{A}_n^{\alpha,\beta}(\xi;x) &= x+\frac{\alpha}{n(1+e\beta)};\\
\mathcal{A}_n^{\alpha,\beta}(\xi^2;x) &= x^2+\frac{(1+2\alpha+e\beta)}{n(1+e\beta)}x+\frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{n^2(1+e\beta)^2};\\
\mathcal{A}_n^{\alpha,\beta}(\xi^3;x) &= x^3+\frac{(3+3\alpha+3e\beta)}{n(1+e\beta)}x^2+\frac{\left(3\alpha^2+3\alpha+e^2\beta^2-3\alpha e^2\beta^2-3\alpha e\beta+2e\beta+1\right)}{n^2(1+e\beta)^2}x\\
&\quad+\frac{\left(\alpha^3-6\alpha^2e\beta-3\alpha^2e^2\beta^2-5\alpha e\beta-4\alpha e^2\beta^2-\alpha e^3\beta^3\right)}{n^3(1+e\beta)^3};\\
\mathcal{A}_n^{\alpha,\beta}(\xi^4;x) &= x^4+\frac{(3+2\alpha+3e\beta)}{n(1+e\beta)}x^3+\frac{\left(-6\alpha^2-25e^2\beta^2-50e\beta+6\alpha e^2\beta^2-12\alpha-25\right)}{n^2(1+e\beta)^2}x^2\\
&\quad+\frac{x}{n^3(1+e\beta)^3}\Big[2\alpha^3+7e^3\beta^3-5\alpha e^3\beta^3+21e^2\beta^2+3\alpha^2-6\alpha^2e^2\beta^2+3\alpha e^2\beta^2\\
&\quad-9\alpha^2e\beta+21e\beta+20\alpha+24\alpha e\beta\Big].
\end{aligned}$$

2.2.2 Main Theorems

The Bohman-Korovkin-Popoviciu theorem [99] is a powerful mathematical tool used to prove uniform convergence. In this context, it has been applied to the Apostol-Genocchi-Jain-Durremyer operators (2.21) to establish their uniform convergence.

Theorem 2.2.4 *Let us $f\in C[0,\infty)\cap\mho$ and this function also belongs to the class*

$$\mho:=\left\{f: x\in[0,\infty),\ \frac{f(x)}{1+x^2}\text{is convergent as }\ x\to\infty\right\},$$

where $C[0,\infty)$ is the space of all real-valued continuous functions on $[0,\infty)$.
Then, we have

$$\lim_{n\to\infty}V_n^{(\alpha)}(f;x)=f(x),$$

uniformly on each compact subset of $[0,\infty)$, where $\alpha(n)$ be such that $\alpha\to0$ as $n\to\infty$.
Proof: *As $\alpha\to0$ as $n\to\infty$ and from the Lemma 2.2.3, we have*

$$\lim_{n\to\infty}V_n^{(\alpha)}(f;x)-f(x)=x^i,\quad i=0,1,2.$$

uniformly on each compact subset of the non-negative half-line real axis. Hence, we get the desired result by applying the well-known Korovkin-type theorem [21] regarding the convergence of a sequence of positive linear operators.

Let $C_\beta^2[0,\infty)$ denote the space of real valued contiuous and bounded function f on the interval $[0,\infty)$, endowed with the norm,

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|.$$

For any $\delta > 0$, the Peetre's K-functional is defined by:

$$K_2(f;\delta) = \inf_{l\in C_\beta^2[0,\infty)} \{\|f-l\| + \delta\|l''\|\},$$

where

$$C_\beta^2[0,\infty) = \left\{g \in C_\beta[0,\infty) : l', l'' \in C_\beta[0,\infty)\right\}.$$

By DeVore and Lorentz [49], there exist an absolute constant $C > 0$, such that

$$K_2(f;\delta) \leqslant C\omega_2\left(f;\sqrt{\delta}\right),$$

where second order modulus of continuity for $f \in C_\beta[0,\infty)$ is defined as:

$$\omega_2\left(f;\sqrt{\delta}\right) = \sup_{x\in[0,\infty)} \sup_{0<h\leqslant\sqrt{\delta}} |f(x+2h) - 2f(x+h) + f(x)|$$

Also, we denote the usual modulus of continuity by $\omega(f;\delta)$.
The following auxiliary operators are defined in order to prove our next theorem.

$$\tilde{V}_n(l;x) = V_n^{(\alpha)}(f;x) - f\left(\frac{nx + (\alpha+1+e\beta)}{(n-1)}\right) + f(x).$$

Lemma 2.2.5 *Let $g \in C_\beta^2[0,\infty)$. Then for all $x \geqslant 0$ and $n > 2$, we have*

$$\left|\tilde{V}_n(l;x) - l(x)\right| \leqslant \phi_n^a(x)\|l''\|$$

where

$$\phi_n^a(x) = V_n^{(\alpha)}\left[(t-x)^2, x\right] + \left(\frac{x+\alpha+1+e\beta}{(n-1)}\right)^2$$

$$\tilde{V}_n(l;x) = V_n^{(\alpha)}(f;x) - f\left(\frac{nx + (\alpha+1+e\beta)}{(n-1)}\right) + f(x).$$

Proof: *By the definition of $\tilde{V}_n$, it is obvious that*

$$\tilde{V}_n(t-x;x) = 0, \tag{2.22}$$

let $l^r \in C_\beta^2[0,\infty)$, from Taylor expansion of l

$$l(t) - l(x) = (t-x)l'(x) + \int_x^t (t-u)l''(u)\,du.$$

We can write

$$\tilde{V}_n(l;x) - l(x) = l'(x)\,\tilde{V}_n(t-x;x) + \tilde{V}_n\left(\int_x^t (t-u)l''(u)\,du; x\right)$$
$$= \tilde{V}_n\left(\int_x^t (t-u)l''(u)\,du; x\right)\tilde{V}_n\left(\int_x^t (t-u)l''(u)\,du; x\right)$$
$$- \int_x^{\left(\frac{nx+(\alpha+1+e\beta)}{(n-1)}\right)} \left(\frac{nx+(\alpha+1+e\beta)}{(n-1)} - u\right) l''(u)\,dt.$$

Since

$$\tilde{V}_n\left(\int_x^t (t-u)l''(u)\,du; x\right) \leqslant (t-x)^2 \|l''\|,$$

and

$$\left|\int_x^{\left(\frac{nx+(\alpha+1+e\beta)}{(n-1)}\right)} \left(\frac{nx+(\alpha+1+e\beta)}{(n-1)} - u\right) l''(u)\,dt\right| \leqslant \left[\frac{x+(\alpha+1+e\beta)}{(n-1)}\right]^2 \|l''\|$$
$$\Rightarrow \tilde{V}_n\left|(l;x) - l(x)\right| \leqslant \left\{V_n^{(\alpha)}\left((t-x)^2;x\right) + \left[\frac{x+(\alpha+1+e\beta)}{(n-1)}\right]^2\right\}\|l''\|.$$

By Lemma 2.2.2, we may conclude that

$$\phi_n^a(x) = V_n^{(\alpha)}\left[(t-x)^2, x\right] + \left(\frac{x+\alpha+1+e\beta}{(n-1)}\right)^2]\left|\tilde{V}_n(l;x) - l(x)\right| \leqslant \phi_n^a(x)\|l''\|.$$

Theorem 2.2.6 *Let $f \in C_\beta[0,\infty)$. Then for all $x \geqslant 0$ and $n > 2$, there exist a constant $C > 0$ such that*

$$|V_n(f;x) - f(x)| \leqslant C\omega_2\left(f; \sqrt{\phi_n^a(x)}\right) + \omega\left(f; \left(\frac{nx+(\alpha+1+e\beta)}{(n-1)}\right)\right),$$

where $\phi_n^a(x)$ defined as in above Lemma 2.2.3.

Proof: *For $f \in C_\beta[0,\infty)$ and $g \in C_\beta^2[0,\infty)$, by the definition of the operators $\tilde{V}_n$, one has*

$$|V_n(f;x) - f(x)| \leqslant \left|\tilde{V}_n(f-g;x)\right| + |(f-g)(x)|$$
$$+ \left|\tilde{V}_n(l;x) - l(x)\right| + \left|f\left(\frac{nx+(\alpha+1+e\beta)}{(n-1)}\right) - f(x)\right|,$$

and

$$\left|\tilde{V}_n(l;x)\right| \leqslant 3\|f\|.$$

Therefore, we can get

$$|V_n(f;x) - f(x)| \leqslant \|(f-g)\| + \left|\tilde{V}_n(l;x) - l(x)\right| + \omega\left|f;\left(\frac{nx+(\alpha+1+e\beta)}{(n-1)}\right)\right|.$$

Using Lemma 2.2.5, the above inequality leads to

$$|V_n(f;x) - f(x)| \leqslant \|(f-g)\| + \phi_n^a(x)\|l''\| + \omega\left|f;\left(\frac{nx+(\alpha+1+e\beta)}{(n-1)}\right)\right|.$$

Thus, taking infimum over all $g \in C_\beta^2[0,\infty)$ *on the right-hand side of the last equality and considering the definition of K-functional, we get the required result.*

Now consider the Lipschitz-type space:

$$lip_M^*(r) := \left\{f \in C_\beta[0,\infty) : |f(t) - f(x)| \leqslant M\frac{|t-x|^r}{(t-x)^{\frac{r}{2}}}; x,t \in (0,\infty)\right\},$$

where M *is a positive constant and* $r \in (0,1]$.

Lemma 2.2.7 *For all* $x \leq 0$ *and* $n > 2$*, we have*

$$V_n(|t-x|;x) \leqslant \sqrt{\delta_n(x)},$$

where

$$\delta_n(x) = V_n\left((t-x)^2;x\right).$$

Proof: *By the definition of the proposed operators, we get*

$$V_n(|t-x|;x) = \sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x)\frac{1}{\beta(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}|t-x|dt.$$

After applying the Cauchy-Schwarz inequality on the right-hand side of the above inequality, we get

$$V_n(|t-x|;x) = \left\{\sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x)\left(\frac{1}{\beta(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}|t-x|dt\right)^2\right\}^{\frac{1}{2}}. \qquad (2.23)$$

Again applying the Cauchy-Schwarz inequality to the integral in the right-hand side of (5.2.2), we have

$$\begin{aligned} V_n(|t-x|;x) &= \left\{\sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x)\left(\frac{1}{\beta(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}(t-x)^2dt\right)\right\}^{\frac{1}{2}} \\ &= \sqrt{V_n\left((t-x)^2;x\right)} = \sqrt{\delta_n(x)}. \end{aligned}$$

Theorem 2.2.8 *Let $f \in lip_M^*(r)$. Then for all $x > 0$ and $n > 2$ we have*

$$|V_n(f;x) - f(x)| \leqslant M\left(\frac{\delta_n(x)}{x}\right)^{\frac{r}{2}},$$

where $\delta_n(x)$ defined as in Lemma 2.2.5.

Proof: *Firstly, we suppose that $r = 1$. Then for $f \in lip_M^*(1)$ we may write:*

$$\begin{aligned}|V_n(f;x) - f(x)| &\leqslant \sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x)\frac{1}{\beta(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}|f(t) - f(x)|dt\\ &\leqslant M\sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x)\frac{1}{\beta(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}\frac{|t-x|}{\sqrt{t-x}}dt.\end{aligned}$$

Using the fact $\frac{1}{\sqrt{t-x}} < \frac{1}{\sqrt{x}}$ and Lemma 2.2.5 in the last inequality, we get

$$\begin{aligned}|V_n(f;x) - f(x)| &\leqslant \frac{M}{\sqrt{x}}\sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x)\frac{1}{\beta(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}|t-x|dt\\ &= \frac{M}{\sqrt{x}}(V_n|t-x|;x) \leqslant M\sqrt{\frac{\delta_n(x)}{x}}.\end{aligned}$$

This is the required result for $r = 1$. Let $r \in (0,1)$. Then application of the Holder inequality two times with $p=\frac{1}{r}$ and $q = \frac{1}{1-r}$ gives

$$\begin{aligned}|V_n(f;x) - f(x)| &\leqslant \sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x)\frac{1}{\beta(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}|f(t) - f(x)|dt\\ &\leqslant \left\{\sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x)\left(\frac{1}{\beta(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}|f(t) - f(x)|dt\right)^{\frac{1}{r}}\right\}^r\\ &\leqslant \left\{\sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x)\frac{1}{\beta(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}|f(t) - f(x)|^{\frac{1}{r}}dt\right\}^r.\end{aligned}$$

Since $f \in lip_M^*(r)$, this equality leads to

$$\begin{aligned}|V_n(f;x) - f(x)| &\leqslant M\left\{\sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x)\frac{1}{\beta(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}\left|\frac{|t-x|}{\sqrt{t-x}}\right|dt\right\}^r\\ &\leqslant \frac{M}{x^{\frac{r}{2}}}\left\{\sum_{k=0}^{\infty} v_{n,k}^{(\alpha)}(x)\frac{1}{\beta(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}||t-x||dt\right\}^r\\ &= \frac{M}{x^{\frac{r}{2}}}\{V_n(|t-x|;x)\}^r\end{aligned}$$

Therefore, by Lemma 2.2.5, we may conclude that

$$|V_n(f;x) - f(x)| \leqslant M\left(\frac{\delta_n(x)}{x}\right)^{\frac{r}{2}},$$

which completes the proof.

Let $H_{x^2}[0,\infty)$ be set of all functions f defined on $[0,\infty)$ having the property

$$|f(x)| \leqslant M_f\left(1+x^2\right),$$

where M_f is a constant depending only on f. By $C_{x^2}[0,\infty)$, we denotes the subspace of all continuous functions belonging to $H_{x^2}[0,\infty)$. For any positive b, by

$$\omega_b(f;\delta) = \sup_{|t-x|\leqslant\delta}\ \sup_{x,t\in[0,b]} |f(t)-f(x)|,$$

we denote the usual modulus of continuity of f on the closed interval $[0,b]$.

Theorem 2.2.9 *let $f \in C_{x^2}[0,\infty)$ and $\omega_{b+1}(f;s)$ be its modulus of continuity on the finite interval $[0,b+1] \subset [0,\infty)$ with $b>0$.Then for all $n>2$, we have*

$$\|V_n(f;x)-f(x)\|_{C[0,b]} \leqslant 6M_f\left(1+b^2\right)\delta_n(b) + 2\omega_{b+1}(f;\delta),$$

where $\delta_n(x)$ defined as in Lemma 2.2.5.

Proof:

$$|f(t)-f(x)| \leqslant 6M_f\left(1+b^2\right)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f;\delta),$$

$$|V_n(f;x)-f(x)| \leqslant 6M_f\left(1+b^2\right)M_n\left((t-x)^2;x\right)$$

$$+\,\omega_{b+1}(f;\delta)\left(1+\frac{1}{\delta_n}M_n(t-x);x\right).$$

From Lemma 2.2.5, for $x\in[0,b]$, we get

$$|V_n(f;x)-f(x)| \leqslant 6M_f\left(1+b^2\right)\delta_n(x) + \omega_{b+1}(f;\delta)\left(1+\frac{1}{\delta_n}\sqrt{\delta_n(x)}\right)$$

$$\leqslant 6M_f\left(1+b^2\right)\delta_n(b) + \omega_{b+1}(f;\delta)\left(1+\frac{1}{\delta_n}\sqrt{\delta_n(b)}\right).$$

Finally, by choosing $\delta=\delta_n(b)$, we arrive at the desired result.

Theorem 2.2.10 *If $f\in C_B[0,\infty)$ then for $x\in[0,\infty)$, we have*

$$\left|V_n^{(\alpha)}(f;x)-f(x)\right| \leq 2\omega f;\sqrt{V_n^{(\alpha)}}\left((\xi-x)^2;x\right),$$

where ω is the modulus of continuity of f [49] defined as:

$$\omega(f;x) := \sup_{\substack{x,y\in[0,\infty)\\ |x-y|\leq\delta}} |f(x)-f(y)|.$$

Proof: *Applying the well-known property of $\omega(f;x)$, Lemma 2.2.3, we have*

$$\begin{aligned}
\left|V_n^{(\alpha)}(f;x)-f(x)\right| &= \sum_{k=0}^{\infty}[\beta(k+1,n)]^{-1}b_{n,k}(x)\int_0^{\infty}V_{n,k}(t)f(t)\,dt \\
&= \left|\sum_{k=0}^{\infty}[\beta(k+1,n)]^{-1}b_{n,k}^{(\alpha)}(x)\int_0^{\infty}V_{n,k}(f(t)-f(x))\,dt\right| \\
&= \sum_{k=0}^{\infty}[\beta(k+1,n)]^{-1}b_{n,k}(x)\int_0^{\infty}V_{n,k}(f(t)-f(x))\,dt \\
&= \sum_{k=0}^{\infty}[\beta(k+1,n)]^{-1}b_{n,k}^{(\alpha)}(x)\int_0^{\infty}V_{n,k}(t)\left(1+\frac{1}{\delta}|\xi-x|\right)dt \\
&= \left[1+\frac{1}{\delta}\sum_{k=0}^{\infty}[\beta(k+1,n)]^{-1}b_{n,k}^{(\alpha)}(x)\int_0^{\infty}V_{n,k}(t)|t-x|\,dt\right]\omega(f;\delta).
\end{aligned}$$

For the integration, the following result holds by using Cauchy-Schwarz inequality

$$\begin{aligned}
\left|V_n^{(\alpha)}(f;x)-f(x)\right| &\leqslant [1+\frac{1}{\delta}\sum_{k=0}^{\infty}(\beta(k+1,n)\,d\xi)^{-1}b_{n,k}^{(\alpha)}(x)(\beta(k+1,n)\,d\xi)^{1/2} \\
&\quad\times\left(\int_0^{\infty}\beta(k+1,n)(\xi-x)^2d\xi\right)^{1/2}d\xi]\omega(f;\delta).
\end{aligned}$$

Now using the last inequality for infinite sum and we have

$$\begin{aligned}
\left|V_n^{(\alpha)}(f;x)-f(x)\right| &\leqslant \Bigg[1+\frac{1}{\delta}\Bigg\{\sum_{k=0}^{\infty}(\beta(k+1,n)\,d\xi)^{-1}b_{n,k}^{(\alpha)}(x)\int_0^{\infty}\beta(k+1,n)d\xi\Bigg\}^{1/2} \\
&\quad\times\sum_{k=0}^{\infty}\left(\int_0^{\infty}\beta(k+1,n)\,d\xi\right)^{-1}b_{n,k}^{(\alpha)}(x) \\
&\quad\times\int_0^{\infty}\beta(k+1,n)(\xi)(\xi-x)^2d\xi\}^{1/2}\Bigg]\omega(f;\delta) \\
&= \left[1+\frac{1}{\delta}V_n^{(\alpha)}(1;x)^{1/2}\right]V_n^{(\alpha)}\Big((\xi-x)^2;x\Big)^{1/2}\omega(f;\delta).
\end{aligned}$$

By taking

$$\delta=\left\{V_n^{(\alpha)}((\xi-x)^2;x\right)^{1/2},$$

we get the required result.

2.2.3 Graphical Comparisons

Example 2.2.11 *We have estimated the rate of convergence for the operators $V_m^{(\mu,\upsilon)}(f;x)$ to the function $f(x)=9/2x^2-2/9x+1$. In table, we estimated the absolute error $E_m =$*

$\left|V_m^{(\mu,\upsilon)}(f;x) - f(x)\right|$ *for different values of* $m = [20, 40, 60, 80]$, *while keeping* μ, υ*=1. As the conclusion comes from the error table for a large value of n the proposed operator (2.21) converges to* $f(x)$.

Table 2.2: Table for absolute error $E_m = \left|V_m^{(\mu,\upsilon)}(f;x) - f(x)\right|$ with $m = [20, 40, 60, 80]$.

x	E_{20}	E_{40}	E_{60}	E_{80}
0	0.0365846	0.0225925	0.0159124	0.0122399
0.4	0.319678	0.142688	0.0916417	0.067467
0.8	0.920152	0.422625	0.274099	0.202792
1.2	1.76484	0.817217	0.531461	0.393736
1.6	2.85373	1.32647	0.863725	0.640298
2	4.18683	1.95037	1.27089	0.942478

Our postulated operators (2.21) is an enhanced operator that enables faster convergence and better approximation. To emphasise our claim, we present a figure based on a numerical example to demonstrate a quicker rate of convergence for our operators.

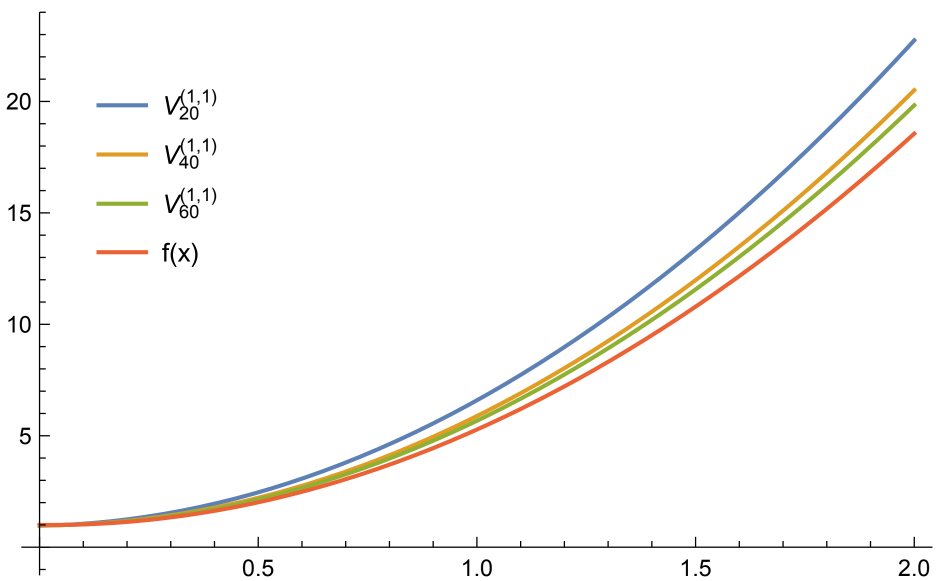

Figure 2.3: Considering $m = [20, 40, 60, 80]$, the convergence of operators $V_m^{(\mu,\upsilon)}(f;x)$ towards the function $f(x) = 9/2x^2 - 2/9x + 1$.

Chapter 3

An Approach to Preserve Functions with Exponential Growth by Modified Lupaş-Kantrovich Operators

We propose a modification of the so-known Lupaş-Kantrovich operators that preserve exponential function e^{-x}. To support this claim, we estimate the convergence rate of the operators in terms of both the usual and exponential modulus of continuity. The idea of the moment-generating function is used to determine the moment and central moment for modified operators. Our analysis also includes a global estimate and quantitative Voronovskaya results to examine the asymptotic behaviour. At the end of this chapter, we present a result and accompanying graphs to illustrate the efficiency of our modified operators.

3.1 Introduction

King [98] received recognition in 2003 for his modification of Bernstein operators that preserve test functions e_0 and e_2 on $[0, 1]$. Later, King drew the attention of scholars to this topic, and they presented numerous related studies. Many researchers have done exceptional work in this area by defining the operators that preserve e_0 and e_2, $e_2 + ae_1$ for $a > 0$, exponential functions, linear functions, and so on. According to the overall purpose of this paper, we will limit the scope of our investigation to only preserving the exponential functions. We know, studying how to preserve exponential functions is still at its dawn. Let us provide some of the most recent sources for this subject here. Acar et al. [13], published the modification of Szász-Mirakyan operators that preserves e^{2ax}, for $a \in (0, \infty]$. They discussed shape preservation properties and compared modified operators to Szász-Mirakyan operators. They also used a natural transformation to estimate the error in terms of the modulus of continuity of first order. Further, these modified operators [13] were studied in detail by Aral et al. [25], who demonstrated their importance from a computational perspective. Acar et al. [12] calculated the approximation order using a new proposed weighted modulus of smoothness and introduced the Szász-Mirakyan operators that fix $e^{\alpha x}$ and $e^{2\alpha x}$ where $\alpha > 0$ concurrently. Furthermore, they presented some saturation results for their modified operators to illustrate the accuracy of their estimates. Reader can also refer to the article [51; 73; 102] and references therein.

Many operators have been modified during the last four years using the same idea, including Bernstein [25], Baskakov-Szász-Mirakyan [71], Stancu Szász-Mirakyan-Durrmeyer operators [93], Phillips operators [76; 144], Baskakov-Schurer-Szász-Stancu operators[136], and Baskakov-Schurer-Szász operators [148]. Also, Deo et al. [41; 47] suggested the operators based on King's technique that outperforms the Baskakov Durrmeyer operators and Szász-Mirakyan Durrmeyer in terms of convergence rate. Yilmaz et al. [149] altered the Baskakov-Kantorovich operators in such a way that we get a sequence of operators that preserve constant functions and e^x. The whole purpose of our paper is to give an alteration of the operators [98] that preserves constant functions and e^{-x}.

3.1.1 Construction of Operators

In the year 1995, Lupaş [109] proposed discrete operators and established some direct results. For $f \in C[0, \infty)$, the Lupaş operators are defined as

$$\mathcal{G}_m(f; x) = \sum_{k=0}^{\infty} g_{m,k}(x) f\left(\frac{k}{m}\right),$$

where

$$g_{m,k}(x) = 2^{-mx} \frac{(mx)_k}{2^k k!}.$$

In [[77], Page No. 210], the modified form of Lupaş Kantorovich operators for the objective of approximating integral functions is defined as:

$$G_m(f;x) = (m+1)\sum_{k=0}^{\infty} 2^{-mx}\frac{(mx)_k}{2^k k!}\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(u)du. \tag{3.1}$$

We aim to construct operators that preserve 1 and e^{-x}. Let us assume operators (3.1) for some function $\lambda_m(x)$

$$G_m(e^{-u};\lambda_m(x)) = (m+1)\sum_{k=0}^{\infty} 2^{-m\lambda_m(x)}\frac{(m\lambda(x))_k}{2^k k!}\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} e^{-u}du,$$

therefore

$$G_m(e^{-u};\lambda_m(x)) = -(m+1)\,2^{-m\lambda_m(x)}\left(e^{-\frac{1}{m+1}}-1\right)\sum_{k=0}^{\infty}\frac{(m\lambda(x))_k}{k!}\left(\frac{e^{-\frac{1}{m+1}}}{2}\right)^k.$$

Now using the binomial series $\sum\limits_{k=0}^{\infty}\frac{(a)_k}{k!}z^k = (1-z)^{-a}$ $|z|<1$, we obtain

$$\begin{aligned} G_m(e^{-u};\lambda_m(x)) &= -(m+1)\,2^{-m\lambda_m(x)}\left(e^{-\frac{1}{m+1}}-1\right)\left(1-\frac{e^{-\frac{1}{m+1}}}{2}\right)^{-m\lambda_m(x)} \\ &= -(m+1)\left(e^{-\frac{1}{m+1}}-1\right)\left(2-e^{-\frac{1}{m+1}}\right)^{-m\lambda_m(x)}. \end{aligned}$$

Taking $G_m(e^{-u};\lambda_m(x)) = e^{-x}$ into account, we can calculate

$$\lambda_m(x) = \frac{x+\log\left((1+m)\left(1-e^{\frac{-1}{m+1}}\right)\right)}{m\log\left(2-e^{-\frac{1}{m+1}}\right)}. \tag{3.2}$$

We analyze the modified form of the operators (3.1) is

$$\tilde{\mathcal{K}}_m(f;x) = (m+1)\sum_{k=0}^{\infty} 2^{-m\lambda_m(x)}\frac{(m\lambda(x))_k}{2^k k!}\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(u)du, \tag{3.3}$$

where$\lambda_m(x)$ is given in (3.2).

3.2 Preliminaries

Performing basic calculations, the moment-generating function of the operators (3.3) can be expressed as:

$$\tilde{\mathcal{K}}_m\left(e^{\theta u};x\right) = \frac{1}{\theta}(m+1)\left(e^{\frac{\theta}{m+1}}-1\right)\left(2-e^{\frac{\theta}{m+1}}\right)^{-m\lambda_m(x)}. \tag{3.4}$$

Since the moments are linked to the moment-generating function, the $r-$th moment $\tilde{\mathcal{K}}_m(e_r;x)$, $e_r(u)=u^r$ $(r\in\mathbb{N}\cup\{0\})$ can be calculated by employing the following relation:

$$\tilde{\mathcal{K}}_m(e_r;x)=\left[\frac{\partial^r}{\partial\theta^r}\tilde{\mathcal{K}}_m\left(e^{\theta u};x\right)\right]_{\theta=0}=\left[\frac{\partial^r}{\partial\theta^r}\frac{1}{\theta}(m+1)\left(e^{\frac{\theta}{m+1}}-1\right)\left(2-e^{\frac{\theta}{m+1}}\right)^{-m\bar{\lambda}_m(x)}\right]_{\theta=0}.$$

Employing Mathematica software, the expansion of the above statement in powers of θ would be as follows:

$$\begin{aligned}
&\tilde{\mathcal{K}}_m\left(e^{\theta u};x\right)\\
&=\frac{1}{\theta}(m+1)\left(e^{\frac{\theta}{m+1}}-1\right)\left(2-e^{\frac{\theta}{m+1}}\right)^{-m\bar{\lambda}_m(x)}\\
&=1+\frac{\theta(2m\bar{\lambda}_m(x)+1)}{2(m+1)}+\frac{\theta^2\left(3m^2(\bar{\lambda}_m(x))^2+9m\bar{\lambda}_m(x)+1\right)}{6(m+1)^2}\\
&+\frac{\theta^3\left(4m^3(\bar{\lambda}_m(x))^3+30m^2(\bar{\lambda}_m(x))^2+40m\bar{\lambda}_m(x)+1\right)}{24(m+1)^3}\\
&+\frac{\theta^4\left(5m^4(\bar{\lambda}_m(x))^4+70m^3(\bar{\lambda}_m(x))^3+250m^2(\bar{\lambda}_m(x))^2+215m\bar{\lambda}_m(x)+1\right)}{120(m+1)^4}+O\left(\theta^5\right).
\end{aligned}\tag{3.5}$$

Moreover, by changing the scale attribute of moment-generating function, if we can expand $e^{-\theta x}\tilde{\mathcal{K}}_m\left(e^{\theta u};x\right)$ in powers of θ, the r-th order central moment $\psi_r(x)=\tilde{\mathcal{K}}_m((u-x)^r;x)$ may be produced by accumulating the coefficient of $\frac{\theta^r}{r!}$.

$$\begin{aligned}
&e^{-\theta x}\tilde{\mathcal{K}}_m\left(e^{\theta u};x\right)\\
&=e^{-\theta x}\frac{1}{\theta}(m+1)\left(e^{\frac{\theta}{m+1}}-1\right)\left(2-e^{\frac{\theta}{m+1}}\right)^{-m\bar{\lambda}_m(x)}\\
&=1+\frac{\theta(2m\bar{\lambda}_m(x)-2mx-2x+1)}{2(m+1)}\\
&+\frac{\theta^2}{6(m+1)^2}(3m^2(\bar{\lambda}_m(x))^2+3m^2x^2-6m^2\bar{\lambda}_m(x)x+9m\bar{\lambda}_m(x)+6mx^2-6m\bar{\lambda}_m(x)x\\
&-3mx+3x^2-3x+1)\\
&+\frac{\theta^3}{24(m+1)^3}(4m^3(\bar{\lambda}_m(x))^3-m^3x^3+12m^3\bar{\lambda}_m(x)x^2-12m^3(\bar{\lambda}_m(x))^2x+30m^2(\bar{\lambda}_m(x))^2\\
&-12m^2x^3+24\bar{\lambda}_m(x)m^2x^2+6m^2x^2-12m^2(\bar{\lambda}_m(x))^2x-36m^2\bar{\lambda}_m(x)x+40m\bar{\lambda}_m(x)\\
&-12mx^3+12m\bar{\lambda}_m(x)x^2+12mx^2-36m\bar{\lambda}_m(x)x-4mx-4x^3+6x^2-4x+1)\\
&+\frac{\theta^4}{120(m+1)^4}(5m^4(\bar{\lambda}_m(x))^4+5m^4x^4-20m^4\bar{\lambda}_m(x)x^3+30m^4(\bar{\lambda}_m(x))^2x^2-20m^4(\bar{\lambda}_m(x))^3x\\
&+70m^3(\bar{\lambda}_m(x))^3+20m^3x^4-60m^3\bar{\lambda}_m(x)x^3-10m^3x^3+60m^3(\bar{\lambda}_m(x))^2x^2+90m^3\bar{\lambda}_m(x)x^2\\
&-20m^3(\bar{\lambda}_m(x))^3x-150m^3(\bar{\lambda}_m(x))^2x+250m^2(\bar{\lambda}_m(x))^2+30m^2x^4-60m^2\bar{\lambda}_m(x)x^3-30m^2x^3\\
&+30m^2(\bar{\lambda}_m(x))^2x^2+180m^2\bar{\lambda}_m(x)x^2+10m^2x^2-150m^2(\bar{\lambda}_m(x))^2x-200m^2\bar{\lambda}_m(x)x\\
&+215m\bar{\lambda}_m(x)+20mx^4-20m\bar{\lambda}_m(x)x^3-30mx^3+90m\bar{\lambda}_m(x)x^2+20mx^2-200m\bar{\lambda}_m(x)x\\
&-5mx+5x^4-10x^3+10x^2-5x+1)+O\left(\theta^5\right).
\end{aligned}$$

Lemma 3.2.1 *We calculate the moments of the proposed operator (3.3) using (3.5) as described below:*

$$\tilde{\mathcal{K}}_m(e_0;x) = 1;$$
$$\tilde{\mathcal{K}}_m(e_1;x) = \frac{(2m\tau_m(x)+1)}{2(m+1)};$$
$$\tilde{\mathcal{K}}_m(e_2;x) = \frac{\left(3m^2(\tau_m(x))^2 + 9m\tau_m(x) + 1\right)}{3(m+1)^2};$$
$$\tilde{\mathcal{K}}_m(e_3;x) = \frac{\left(4m^3(\tau_m(x))^3 + 30m^2(\tau_m(x))^2 + 40m\tau_m(x) + 1\right)}{4(m+1)^3};$$
$$\tilde{\mathcal{K}}_m(e_4;x) = \frac{\left(5m^4(\tau_m(x))^4 + 70m^3(\tau_m(x))^3 + 250m^2(\tau_m(x))^2 + 215m\tau_m(x) + 1\right)}{5(m+1)^4}.$$

Lemma 3.2.2 *As a result of the above analysis, the first four central moments are as follows:*

$$\psi_1(x) = \frac{1}{2(m+1)}(2m\tau_m(x) - 2mx - 2x + 1);$$
$$\psi_2(x) = \frac{1}{3(m+1)^2}(3m^2(\tau_m(x))^2 + 3m^2x^2 - 6m^2\tau_m(x)x + 9m\tau_m(x) + 6mx^2$$
$$- 6m\tau_m(x)x - 3mx + 3x^2 - 3x + 1);$$
$$\psi_3(x) = \frac{1}{4(m+1)^3}(4m^3(\tau_m(x))^3 - 4m^3x^3 + 12m^3\tau_m(x)x^2 - 12m^3(\tau_m(x))^2x$$
$$+ 30m^2(\tau_m(x))^2 - 12m^2x^3 + 24m^2\tau_m(x)x^2 + 6m^2x^2 - 12m^2(\tau_m(x))^2x$$
$$- 36m^2\tau_m(x)x + 40m\tau_m(x) - 12mx^3 + 12m\tau_m(x)x^2 + 12mx^2 - 36m\tau_m(x)x$$
$$- 4mx - 4x^3 + 6x^2 - 4x + 1);$$
$$\psi_4(x) = \frac{1}{5(m+1)^4}(5m^4(\tau_m(x))^4 + 5m^4x^4 - 20m^4\tau_m(x)x^3 + 30m^4(\tau_m(x))^2x^2$$
$$- 20m^4(\tau_m(x))^3x + 70m^3(\tau_m(x))^3 + 20m^3x^4 - 60m^3\tau_m(x)x^3 - 10m^3x^3$$
$$+ 60m^3(\tau_m(x))^2x^2 + 90m^3\tau_m(x)x^2 - 20m^3(\tau_m(x))^3x - 150m^3(\tau_m(x))^2x$$
$$+ 250m^2\tau_m(x)^2 + 30m^2x^4 - 60m^2\tau_m(x)x^3 - 30m^2x^3 + 30m^2(\tau_m(x))^2x^2$$
$$+ 180m^2\tau_m(x)x^2 + 10m^2x^2 - 150m^2(\tau_m(x))^2x - 200m^2\tau_m(x)x + 215n\tau_m(x)$$
$$+ 20nx^4 - 20n\tau_m(x)x^3 - 30nx^3 + 90n\tau_m(x)x^2 + 20nx^2 - 200n\tau_m(x)x$$
$$- 5nx + 5x^4 - 10x^3 + 10x^2 - 5x + 1).$$

Also,

$$\lim_{m\to\infty} m\psi_1(x) = \lim_{m\to\infty} m\left[\frac{(2m\tau_m(x) - 2mx - 2x + 1)}{2(m+1)}\right] = x$$

and

$$\lim_{m\to\infty} m\psi_2(x) = \lim_{m\to\infty} \frac{m}{3(m+1)^2}\Big[3m^2(\daleth_m(x))^2 + 3m^2x^2 - 6m^2\daleth_m(x)\,x + 9m\daleth_m(x)$$
$$+6mx^2 - 6m\daleth_m(x)\,x - 3mx + 3x^2 - 3x + 1\Big] = 2x.$$

3.3 Main Results

The subspace of real-valued continuous functions with uniform norms that have finite limits at infinity can be represented as $C^*[0,\infty)$. For a function over an infinite interval, Boyanov [35] described its approximation properties. Holhoş [82] later proved the next theorem to figure out how fast a function will converge.

Theorem 3.3.1 *Let $Q : C^*[0,\infty) \to C^*[0,\infty)$ be the sequence of linear positive operators and*

$$\|Q(e_0) - 1\|_{[0,\infty)} = \alpha_1(m);$$
$$\left\|Q(e^{-u}) - e^{-x}\right\|_{[0,\infty)} = \alpha_2(m);$$
$$\left\|Q\left(e^{-2u}\right) - e^{-2x}\right\|_{[0,\infty)} = \alpha_3(m).$$

Then

$$\|Q_m f - f\|_{[0,\infty)} \le \alpha_1(m)\|f\|_{[0,\infty)} + (2 + \alpha_1(m))\omega^*(f;\ \sqrt{\alpha_1(m) + 2\alpha_2(m) + \alpha_3(m)}).$$

The modulus of continuity can be expressed as:

$$\omega^*(\tilde{v};\delta) = \sup_{\substack{|e^{-x}-e^{-u}|\le\delta \\ x,u>0}} |\tilde{v}(x) - \tilde{v}(u)|,$$

with the property

$$|\tilde{v}(u) - \tilde{v}(x)| \le \left(1 + \frac{(e^{-u} - e^{-x})^2}{\delta^2}\right)\omega^*(\tilde{v};\delta), \quad \delta > 0. \tag{3.6}$$

In the presented theorem, we present quantitative estimates for proposed operators as an application of the previous theorem.

Theorem 3.3.2 *For $f \in C^*[0,\infty)$, the inequality*

$$\left\|\tilde{\mathcal{K}}_m f - f\right\|_{[0,\infty)} \le 2\omega^*\left(f,\ \sqrt{\alpha_3(m)}\right),$$

holds true. Here $\tilde{\mathcal{K}}_m f$ converges to f uniformly and $\alpha_3(m) \to 0$ as $m \to \infty$.

Proof: *The operators preserve constant functions as well as e^{-x}, so $\alpha_1(m) = \alpha_2(m) = 0$.*

Now, we simply need to evaluate $\alpha_3(m)$.

From (3.4), we obtain

$$\tilde{\mathcal{K}}_m\left(e^{-2u};x\right)=\frac{-1}{2}(m+1)\left(e^{\frac{-2}{m+1}}-1\right)\left(2-e^{\frac{-2}{m+1}}\right)^{-\lambda_m(x)},$$

where

$$\lambda_m(x)=\frac{x+\log\left((1+m)\left(1-e^{\frac{-1}{m+1}}\right)\right)}{m\log\left(2-e^{-\frac{1}{m+1}}\right)}.$$

Employing Mathematica software, we obtain

$$\tilde{\mathcal{K}}_m\left(e^{-2u};x\right)=e^{-2x}+\frac{2xe^{-2x}}{m}+\frac{\left(24x^2-72x-11\right)e^{-2x}}{12m^2}+O\left(\left(\frac{1}{m}\right)^3\right).$$

Since

$$\underset{x\in[0,\infty)}{sup}\ xe^{-2x}=\frac{1}{2e},\ \underset{x\in[0,\infty)}{sup}\ x^2e^{-2x}=\frac{1}{e^2}.$$

So, we obtain

$$\begin{aligned}\alpha_3(m)&=\left\|\tilde{\mathcal{K}}_m\left(e^{-2u}\right)-e^{-2x}\right\|_{[0,\infty)}\\&=\underset{x\in[0,\infty)}{sup}\left|\tilde{\mathcal{K}}_m\left(e^{-2u}\right)-e^{-2x}\right|\\&\le\frac{1}{m}\left(\frac{1}{e}\right)+\frac{1}{m^2}\left(\frac{2}{e^2}+\frac{3}{e}+\frac{11}{12}\right)+O\left(\left(\frac{1}{m}\right)^3\right)\\&\le O\left(\frac{1}{m}\right)\to 0\ as\ m\to\infty.\end{aligned}$$

Remark 3.3.3 *With the aid of Mathematica and Lemma 3.2.2, we find*

$$\begin{aligned}\lim_{n\to\infty}m^2\psi_4(x)=\lim_{n\to\infty}\frac{m^2}{5(n+1)^4}\Big[&5m^4(\lambda_m(x))^4+5m^4x^4-20m^4\lambda_m(x)x^3+30m^4(\lambda_m(x))^2x^2\\&-20m^4(\lambda_m(x))^3x+70m^3(\lambda_m(x))^3+20m^3x^4-60m^3\lambda_m(x)x^3-10m^3x^3\\&+60m^3(\lambda_m(x))^2x^2+90m^3\lambda_m(x)x^2-20m^3(\lambda_m(x))^3x-150m^3(\lambda_m(x))^2x\\&+250m^2\lambda_m(x)^2+30m^2x^4-60m^2\lambda_m(x)x^3-30m^2x^3+30m^2(\lambda_m(x))^2x^2\\&+180m^2\lambda_m(x)x^2+10m^2x^2-150m^2(\lambda_m(x))^2x-200m^2\lambda_m(x)x\\&+215m\lambda_m(x)+20mx^4-20m\lambda_m(x)x^3-30mx^3+90m\lambda_m(x)x^2+20mx^2\\&-200m\lambda_m(x)x-5mx+5x^4-10x^3+10x^2-5x+1\Big]\\=\ &12x^2,\end{aligned}$$

and

$$\lim_{m\to\infty}m^2\tilde{\mathcal{K}}_m\left((e^{-x}-e^{-u})^4;x\right)=12e^{-4x}x^2.$$

Theorem 3.3.4 *For $x \in [0,\infty)$, and $f, f'' \in C^*[0,\infty)$, we obtain*

$$\left|m\left[\tilde{\mathcal{K}}_m(f;x) - f(x)\right] - x\left[f'(x) + f''(x)\right]\right|$$
$$\leq |\alpha_m(x)|\,|f'(x)| + |\beta_m(x)|\,|f''(x)| + 4\omega^*\left(f'', \frac{1}{\sqrt{m}}\right)((\beta_m(x) + x) + \gamma_m(x)).$$

Proof: *Using Taylor's expansion, we obtain*

$$f(u) = f(x) + (u-x)f'(x) + \frac{1}{2}(u-x)^2 f''(x) + \tilde{v}(u,x)(u-x)^2,$$

where

$$\tilde{v}(u,x) = \frac{f''(\mu) - f''(x)}{2}, \quad x < \mu < u.$$

Applying the operators $\tilde{\mathcal{K}}_m$ to both sides of the aforementioned expression and the Lemma (3.2.2) results in the following expression:

$$\left|\tilde{\mathcal{K}}_m(f;x) - f(x) - \psi_1(x)f'(x) - \frac{1}{2}\psi_2(x)f''(x)\right| \leq \left|\tilde{\mathcal{K}}_m\left(\tilde{v}(u,x)(u-x)^2;x\right)\right|.$$

Using Lemma 3.2.2, we get

$$\left|m\left[\tilde{\mathcal{K}}_m(f;x) - f(x)\right] - xf'(x) - \frac{1}{2}(2x)f''(x)\right|$$
$$\leq |m(\psi_1(x)) - x|\,|f'(x)| + \frac{1}{2}|m(\psi_2(x)) - 2x|\,|f''(x)|$$
$$+ \left|m\tilde{\mathcal{K}}_m\left(\tilde{v}(u,x)(u-x)^2;x\right)\right|.$$

Taking $\alpha_m(x) = m(\psi_1(x)) - x$ and $\beta_m(x) = \frac{1}{2}|m(\psi_2(x)) - 2x|$, we get

$$\left|m\left[\tilde{\mathcal{K}}_m(f;x) - f(x)\right] - x\left[f'(x) + f''(x)\right]\right|$$
$$\leq |\alpha_m(x)|\,|f'(x)| + |\beta_m(x)|\,|f''(x)| + \left|m\tilde{\mathcal{K}}_m\left(\tilde{v}(u,x)(u-x)^2;x\right)\right|.$$

To conclude our proof, we need to find the estimate of $\left|n\tilde{\mathcal{K}}_m\left(\tilde{v}(u,x)(u-x)^2;x\right)\right|$. Employing inequality (3.6), we arrive at

$$|\tilde{v}(u,x)| \leq \left(1 + \frac{(e^{-u} - e^{-x})^2}{\delta^2}\right)\omega^*(f'';\delta).$$

For the cases $|e^{-u} - e^{-x}| \leq \delta$ and $|e^{-u} - e^{-x}| > \delta$ the two inequalities

$$|\tilde{v}(u,x)| \leq 2\omega^*(f'';\delta) \quad \text{and} \quad |\tilde{v}(u,x)| \leq \frac{2(e^{-u} - e^{-x})^2}{\delta^2}\omega^*(f'';\delta),$$

holds true respectively.

Thus

$$|\tilde{v}(u,x)| \leq 2\left(1 + \frac{(e^{-u} - e^{-x})^2}{\delta^2}\omega^*(f'';\delta)\right).$$

Making use of the preceding argument and Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
& m\tilde{\mathcal{K}}_m\left(\tilde{v}(u,x)(u-x)^2;x\right) \\
& \leq m\tilde{\mathcal{K}}_m\left(2\left(1+\frac{(e^{-u}-e^{-x})^2}{\delta^2}\omega^*(f'',\delta)\right)(u-x)^2;x\right) \\
& = 2m(\psi_2(x))\,\omega^*(f'',\delta)+\frac{2m}{\delta^2}\omega^*(f'',\delta)\,\tilde{\mathcal{K}}_m\left((e^{-u}-e^{-x})^2(u-x)^2;x\right) \\
& = 2\omega^*(f'',\delta)\left[m(\psi_2(x))+\frac{m}{\delta^2}(\psi_4(x))^{1/2}\left(\tilde{\mathcal{K}}_m\left((e^{-u}-e^{-x})^2;x\right)\right)^{1/2}\right].
\end{aligned}
$$

Conclude the result by selecting $\delta=\frac{1}{\sqrt{m}}$ *and* $\gamma_m(x)=(\psi_4(x))^{1/2}\left(\tilde{\mathcal{K}}_m\left((e^{-u}-e^{-x})^2;x\right)\right)^{1/2}$.

Theorem 3.3.5 *For* $f, f''\in C^*[0,\infty)$ *and* $x\in[0,\infty)$, *the following equality holds*

$$\lim_{m\to\infty} m\left[\tilde{\mathcal{K}}_m(f;x)-f(x)\right]=x\left[f'(x)+f''(x)\right].$$

Proof: *With the help of Taylor's expansion of f, we obtain*

$$f(u)=f(x)+(u-x)f'(x)+\frac{1}{2}(u-x)^2f''(x)+\tilde{v}(u,x)(u-x)^2, \tag{3.7}$$

where

$$\lim_{u\to x}\tilde{v}(u,x)=0.$$

Applying $\tilde{\mathcal{K}}_m$ *to both sides of (3.7) and from Lemma 3.2.2, we obtain*

$$\tilde{\mathcal{K}}_m(f;x)-f(x)=\psi_1(x)f'(x)+\frac{1}{2}\psi_2(x)f''(x)+\tilde{\mathcal{K}}_m\left(\tilde{v}(u,x)(u-x)^2;x\right).$$

The Cauchy-Schwarz inequality can be used to determine

$$\tilde{\mathcal{K}}_m\left(\tilde{v}(u,x)(u-x)^2;x\right)\leq\sqrt{\tilde{\mathcal{K}}_m\left(\tilde{v}^2(u,x);x\right)\tilde{\mathcal{K}}_m\left((u-x)^4;x\right)}. \tag{3.8}$$

Also, we have

$$\lim_{m\to\infty}\tilde{\mathcal{K}}_m\left(\tilde{v}^2(u,x);x\right)=0. \tag{3.9}$$

From (3.8) and (3.9), we obtain

$$\lim_{m\to\infty} m\tilde{\mathcal{K}}_m\left(\tilde{v}(u,x)(u-x)^2;x\right)=0.$$

Thus we achieve

$$
\begin{aligned}
& \lim_{m\to\infty} m\left[\tilde{\mathcal{K}}_m(f;x)-f(x)\right] \\
& = \lim_{n\to\infty} m\left[\psi_1(x)f'(x)+\frac{1}{2}\psi_2(x)f''(x)+\tilde{\mathcal{K}}_m\left(\tilde{v}(u,x)(u-x)^2;x\right)\right] \\
& = x\left[f'(x)+f''(x)\right].
\end{aligned}
$$

Let $C_B[0,\infty)$ denote the space of real-valued continuous and bounded function f on the interval $[0,\infty)$, endowed with the norm,

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|.$$

For any $\delta > 0$, the Peetre's K-functional is defined by:

$$K_2(f;\delta) = \inf_{l\in C_B[0,\infty)} \{\|f-l\| + \delta\|l''\|\},$$

where

$$C_B^2[0,\infty) = \{g \in C_B[0,\infty) : l', l'' \in C_B[0,\infty)\}.$$

By DeVore and Lorentz [49] there exist an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leqslant C\omega_2\left(f;\ \sqrt{\delta}\right),$$

where the modulus of continuity of second order is denoted by

$$\omega_2\left(f;\ \sqrt{\delta}\right) = \sup_{x\in[0,\infty)} \sup_{0<h\leqslant\sqrt{\delta}} |f(x+2h) - 2f(x+h) + f(x)|.$$

Theorem 3.3.6 *Let $f \in C_B[0,\infty)$, then the inequality*

$$\left|\tilde{\mathcal{K}}_m(f;x) - f(x)\right| \leq M\omega_2\left(f;\frac{1}{2}\sqrt{\psi_2(x) + \frac{\left(\frac{(2m\tau_m(x)+1)}{2(m+1)} - x\right)^2}{2}}\right) + \omega\left(f;\left(\frac{(2m\tau_m(x)+1)}{2(m+1)} - x\right)\right),$$

holds true for all $x \in [0,\infty)$ and a positive constant M.

Proof: *We establish the auxiliary operators $\mathbb{K}_m : C_B[0,\infty) \to C_B[0,\infty)$*

$$\mathbb{K}_m(f;x) = \tilde{\mathcal{K}}_m(v;x) + f(x) - f\left(\frac{(2m\tau_m(x)+1)}{2(m+1)}\right).$$

For the operators (3.3), we have

$$\left\|\tilde{\mathcal{K}}_m(f;x)\right\| \leq \|f\|,$$

implies

$$\|\mathbb{K}_m(f;x)\| \leq 3\|f\|. \tag{3.10}$$

Additionally, for $\hbar \in C_B^2[0,\infty)$, Taylor expansion is presented as:

$$\tilde{v}(u) - \tilde{v}(x) = (u-x)\tilde{v}'(x) + \int_x^u (u-\mu)\tilde{v}''(\mu)\,d\mu, \qquad x \in \mathbb{R}[0,\infty).$$

By applying operators $\mathbb{K}_m(f;x)$ on both sides of the above equation and utilizing the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|\mathbb{K}_m(\tilde{v};x)-\tilde{v}(x)| &= \left|\mathbb{K}_m\left(\int_x^u (u-\mu)\tilde{v}''(\mu)\,d\mu;x\right)\right| \\
&\leq \left|\tilde{\mathcal{K}}_m\left(\int_x^u (u-\mu)\tilde{v}''(\mu)\,d\mu;x\right)\right| + \left|\int_x^{\frac{(2m\mathfrak{l}_m(x)+1)}{2(m+1)}} \left(\frac{(2m\mathfrak{l}_m(x)+1)}{2(m+1)}-\mu\right)\tilde{v}''(\mu)\,d\mu\right| \\
&\leq \|\tilde{v}''\|\left(\psi_2(x)+\frac{\left(\frac{(2m\mathfrak{l}_m(x)+1)}{2(m+1)}-x\right)^2}{2}\right). \qquad (3.11)
\end{aligned}$$

Using the estimates from equations (3.10) and (3.11), we can finally conclude that

$$\begin{aligned}
&\left|\tilde{\mathcal{K}}_m(f;x)-f(x)\right| \\
&\leq |\mathbb{K}_m(f-\tilde{v};x)-(f-\tilde{v})(x)| + \left|f\left(\frac{(2m\mathfrak{l}_m(x)+1)}{2(m+1)}\right)-f(x)\right| + |\mathbb{K}_m(\tilde{v};x)-\tilde{v}(x)| \\
&\leq 4\|f-\tilde{v}\| + \left(\psi_2(x)+\frac{\left(\frac{(2m\mathfrak{l}_m(x)+1)}{2(m+1)}-x\right)^2}{2}\right)\|\tilde{v}''\| + \left|v\left(\frac{(2m\mathfrak{l}_m(x)+1)}{2(m+1)}\right)-f(x)\right| \\
&\leq 4K_2\left(f,\frac{1}{4}\left(\psi_2(x)+\frac{\left(\frac{(2m\mathfrak{l}_m(x)+1)}{2(m+1)}-x\right)^2}{2}\right)\right) + \left|f\left(\frac{(2m\mathfrak{l}_m(x)+1)}{2(m+1)}\right)-f(x)\right| \\
&\leq M\omega_2\left(f,\frac{1}{2}\sqrt{\left(\psi_2(x)+\frac{\left(\frac{(2m\mathfrak{l}_m(x)+1)}{2(m+1)}-x\right)^2}{2}\right)}\right) + \omega\left(f,\left(\frac{(2m\mathfrak{l}_m(x)+1)}{2(m+1)}-x\right)\right).
\end{aligned}$$

3.4 Graphical Comparisions

Example 3.4.1 *The function $f(x) = cos(x)$ is approximated by the operators $\tilde{\mathcal{K}}_m$ in Figure 3.1 for n $= 10, 20, 40$. The operators (3.3) clearly converges to the function $f(x)$ as n is increased. We calculate the error $|\tilde{\mathcal{K}}_m(f;x)-f(x)|$ in Table 3.1 and also give a graph depicting the error in Figure 3.2.*

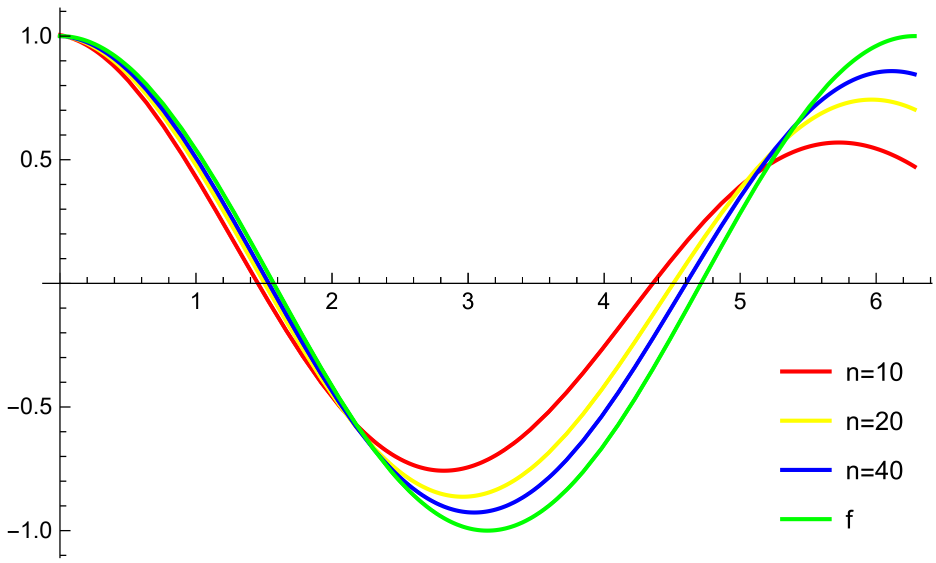

Figure 3.1: Approximation behaviour of $\tilde{\mathcal{K}}_m$ for the function $f(x) = cos(x)$.

Table 3.1: Estimation of absolute error $|\tilde{\mathcal{K}}_m(f; x) - f(x)|$ for the function $f(x) = cos(x)$ at different values of $n = 10, 20, 40, 60$.

x	$n = 10$	$n = 20$	$n = 40$	$n = 60$
$\frac{\pi}{2}$	0.107714	0.0648587	0.0356679	0.0245513
π	0.286407	0.152325	0.0777487	0.0520773
$\frac{3\pi}{2}$	0.236268	0.166841	0.0992171	0.0700621
2π	0.528039	0.296931	0.154454	0.103863

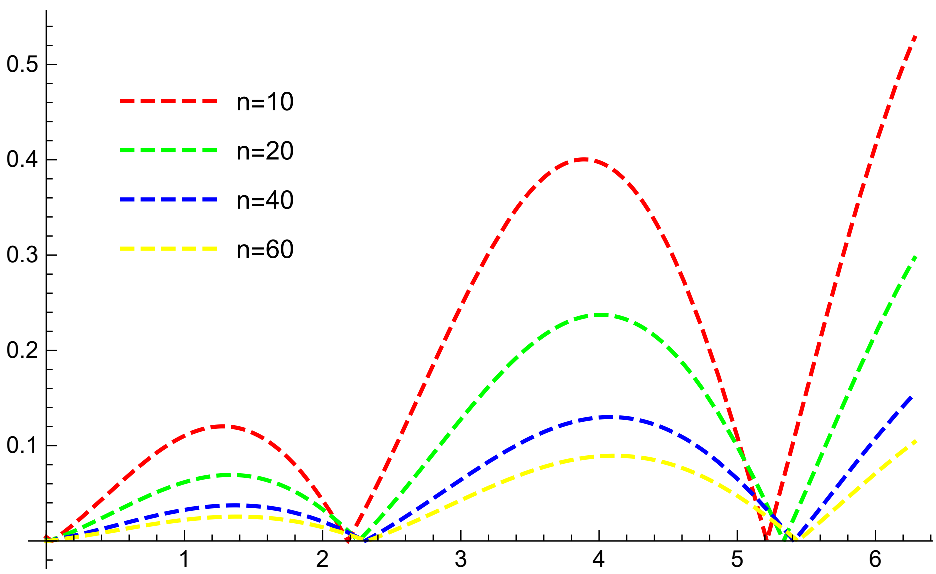

Figure 3.2: Absolute error $|\tilde{\mathcal{K}}_m(f; x) - f(x)|$ of the proposed operators for $f(x) = cos(x)$ with $n = 10, 20, 40, 60$.

Chapter 4

Study of Operators Associated with Inverse Pólya-Eggenberger Distribution

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Researchers have spent the last few decades studying a large array of approximation operators due to the development of the theory of inverse Pólya-Eggenberger distribution. This chapter focuses mainly on the investigation of a modification of certain inverse Pólya-Eggenberger distribution. In the first section, we introduce the Bézier variant of a sequence of summation-integral type operators involving inverse Pólya-Eggenberger distribution and Păltănea operators [126]. For these operators, we estimate the approximation behaviour including first and second-order modulus of smoothness. Lastly, we establish the rate of convergence for a class of functions with derivatives of bounded variation. In the second section, we explore the approximation properties of a non-negative real parametric generalization of the Baskakov operators using the inverse Pólya-Eggenberger (I-P-E) distribution. As a result of this study, we can obtain some approximation results, including the Voronovskaya type asymptotic formula, error estimate in terms of modulus of continuity and in the sense of k-functional, and weighted approximation.

4.1 Bézier Variant of Summation-Integral Type Operators

4.1.1 Introduction

In 1970, Stancu [142] presented the generalization of Baskakov operators using the concept of inverse Pólya-Eggenberger distribution (IPED) with subject to parameter

$\beta \geq 0$. For $\varphi \in C[0,\infty)$, these operators are defined as follows:

$$L_n^{(\beta)}(\varphi;x) = \sum_{r=0}^{\infty} \begin{pmatrix} n+r-1 \\ r \end{pmatrix} \frac{\prod_{i=0}^{r-1}(x+i\beta)\prod_{j=0}^{n-1}(1+j\beta)}{\prod_{k=0}^{n+r-1}(x+1+k\beta)} f\left(\frac{r}{n}\right)$$
$$= \sum_{r=0}^{\infty} w_{n,r}^{(\beta)}(x) f\left(\frac{r}{n}\right), \quad x \in [0,\infty). \tag{4.1}$$

In the current decade, several researchers proposed various operators and their variants based on PED as well as IPED and studied their approximation properties and interesting results. For more literature on this topic, one can refer to articles [5; 44; 46; 50; 52; 53; 78; 85; 95].

In 2003, Srivastava-Gupta [139] introduced a widespread family of positive linear operators and contemplated their estimated properties and interesting results. In light of this work, we consider summation-integral type operators involving inverse Pólya-Eggenberger distribution [142] and Păltănea operators [126].

For $\rho \geq 0$, we consider the integral-type operators, which is a generalisation of the operators (4.1):

$$\hat{L}_n^{(\beta)}(\varphi;x) = \sum_{r=1}^{\infty} w_{n,r}^{(\beta)}(x) \int_0^{\infty} \chi_{n,r}^{\rho}(t)\,\varphi(t)dt + w_{n,0}^{(\beta)}(x)\varphi(0), \tag{4.2}$$

where

$$\chi_{n,r}^{\rho}(t) = \begin{cases} \frac{n\rho}{\Gamma(r\rho)} e^{-n\rho t}(n\rho t)^{r\rho-1}, & c=0 \\ \frac{\Gamma\left(\frac{n\rho}{c}+r\rho\right)}{\Gamma(r\rho)\Gamma\left(\frac{n\rho}{c}\right)} \frac{c^{r\rho}t^{r\rho-1}}{(1+ct)^{\frac{n\rho}{c}+r\rho}}, & c \in \mathbb{N}, \end{cases} \tag{4.3}$$

and

$$\int_0^{\infty} \chi_{n,r}^{\rho}(t)t^j dt = \begin{cases} \frac{\Gamma(r\rho+j)}{\Gamma(r\rho)} \frac{1}{\prod_{i=1}^{j}(n\rho-ic)}, & j \neq 0 \\ 1, & j=0. \end{cases} \tag{4.4}$$

The operators (4.2) were also studied by Kajla et al. [95] for $c=0$. Now we introduce the Bézier variant of (4.2) for $\theta \geq 1$ as follows:

$$\hat{L}_{n,\theta}^{(\beta)}(\varphi;x) = \sum_{r=1}^{\infty} w_{n,r}^{\theta,\beta}(x) \int_0^{\infty} \chi_{n,r}^{\rho}(t)\,\varphi(t)dt + w_{n,0}^{\theta,\beta}(x)\varphi(0), \tag{4.5}$$

where $w_{n,r}^{\theta,\beta}(x) = \left(J_{n,r}^{\beta}(x)\right)^{\theta} - \left(J_{n,r+1}^{\beta}(x)\right)^{\theta}$ and $J_{n,r}^{\beta}(x,c) = \sum_{j=r}^{\infty} w_{n,j}^{(\beta)}(x)$. It is obvious that the operators $L_{n,\theta}^{(\beta)}(.;x)$ are the linear positive operators.

Special Cases:

(1) For $c = 0$ and $\theta = 0$, the operators (4.5) reduce to Baskakov-Szász type operators based on inverse Polya-Eggenberger-distribution [95].

(2) For $c = \beta = 0$ and $\theta = \rho = 1$, the operators (4.5) include Baskakov-Szász operators (see [2; 80]).

(3) For $c = \beta = 0$, $\theta = 1$ and $\rho \to \infty$, the operators (4.5) reduce to Baskakov operators [30].

(4) For $c = 0$, $\theta = 1$, $\beta > 0$ and $\rho \to \infty$, the operators (4.5) include Stancu operators [142].

A lot of work has already been done on summation-integral type operators involving various linear positive operators and analysis of their convergence results. We refer to the readers some interesting articles (see [7; 37; 39; 81; 118; 130]) for more information.

The vital target of the paper is to contemplate the approximation properties of operators (4.5) including K-functional and second-order modulus of smoothness. Lastly, we set up the rate of convergence for functions with a class of derivative of bounded variation.

4.1.2 Preliminaries

Lemma 4.1.1 *For $e_k(x) = x^k,\ k = 0, 1, 2, \ldots,$ the moments of positive linear operators* (4.2) *are as follows:*

$$\hat{L}_n^{(\beta)}(e_0; x) = 1;$$
$$\hat{L}_n^{(\beta)}(e_1; x) = \frac{n\rho x}{(1-\beta)(n\rho - c)};$$
$$\hat{L}_n^{(\beta)}(e_2; x) = \frac{n\rho x}{(1-\beta)(n\rho - c)(n\rho - 2c)}\left[\rho\frac{\{(n+1)(x+\beta) + (1-2\beta)\}}{1-2\beta} + 1\right].$$

Lemma 4.1.2 *For $n \in \mathbb{N}$ and with the help of Lemma 4.1.1, the central moments of the operators* (4.2) *are as follows:*

$$\hat{L}_n^{(\beta)}((e_1 - x); x) = \frac{(1-\beta)c + n\beta\rho}{(1-\beta)(n\rho - c)}x;$$
$$\hat{L}_n^{(\beta)}((e_1 - x)^2; x) = \frac{n\rho\left[1 + \rho + \beta\{(n-1)\rho - 2\}\right]}{(1-\beta)(1-2\beta)(n\rho - c)(n\rho - 2c)}x + \frac{2c^2(1 - 3\beta + 2\beta^2) + c(1 + \beta - 6\beta^2)n\rho + \{1 + \beta(1+2\beta)n\}n\rho^2}{(1-\beta)(1-2\beta)(n\rho - c)(n\rho - 2c)}x^2.$$

Further, for every $x \in [0,\infty)$, $\hat{L}_n^{(\beta)}((e_1-x)^m;x)$ is a polynomial in x of degree m with

$$\hat{L}_n^{(\beta)}((e_1-x)^m;x) = O\left(\frac{1}{n^{-[(m+1)/2]}}\right), \quad as\ n\to\infty.$$

Lemma 4.1.3 *If $\beta=\beta(n)\to 0$ as n is sufficient large and $\lim_{n\to\infty} n\beta(n) = s \in \mathbb{R}$, then*

(i) $\lim_{n\to\infty} n\hat{L}_n^{(\beta)}((e_1-x);x) = \left(s+\frac{c}{\rho}\right)x;$

(ii) $\lim_{n\to\infty} n\hat{L}_n^{(\beta)}((e_1-x)^2;x) = \left(1+s+\frac{1}{\rho}\right)x + \left(1+s+\frac{c}{\rho}\right)x^2 \leqslant \left(1+s+\frac{c}{\rho}\right)(1+x)\,x.$

Lemma 4.1.4 *For every $\varphi \in C_B[0,\infty)$, we have*

$$\left|\hat{L}_{n,\theta}^{(\beta)}(\varphi(t);x)\right| \le \theta\hat{L}_n^{(\beta)}(\|\varphi\|;x) \le \theta\|\varphi\|.$$

Proof: *For $0\le l_1\le l_2\le 1$, $\theta\ge 1$, using the inequality*

$$\left|l_1^\beta - l_2^\beta\right| \le \theta\,|l_1-l_2|,$$

and from the definition of $w_{n,r}^{\theta,\beta}(x)$, we get

$$\begin{aligned} 0 &< \left(J_{n,r}^\beta(x)\right)^\theta - \left(J_{n,r+1}^\beta(x)\right)^\theta \\ &\le \theta\left(J_{n,r}^\beta(x) - J_{n,r+1}^\beta(x)\right) \\ &= \theta J_{n,0}^\beta(x). \end{aligned}$$

Hence

$$\left|\hat{L}_{n,\theta}^{(\beta)}(\varphi(t);x)\right| \le \theta\hat{L}_n^{(\beta)}(\|\varphi\|;x) \le \theta\|\varphi\|.$$

4.1.3 Direct Estimates

Suppose $C_B[0,\infty)$ denotes the class of continuous and bounded functions in $[0,\infty)$ with norm $\|\varphi\| = \sup_{x\in[0,\infty)} |\varphi(x)|$.
The usual modulus of smoothness for function $\varphi\in C_B[0,\infty)$ is defined as follows:

$$\omega\left(\varphi;\sqrt{\delta}\right) = \sup_{0<h\le\sqrt{\delta}}\ \sup_{x\in[0,\infty)} |\varphi(x+h)-\varphi(x)|,$$

and let

$$\omega_2\left(\varphi;\sqrt{\delta}\right) = \sup_{0<h\le\sqrt{\delta}}\ \sup_{x\in[0,\infty)} |\varphi(x+2h)-2\varphi(x+h)+\varphi(x)|,$$

be the second order modulus of continuity and corresponding K-functional is defined as follows:

$$K_2(\varphi;\delta) := \inf_{x\in C_B^2[0,\infty)} \left\{\|\varphi-g\| + \delta\|g'\| + \delta^2\|g''\| : g\in C_B^2[0,\infty)\right\}, \quad \delta>0,$$

where $C_B^2[0,\infty)=\{g\in C_B[0,\infty): g', g''\in C_B[0,\infty)\}$.

From DeVore and Lorentz ([49], p.177., Theorem 2.4), the following inequality holds for $M>0$:

$$K_2(\varphi;\delta)\leqslant M\omega_2\left(\varphi;\sqrt{\delta}\right). \tag{4.6}$$

Theorem 4.1.5 *For $\varphi\in C_B[0,\infty)$, we have*

$$\left|\hat{L}_{n,\theta}^{(\beta)}(\varphi;x)-\varphi(x)\right|\le\left\{1+\left(\left(1+s+\frac{c}{\rho}\right)\theta x(1+x)\right)^{\frac{1}{2}}\right\}\omega\left(f;\frac{1}{\sqrt{n}}\right).$$

Proof: *By linearity property of $\hat{L}_{n,\theta}^{(\beta)}(.;x)$, we have*

$$\begin{aligned}\left|\hat{L}_{n,\theta}^{(\beta)}(\varphi;x)-\varphi(x)\right|&\le\int_0^\infty\mathcal{K}_n^\theta(x,t;c)\,|\varphi(t)-\varphi(x)|\,dt\\&\le\int_0^\infty\mathcal{K}_n^\theta(x,t;c)\left(1+\frac{|t-x|}{\delta}\right)dt.\end{aligned}$$

Using Holder's inequality on the right-hand side of the above expression, we have

$$\begin{aligned}\left|\hat{L}_{n,\theta}^{(\beta)}(\varphi;x)-\varphi(x)\right|&\le\left\{1+\frac{1}{\delta}\left(\hat{L}_{n,\theta}^{(\beta)}((t-x)^2;x)\right)^{\frac{1}{2}}\right\}\omega(\varphi;\delta)\\&\le\left\{1+\frac{1}{\delta}\left(\theta\hat{L}_{n}^{(\beta)}((t-x)^2;x)\right)^{\frac{1}{2}}\right\}\omega(\varphi;\delta).\end{aligned}$$

By using Lemma 4.1.3, Lemma 4.1.4 and choosing $\delta=\frac{1}{\sqrt{n}}$, we obtain the result.

Theorem 4.1.6 *For $\varphi\in C_B[0,\infty)$, we have*

$$\left|\hat{L}_{n,\theta}^{(\beta)}(\varphi(t);x)-\varphi(x)\right|\le M\omega_2\left(\varphi;\sqrt{\theta^{\frac{1}{2}}\delta_{n,\rho}^{c,s}(x)}\right),$$

where $\delta_{n,\rho}^{c,s}(x)=\sqrt{\hat{L}_n^{(\beta)}(t-x)^2;x)}$.

Proof: *Using Taylor's expansion for $g\in C_B^2$ and $x,t\in[0,\infty)$, we get*

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-u)g''(u)\,du.$$

Applying $\hat{L}_{n,\theta}^{(\beta)}$ on both sides of the above expansion, we get

$$\hat{L}_{n,\theta}^{(\beta)}(g(t);x)-g(x)=g'(x)\hat{L}_{n,\theta}^{(\beta)}((t-x);x)+\hat{L}_{n,\theta}^{(\beta)}\left(\int_x^t(t-u)g''(u)\,du;x\right).$$

Using Cauchy-Schwarz inequality, Lemma 4.1.3 and Lemma 4.1.4, we get

$$\begin{aligned}\left|\hat{L}^{(\beta)}_{n,\theta}(g(t);x)-g(x)\right| &\leq \left|g'(x)\hat{L}^{(\beta)}_{n,\theta}((t-x);x)\right| + \left|\hat{L}^{(\beta)}_{n,\theta}\left(\int_x^t (t-u)\,g''(u)\,du;x\right)\right|\\ &\leq \|g'\|\,\hat{L}^{(\beta)}_{n,\theta}(|t-x|\,;x) + \frac{\|g''\|}{2}\hat{L}^{(\beta)}_{n,\theta}\left((t-x)^2;x\right)\\ &\leq \|g'\|\left(\hat{L}^{(\beta)}_{n,\theta}\left((t-x)^2;x\right)\right)^{\frac{1}{2}} + \frac{\|g''\|}{2}\hat{L}^{(\beta)}_{n,\theta}\left((t-x)^2;x\right)\\ &\leq \sqrt{\theta}\,\|g'\|\,\delta^{c,s}_{n,\rho}(x) + \frac{\theta\,\|g''\|}{2}\left(\delta^{c,s}_{n,\rho}(x)\right)^2. \end{aligned} \tag{4.7}$$

Using Lemma 4.1.4 and (4.7)*, we obtain*

$$\begin{aligned}\left|\hat{L}^{(\beta)}_{n,\theta}(\varphi(t);x)-\varphi(x)\right| &\leq \left|\hat{L}^{(\beta)}_{n,\theta}(\varphi(t)-g(t);x)\right| + \left|\hat{L}^{(\beta)}_{n,\theta}(g(t);x)-g(x)\right| + |\varphi(x)-g(x)|\\ &\leq (\theta+1)\,\|\varphi-g\| + \sqrt{\theta}\,\|g'\|\,\delta^{c,s}_{n,\rho}(x) + \frac{\theta\,\|g''\|}{2}\left(\delta^{c,s}_{n,\rho}(x)\right)^2.\end{aligned}$$

Taking the infimum on the right-hand side of the above inequality over $g \in C^2_B$, we get

$$\left|\hat{L}^{(\beta)}_{n,\theta}(\varphi(t);x)-\varphi(x)\right| \leq (\theta+1)K_2\left(\varphi;\theta^{\frac{1}{2}}\delta^{c,s}_{n,\rho}(x)\right).$$

Using relation (4.31)*, we obtain the required result.*

4.1.4 Rate of Convergence

In this section, we discuss the rate of convergence by means of the decomposition technique of functions with a derivative of bounded variation (DBV).

Let $\varphi \in DBV_\gamma \in [0,\infty)$, $\gamma \geqslant 0$, be the class of all functions defined on $[0,\infty)$, having a derivative of bounded variation on every finite subinterval of $[0,\infty)$ and $|\varphi(t)| \leqslant Mt^\gamma$, $M > 0$.

The function $\varphi \in DBV_\gamma[0,\infty)$, we may write

$$\varphi(x) = \int_0^x g(t)\,dt + \varphi(0),$$

where $g(t)$ is a function of bounded variation on each finite subinterval of $[0,\infty)$.

We may rewrite the operators (4.2) as:

$$\hat{L}^{(\beta)}_{n,\theta}(\varphi;x) = \int_0^\infty \mathcal{K}^\theta_n(x,t;c)\,\varphi(t)\,dt, \tag{4.8}$$

where $\mathcal{K}^\theta_n(x,t;c)$ is the kernel function given by

$$\mathcal{K}^\theta_n(x,t;c) = \sum_{r=1}^{\infty} w^{\beta,\theta}_{n,r}(x)\chi^\rho_{n,r}(t)\,dt + w^{\beta,\theta}_{n,0}(x)\delta(t),$$

where $\delta(t)$ is the Dirac delta function. We denote the auxiliary function f'_x by

$$\varphi'_x(t) = \begin{cases} \varphi'(t) - \varphi'(x-), & 0 \leqslant t < x, \\ 0, & t = x, \\ \varphi'(t) - \varphi'(x+), & 0 < t < \infty. \end{cases} \tag{4.9}$$

Lemma 4.1.7 *Let $\beta = \beta(n) \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} n\beta(n) = s \in \mathbb{R}$. For all $x \in (0, \infty)$ and n is sufficiently large, we have*

(i) $\varsigma^{\beta,c}_{n,\theta}(x, y; c) = \int_0^y \mathcal{K}^\theta_n(x, t; c)\, dt \leqslant \frac{((1+s)\rho+c)}{n\rho}\frac{\theta x(1+x)}{(x-y)^2}, \quad 0 \leqslant y < x;$

(ii) $1 - \varsigma^{\beta,c}_{n,\theta}(x, z; c) = \int_z^\infty \mathcal{K}^\theta_n(x, t; c)\, dt \leqslant \frac{((1+s)\rho+c)}{n\rho}\frac{\theta x(1+x)}{(z-x)^2}, \quad x < z < \infty.$

Proof: *In view of Lemma 4.1.3 and Lemma 4.1.4, first we prove* (i)

$$\begin{aligned}\int_0^y \mathcal{K}^\theta_n(x, t; c)\, dt &\leqslant \int_0^y \frac{(x-t)^2}{(x-y)^2}\mathcal{K}^\theta_n(x, t; c)\, dt \\ &\leqslant \theta(x-y)^{-2}\, \hat{L}^{(\beta)}_n((e_1 - x)^2; x) \leqslant \frac{((1+s)\rho + c)}{n\rho}\frac{\theta x(1+x)}{(x-y)^2}.\end{aligned}$$

The proof of (ii) *is similar to* (i)*, so we omit the details.*

Theorem 4.1.8 *Let $\varphi \in DBV[0, \infty)$, $\beta = \beta(n) \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} n\beta(n) = s \in \mathbb{R}$. Then for every $x \in (0, \infty)$ and sufficient large n, we have*

$$\begin{aligned}\left|\hat{L}^{(\beta)}_{n,\theta}(\varphi; x) - \varphi(x)\right| \leq &\frac{\theta(\rho s + c)}{(\theta+1)n\rho}\left|\varphi'(x+) + \theta\varphi'(x-)\right| x \\ &+ \frac{\theta}{\theta+1}\left|\varphi'(x+) - \varphi'(x-)\right| \sqrt{\frac{\theta((1+s)\rho + c)\, x(1+x)}{n\rho}} \\ &+ \frac{\theta((1+s)\rho + c)\, x(1+x)}{n\rho x}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x} \varphi_{x'} + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} \varphi_{x'} \\ &+ \frac{\theta((1+s)\rho + c)\, x(1+x)}{n\rho x}\left|\varphi(2x) - \varphi(x) - x\varphi(x+)\right| \\ &+ \frac{\theta((1+s)\rho + c)\, x(1+x)}{n\rho}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{x}{k}} \varphi'_x + \frac{x}{\sqrt{n}} \bigvee_{x}^{x+\frac{x}{\sqrt{n}}} (\varphi'_x) + M(\gamma, c, r, x) \\ &+ \frac{\theta((1+s)\rho + c)\, x(1+x)}{n\rho x}\left|\varphi(x)\right| + \sqrt{\frac{\theta N(\rho, s, c) x(1+x)}{n}}\left|\varphi(x+)\right|.\end{aligned}$$

Proof: *Using the (4.2) of the operators $\hat{L}^{(\beta)}_{n,\theta}$, we have*

$$\begin{aligned}\hat{L}^{(\beta)}_{n,\theta}(\varphi; x) - \varphi(x) &= \int_0^\infty \mathcal{K}^\theta_n(x, t; c)\,(\varphi(t) - \varphi(x))\, dt \\ &= \int_0^\infty \left(\int_x^t \mathcal{K}^\theta_n(x, t; c)\,\varphi'(u)\, du\right) dt. \end{aligned} \tag{4.10}$$

From the relation (5.7) we can write

$$\begin{aligned}\varphi'(u) =& \varphi'_x(u) + \frac{1}{\theta+1}(\varphi'(x+) + \theta\varphi'(x-)) \\ &+ \frac{1}{2}\left(\varphi'(x+) + \theta\varphi'(x-)\right)\left(sgn(u-x) + \frac{\theta-1}{\theta+1}\right) \\ &\times \delta_x(u)\left[\varphi'(u) - (\varphi'(x+) + \varphi'(x-))\right], \qquad (4.11)\end{aligned}$$

where

$$\delta_x(u) = \begin{cases} 1, & x = u \\ 0, & x \neq u. \end{cases}$$

It is easy to say that

$$\int_0^\infty \mathcal{K}_n^\theta(x,t;c)\int_x^t [\varphi'(u) - \frac{1}{2}(\varphi'(x+) + \varphi'(x-)]\delta_x(u)dudt = 0. \qquad (4.12)$$

Now

$$\begin{aligned}P_{n,\theta}^{\rho,\beta}(x) &= \int_0^\infty \mathcal{K}_n^\theta(x,t;c)\int_x^t \frac{1}{\theta+1}(\varphi'(x+) + \theta\varphi'(x-))dudt. \\ &= \frac{1}{\theta+1}(\varphi'(x+) + \theta\varphi'(x-))\int_0^\infty \mathcal{K}_n^\theta(x,t;c)\,(t-x)dt \\ &= \frac{1}{\theta+1}(\varphi'(x+) + \theta\varphi'(x-))\hat{L}_{n,\theta}^{(\beta)}(t-x;x), \qquad (4.13)\end{aligned}$$

and

$$\begin{aligned}Q_{n,\theta}^{\rho,\beta}(x) &= \int_0^\infty \mathcal{K}_n^\theta(x,t;c)\int_x^t \frac{1}{2}(\varphi'(x+) + \theta\varphi'(x-))\left(\text{sgn}(u-x) + \frac{\theta-1}{\theta+1}\right)dudt \\ &= \frac{1}{2}(\varphi'(x+) + \theta\varphi'(x-))\left(-\int_0^x \mathcal{K}_n^\theta(x,t;c)\int_x^t\left(\text{sgn}(u-x) + \frac{\theta-1}{\theta+1}\right)dudt\right. \\ &\quad \left. + \int_x^\infty \mathcal{K}_n^\theta(x,t;c)\int_x^t\left(\text{sgn}(u-x) + \frac{\theta-1}{\theta+1}\right)\right) \\ &\leq \frac{\theta}{\theta+1}(\varphi'(x+) + \theta\varphi'(x-))\int_0^\infty \mathcal{K}_n^\theta(x,t;c)\,|t-x|\,dt \\ &\leq \frac{\theta}{\theta+1}(\varphi'(x+) + \theta\varphi'(x-))\left(\hat{L}_{n,\theta}^{(\beta)}\left((e_1-x)^2;x\right)\right)^{\frac{1}{2}}. \qquad (4.14)\end{aligned}$$

By using Lemma 4.1.3 and Lemma 4.1.7, from (4.10) − (4.14)*, we have*

$$\hat{L}_{n,\theta}^{(\beta)}(\varphi;x)-\varphi(x)\leq\left|A_{n,\theta}^{(\beta)}(\varphi';x)+B_{n,\theta}^{(\beta)}(\varphi';x)\right| + \frac{\theta(\rho s+c)}{(\theta+1)n\rho}\left|\varphi'(x+)+\theta\varphi'(x-)\right|x + \frac{\theta}{\theta+1}\left|\varphi'(x+)-\varphi'(x-)\right|\sqrt{\frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho}}, \tag{4.15}$$

where

$$A_{n,\theta}^{(\beta)}(\varphi';x)=\int_0^x\left(\int_x^t\varphi_x'(u)du\right)\mathcal{K}_n^{\theta}(x,t;c)dt,$$

and

$$B_{n,\theta}^{(\beta)}(\varphi';x)=\int_x^\infty\left(\int_x^t\varphi_x'(u)du\right)\mathcal{K}_n^{\theta}(x,t;c)dt.$$

To estimate $A_{n,\theta}^{(\beta)}(\varphi';x)$, using integration by parts and applying Lemma 4.1.7 with $y = x-\frac{x}{\sqrt{n}}$, we obtain

$$\begin{aligned}A_{n,\theta}^{(\beta)}(\varphi';x)&=\left|\int_0^x\left(\int_x^t\varphi_x'(u)du\right)d_t\zeta_{n,\theta}^{\beta,c}(x;t)\right|\\&=\left|\int_0^x\zeta_{n,\theta}^{\beta,c}(x;t)\varphi_x'(t)dt\right|\\&\leq\int_0^y\left|\varphi_x'(t)\right|\left|\zeta_{n,\theta}^{\beta,c}(x;t)\right|dt+\int_0^y\left|\varphi_x'(t)\right|\left|\zeta_{n,\theta}^{\beta,c}(x;t)\right|dt\end{aligned}$$

$$\begin{aligned}&\leq\frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho}\int_0^y\bigvee_t^x\varphi_x'(x-t)^2dt+\int_y^x\bigvee_t^x\varphi_x'dt\\&\leq\frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho}\int_0^{x-\frac{x}{\sqrt{n}}}\bigvee_t^x\varphi_x'(x-t)^2dt+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x\varphi_x'.\end{aligned} \tag{4.16}$$

Substituting $u=\frac{x}{x-t}$, we get

$$\begin{aligned}\frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho}\int_0^{x-\frac{x}{\sqrt{n}}}\bigvee_t^x\varphi_x'(x-t)^2dt &= \frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho x}\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{u}}^x\varphi_x'du\\ &\leq \frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho x}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\bigvee_{x-\frac{x}{k}}^x\varphi_x'du\\ &\leq \frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^x\varphi_x'.\end{aligned} \tag{4.17}$$

From (4.16) *and* (4.17), *we get*

$$A_{n,\theta}^{(\beta)}(\varphi';x)=\frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^x\varphi_x'+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x\varphi_x'. \tag{4.18}$$

We can write

$$\begin{aligned}B_{n,\theta}^{(\beta)}(\varphi';x)\leq&\left|\int_x^{2x}\left(\int_x^t\varphi_x'(u)du\right)d_t(1-\zeta_{n,\theta}^{\beta,c}(x;t))\right|\\&+\left|\int_{2x}^{\infty}\left(\int_x^t\varphi_x'(u)du\right)d_t\mathcal{K}_n^{\theta}(x,t;c)\right|.\end{aligned}$$

From the second part of the Lemma 4.1.7, we get

$$\mathcal{K}_n^{\theta}(x,t;c)=d_t((1-\zeta_{n,\theta}^{\beta,c}(x;t))\quad for\ \ t>x.$$

Hence

$$B_{n,\theta}^{(\beta)}(\varphi';x)=B_{n,\theta,1}^{(\beta)}(\varphi';x)+B_{n,\theta,2}^{(\beta)}(\varphi';x),$$

where

$$B_{n,\theta,1}^{(\beta)}(\varphi';x)=\left|\int_x^{2x}\left(\int_x^t\varphi_x'(u)du\right)d_t(1-\zeta_{n,\theta}^{\beta,c}(x;t))\right|,$$

and

$$B_{n,\theta,2}^{\beta,c}(f';x)=\left|\int_{2x}^{\infty}\left(\int_x^t\varphi_x'(u)du\right)d_t\mathcal{K}_n^{\theta}(x,t;c)\right|.$$

Using integration by parts, applying Lemma 4.1.7, $1-\zeta_{n,\theta}^{\beta,c}(x;t)\le 1$ *and taking* $t=x+\frac{x}{u}$ *successively,*

$$\begin{aligned}
B_{n,\theta,1}^{(\beta)}(\varphi';x) =& \left|\int_x^{2x}\varphi_x'(u)du(1-\zeta_{n,\theta}^{\beta,c}(x;2x)) - \int_x^{2x}\varphi_x'(t)(1-\zeta_{n,\theta}^{\beta,c}(x;t))dt\right| \\
\le& \left|\int_x^{2x}(\varphi'(u)-\varphi'(x+))du\right|\left|1-\zeta_{n,\theta}^{\beta,c}(x;2x)\right| + \left|\int_x^{2x}\varphi_x'(t)(1-\zeta_{n,\theta}^{\beta,c}(x;t))dt\right| \\
\le& \frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho x}\left|\varphi(2x)-\varphi(x)-x\varphi(x+)\right| \\
&+ \frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{\bigvee_x^t\varphi_x'}{(t-x)^2}dt + \int_x^{x+\frac{x}{\sqrt{n}}}\bigvee_x^t\varphi_x'dt \\
\le& \frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho x}\left|\varphi(2x)-\varphi(x)-x\varphi(x+)\right| \\
&+ \frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{x}{k}}\varphi_x' + \frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}}(\varphi_x'). \qquad (4.19)
\end{aligned}$$

Using Lemma 4.1.7, we have

$$\begin{aligned}
B_{n,\theta,2}^{(\beta)}(\varphi';x) =& \left|\int_{2x}^{\infty}\left(\int_x^t(\varphi'(u)-\varphi'(x+))du\right)\mathcal{K}_n^{\theta}(x,t,c)dt\right| \\
\le& \int_{2x}^{\infty}|\varphi(t)-\varphi(x)|\mathcal{K}_n^{\theta}(x,t,c)dt + \int_{2x}^{\infty}|t-x|\,|\varphi(x+)|\,\mathcal{K}_n^{\theta}(x,t,c)dt \\
\le& \left|\int_{2x}^{\infty}\varphi(t)\mathcal{K}_n^{\theta}(x,t,c)dt\right| + |\varphi(x)|\left|\int_{2x}^{\infty}\mathcal{K}_n^{\theta}(x,t,c)dt\right| \\
&+ |\varphi(x+)|\left(\int_{2x}^{\infty}(e_1-x)^2\mathcal{K}_n^{\theta}(x,t,c)dt\right)^{\frac{1}{2}} \\
\le& M\int_{2x}^{\infty}t^{\gamma}\mathcal{K}_n^{\theta}(x,t,c)dt + |\varphi(x)|\left|\int_{2x}^{\infty}\mathcal{K}_n^{\theta}(x,t,c)dt\right| \\
&+ \sqrt{\frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho}}\left|\varphi(x+)\right|. \qquad (4.20)
\end{aligned}$$

For $t \geq 2x$, we get $t \leq 2(t-x)$ and $x \leq t-x$, applying Hölder's inequality, we have

$$\begin{aligned} B^{(\beta)}_{n,\theta,2}(\varphi';x) &\leq M2^{\gamma}\left(\int_{2x}^{\infty}(e_1-x)^{2r}\mathcal{K}^{\theta}_{n}(x,t,c)dt\right)^{\frac{\gamma}{2k}} \\ &\quad + \frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho x}\left|\varphi(x)\right| + \sqrt{\frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho}}\left|\varphi(x+)\right| \\ &= M(\gamma,c,k,x) + \frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho x}\left|\varphi(x)\right| \\ &\quad + \sqrt{\frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho}}\left|\varphi(x+)\right|. \end{aligned} \tag{4.21}$$

From (4.19) and (4.21), we get

$$\begin{aligned} B^{(\beta)}_{n,\theta}(\varphi';x) =& \frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho x}\left|\varphi(2x)-\varphi(x)-x\varphi(x+)\right| \\ &+ \frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho}\sum_{r=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{r}}\varphi'_x + \frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(\varphi'_x) \\ &+ M(\gamma,c,r,x) + \frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho x}\left|\varphi(x)\right| \\ &+ \sqrt{\frac{\theta\left((1+s)\rho+c\right)x(1+x)}{n\rho}}\left|\varphi(x+)\right|. \end{aligned} \tag{4.22}$$

From (4.15)*,* (4.18) *and* (4.22)*, we get our desired result.*

4.2 Generalization of Parametric Baskakov Operators based on the I-P-E Distribution

4.2.1 Introduction

In literature, the inverse Pólya-Eggenberger (I-P-E) distribution is defined as:

$$P(X=\ddot{\varsigma})=\binom{\ddot{m}+\ddot{\varsigma}-1}{\ddot{\varsigma}}\frac{\prod\limits_{\ddot{\varsigma}}^{\ddot{m}-1}(A+\ddot{\varsigma}S)\prod\limits_{\mathscr{J}=0}^{\ddot{m}-1}(B+\ddot{\varsigma}S)}{\prod\limits_{\ddot{\varsigma}=0}^{\ddot{m}+\ddot{\varsigma}-1}(A+B+\ddot{\varsigma}S)},\quad \ddot{\varsigma}=0,1,...,\ddot{m}, \tag{4.23}$$

shows the probability that $\ddot{\varsigma}$ white balls are selected preceding the $\ddot{m}$-th black ball. The details have been given about this distributions (4.23) in [45; 52; 90].

In 1957, Baskakov [30] presented a sequence of positive linear operators, known as Baskakov operators on the unbounded interval $[0,\infty)$ for appropriate functions specified on the interval $[0,\infty)$. Afterwards, many mathematicians who studied the Baskakov operators came up with various modifications, including [70; 117; 127].

Recall that for every $\mathrm{h}\in C_B[0,\infty)$, classical Baskakov operators are defined as:

$$\mathrm{I}_{\mathscr{z}}(\mathrm{h},x)=\sum_{\mathscr{J}=0}^{\infty}\upsilon_{\mathscr{z},\mathscr{J}}(x)\mathrm{h}\left(\frac{\mathscr{J}}{\mathscr{z}}\right), \tag{4.24}$$

where $\mathscr{z}\geq 1$, $x\in[0,\infty)$ and

$$\upsilon_{\mathscr{z},\mathscr{J}}(x)=\binom{\mathscr{z}+\mathscr{J}-1}{\mathscr{J}}\frac{x^{\mathscr{J}}}{(1+x)^{\mathscr{z}+\mathscr{J}}}.$$

Furthermore, using the inverse Pólya-Eggenberger distribution (4.23), Stancu [142] presented a special class of positive linear operators contextualising the Baskakov operators tied to a real-valued function bounded on $[0,\infty)$ as follows:

$$\begin{aligned}\mathrm{I}_{\mathscr{z}}^{(\hat{\varrho})}(\mathrm{h};x)&=\sum_{\mathscr{J}=0}^{\infty}\upsilon_{\mathscr{z},\mathscr{J}}^{(\hat{\varrho})}(x)\,\mathrm{h}\left(\frac{\mathscr{J}}{\mathscr{z}}\right)\\&=\sum_{\mathscr{J}=0}^{\infty}\binom{\mathscr{z}+\mathscr{J}-1}{\mathscr{J}}\frac{1^{[\mathscr{z},-\hat{\varrho}]}x^{[\mathscr{J},-\hat{\varrho}]}}{(1+x)^{[\mathscr{z}+\mathscr{J},-\hat{\varrho}]}}\mathrm{h}\left(\frac{\mathscr{J}}{\mathscr{z}}\right).\end{aligned} \tag{4.25}$$

For the case where $\hat{\varrho}=0$, the operators (4.25) reduce to the basic Baskakov operators [30].

We came across a manuscript where in 2019, Ali and Hasan [27] introduced α-Baskakov operators, which are non-negative real parametric generalisations of Baskakov operators. The operators are reduced to classical Baskakov operators for $\alpha = 1$. Higher order derivatives are represented as α-Baskakov operators in this paper to obtain their new representation as powers of independent variable x.

Now, for every h $\in C_B[0,\infty)$, the parametric generalization of the Baskakov operators is defined as:

$$\mathcal{L}_{n}^{\alpha}(\mathrm{h};x) = \sum_{k=0}^{\infty} \rho_{n,k}^{\alpha}(x)\mathrm{h}\left(\frac{k}{n}\right), \tag{4.26}$$

where $n \geq 1$, $x \in [0,\infty)$ and

$$\rho_{n,k}^{\alpha}(\mathrm{h};x) = \sum_{k=0}^{\infty} \frac{x^{k-1}}{(1+x)^{n+k-1}} \left\{ \frac{\alpha x}{1+x} \binom{n+k-1}{k} - (1-\alpha)(1+x) \binom{n+k-1}{k-2} \right.$$
$$\left. + (1-\alpha)(1+x) \binom{n+k-1}{k-2} + (1-\alpha)x \binom{n+k-1}{k} \right\} \mathrm{h}\left(\frac{k}{n}\right),$$

we call these operators α-Baskakov operators.

The α-Baskakov operators for h(x) can also expressed as:

$$\mathcal{L}_{n}^{\alpha}(\mathrm{h};x) = (1-\alpha) \sum_{k=0}^{\infty} \binom{n+k-2}{k} \frac{x^{k}}{(1+x)^{n+k-1,}} g_{k}$$
$$+ \alpha \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^{k}}{(1+x)^{n+k,}} \mathrm{h}\left(\frac{k}{n}\right),$$

where

$$g_{k} = \mathrm{h}\left(\frac{k}{n}\right)\left(1 + \frac{k}{n-1}\right) - \mathrm{h}\left(\frac{k+1}{n}\right)\frac{k}{n-1}.$$

We consider generalised α-Baskakov operators (4.26) based on the I-P-E distribution (4.23). This work was prompted by D.D. Stancu [142], who introduced two classes of linear positive operators depending on a non-negative parameter α (this parameter may depend only on natural numbers), and proceeded to establish some of their approximation properties to real-valued functions. In 2019, Deo and Dhamija [46] considered a new modification of the Baskakov operators based on I-P-E distribution, and they also examined various modified forms of the Baskakov operators in the context of the Lupaş operators on the basis of I-P-E distribution. In 2016, Dhamija and Deo [52] provided Jain-Durrmeyer operators connected to the I-P-E distribution and examined the approximation properties of the Jain-Durrmeyer operators based on the I-P-E distribution, including the degree of approximation and uniform convergence of the operators. Baskakov-Szász type operators on the inverse Pólya-Eggenberger distribution were presented by Kajla et al. [94]. The rate of convergence for functions with bounded variation derivatives is established by using a Durrmeyer-type operator with basic functions in summation and integration based on Stancu [142] and Pltánea (2008). Numerous authors contributed to the I-P-E distribution and the Baskakov operators [16; 45; 59].

Now we propose α-Pólya-Baskakov operator based on inverse Pólya-Eggenberger distribution (4.23) as follows:

$$\overset{\rightharpoonup}{Q}_{n}^{(\alpha,\hat{\varrho})}(\mathrm{h};x)=\sum_{j=0}^{\infty} q_{n,j}^{(\alpha,\hat{\varrho})}(x)\mathrm{h}\left(\frac{j}{n}\right), \tag{4.27}$$

where α is a non-negative parameter, which may depend only on the natural number n, with $\alpha \to 0$ when $n \to \infty$, $n \geq 1$, $x \in [0,\infty)$, we have

$$\begin{aligned} q_{n,j}^{(\alpha,\hat{\varrho})}(x) &= \alpha \begin{pmatrix} n+j-1 \\ j \end{pmatrix} \frac{1^{[n,-\hat{\varrho}]}x^{[j,-\hat{\varrho}]}}{(1+x)^{[n+j,-\hat{\varrho}]}} \\ &\quad -(1-\alpha)\begin{pmatrix} n+j-3 \\ j-2 \end{pmatrix} \frac{1^{[n-1,-\hat{\varrho}]}x^{[j-1,-\hat{\varrho}]}}{(1+x)^{[n+j-2,-\hat{\varrho}]}} \\ &\quad +(1-\alpha)\begin{pmatrix} n+j-1 \\ j \end{pmatrix} \frac{1^{[n-1,-\hat{\varrho}]}x^{[j,-\hat{\varrho}]}}{(1+x)^{[n+j-1,-\hat{\varrho}]}}. \end{aligned}$$

Another way to express the above operators is as follows:

$$\begin{aligned} \overset{\rightharpoonup}{Q}_{n}^{(\alpha,\hat{\varrho})}(\mathrm{h};x) &= (1-\alpha)\sum_{j=0}^{\infty}\begin{pmatrix} n+j-2 \\ j \end{pmatrix} \frac{1^{[n-1,-\hat{\varrho}]}x^{[j,-\hat{\varrho}]}}{(1+x)^{[n+j-1,-\hat{\varrho}]}} g_{j} \\ &\quad +\alpha\sum_{j=0}^{\infty}\begin{pmatrix} n+j-1 \\ j \end{pmatrix} \frac{1^{[n,-\hat{\varrho}]}x^{[j,-\hat{\varrho}]}}{(1+x)^{[n+j,-\hat{\varrho}]}}\mathrm{h}\left(\frac{j}{n}\right), \end{aligned}$$

where

$$g_{j}=\mathrm{h}\left(\frac{j}{n}\right)\left(1+\frac{j}{n-1}\right)-\mathrm{h}\left(\frac{j+1}{n}\right)\frac{j}{n-1}.$$

The purpose of this note is to investigate the approximation behaviour of the proposed operators.

4.2.2 Premiinary

Lemma 4.2.1 *Deo and Dhamija [44] use the Vandermonde convolution formula and the substitutions* $\lambda = 0,\ p = 0$ *to derive the moments of I-P-E distribution based operators* (4.25). *Stancu [142] however, already calculated these moments using hypergeometric series.*

$$
\begin{aligned}
\mathrm{I}_n^{\hat{\varrho}}(1;x) =&1;\\
\mathrm{I}_n^{\hat{\varrho}}(\mathrm{v};x) =&\frac{x}{1-\hat{\varrho}};\\
\mathrm{I}_n^{\hat{\varrho}}\left(\mathrm{v}^2;x\right) =&\frac{x(\hat{\varrho}(n-1)+nx+x+1)}{(1-\hat{\varrho})(1-2\hat{\varrho})n};\\
\mathrm{I}_n^{\hat{\varrho}}\left(\mathrm{v}^3;x\right) =&\frac{(n+1)x((n+2)x+3)-x\left(\hat{\varrho}^2((3-2n)n-1)-\hat{\varrho}(-3n(nx+1)+3x+2)-1\right)}{(1-\hat{\varrho})(1-2\hat{\varrho})(1-3\hat{\varrho})n^2};\\
\mathrm{I}_n^{\hat{\varrho}}\left(\mathrm{v}^4;x\right) =&\frac{1}{(1-\hat{\varrho})(1-2\hat{\varrho})(1-3\hat{\varrho})(1-4\hat{\varrho})n^3}\left[(1-2\hat{\varrho})(1-3\hat{\varrho})(1-4\hat{\varrho})\right.\\
&-(n+1)(\hat{\varrho}+x)\left\{6\hat{\varrho}^2((n-3)n+4)+\hat{\varrho}((n+2)(5n-9)x+12n-25)\right.\\
&\left.+(n+2)x((n+3)x+6)+7\}\right].
\end{aligned}
$$

Lemma 4.2.2 *The moments of* $\overset{\rightharpoonup}{Q}_n^{(\alpha,\hat{\varrho})}(f;x)$ *are given as follows:*

$$
\begin{aligned}
\overset{\rightharpoonup}{Q}_n^{(\alpha,\hat{\varrho})}(1;x) =&(1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\\
&+\alpha\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{1^{[n,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k,-\hat{\varrho}]}}\\
=&1.\\
\overset{\rightharpoonup}{Q}_n^{(\alpha,\hat{\varrho})}(\mathrm{v};x) =&(1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k(n-2)}{n(n-1)}\right]\\
&+\alpha\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{1^{[n,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k,-\hat{\varrho}]}}\left[\frac{k}{n}\right]\\
=&(1-\alpha)\frac{(n-2)}{n}\mathrm{I}_{n-1}^{\hat{\varrho}}(t;x)+\alpha\mathrm{I}_n^{\hat{\varrho}}(t;x)\\
=&(1-\alpha)\frac{(n-2)}{n}\frac{x}{1-\hat{\varrho}}+\frac{\alpha x}{1-\hat{\varrho}}\\
=&\frac{2\alpha+n-2}{(1-\hat{\varrho})n}x.
\end{aligned}
$$

$$
\begin{aligned}
\overset{\rightharpoonup}{Q}_{n}^{(\alpha,\hat{\varrho})}\left(\mathrm{v}^{2};x\right) &= (1-\alpha)\sum_{k=0}^{\infty}\begin{pmatrix} n+k-2 \\ k \end{pmatrix}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k^{2}(n-3)}{n^{2}(n-1)}\right] \\
&\quad -(1-\alpha)\sum_{k=0}^{\infty}\begin{pmatrix} n+k-2 \\ k \end{pmatrix}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k}{n^{2}(n-1)}\right] \\
&\quad +\alpha\sum_{k=0}^{\infty}\begin{pmatrix} n+k-1 \\ k \end{pmatrix}\frac{1^{[n,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k,-\hat{\varrho}]}}\left[\frac{k^{2}}{n^{2}}\right] \\
&= \frac{(1-\alpha)(n-3)(n-1)}{n^{2}}\mathrm{I}_{n-1}^{\hat{\varrho}}\left(t^{2};x\right)-\frac{(1-\alpha)}{n^{2}}\mathrm{I}_{n-1}^{\hat{\varrho}}(t;x)+\alpha\mathrm{I}_{n}^{\hat{\varrho}}\left(t^{2};x\right) \\
&= \frac{(4\alpha n+(n-3)n)}{(1-\hat{\varrho})(1-2\hat{\varrho})n^{2}}x^{2}+\frac{4\alpha\hat{\varrho}(n-2)+4\alpha+\hat{\varrho}((n-5)n+8)+n-4}{(1-\hat{\varrho})(1-2\hat{\varrho})n^{2}}x.
\end{aligned}
$$

$$
\begin{aligned}
\overset{\rightharpoonup}{Q}_{n}^{(\alpha,\hat{\varrho})}\left(\mathrm{v}^{3};x\right) &= (1-\alpha)\sum_{k=0}^{\infty}\begin{pmatrix} n+k-2 \\ k \end{pmatrix}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k^{3}(n-4)}{n^{3}(n-1)}\right] \\
&= -3(1-\alpha)\sum_{k=0}^{\infty}\begin{pmatrix} n+k-2 \\ k \end{pmatrix}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k^{2}}{n^{3}(n-1)}\right] \\
&\quad -(1-\alpha)\sum_{k=0}^{\infty}\begin{pmatrix} n+k-2 \\ k \end{pmatrix}\frac{1^{[n-1,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k-1,-\hat{\varrho}]}}\left[\frac{k}{n^{3}(n-1)}\right] \\
&\quad +\alpha\sum_{k=0}^{\infty}\begin{pmatrix} n+k-1 \\ k \end{pmatrix}\frac{1^{[n,-\hat{\varrho}]}x^{[k,-\hat{\varrho}]}}{(1+x)^{[n+k,-\hat{\varrho}]}}\left[\frac{k^{3}}{n^{3}}\right] \\
&= \frac{(1-\alpha)(n-4)(n-1)^{2}}{n^{3}}\mathrm{I}_{n-1}^{\hat{\varrho}}\left(t^{3};x\right)-\frac{(1-\alpha)(n-1)}{n^{3}}\mathrm{I}_{n-1}^{\hat{\varrho}}\left(t^{2};x\right) \\
&\quad -\frac{1}{n^{3}}\mathrm{I}_{n-1}^{\hat{\varrho}}(t;x)+\alpha\mathrm{I}_{n}^{\hat{\varrho}}\left(t^{3};x\right) \\
&= \frac{(n+1)(6\alpha+n-4)}{(1-\hat{\varrho})(1-2\hat{\varrho})(1-3\hat{\varrho})n^{2}}x^{3} \\
&\quad +\frac{3(6\alpha(\hat{\varrho}(n-2)+1)+\hat{\varrho}((n-6)n+11)+n-5)}{(1-\hat{\varrho})(1-2\hat{\varrho})(1-3\hat{\varrho})n^{2}}x^{2} \\
&\quad -\frac{1}{(1-\hat{\varrho})(1-2\hat{\varrho})(1-3\hat{\varrho})n^{3}}\left[\begin{array}{c}(-\hat{\varrho}(n-3)-1)(\hat{\varrho}(n(2n-9)+16)+n-8) \\ -2\alpha(\hat{\varrho}(3\hat{\varrho}(n(2n-7)+8)+9n-20)+4)\end{array}\right]x.
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{Q}_n^{(\alpha,\varrho)}\left(\mathrm{v}^4;x\right) &= (1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{1^{[n-1,-\varrho]}x^{[k,-\varrho]}}{(1+x)^{[n+k-1,-\varrho]}}\left[\frac{k^4\,(n-5)}{n^4\,(n-1)}\right]\\
&-6\,(1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{1^{[n-1,-\varrho]}x^{[k,-\varrho]}}{(1+x)^{[n+k-1,-\varrho]}}\left[\frac{k^3}{n^4\,(n-1)}\right]\\
&-4\,(1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{1^{[n-1,-\varrho]}x^{[k,-\varrho]}}{(1+x)^{[n+k-1,-\varrho]}}\left[\frac{k^2}{n^4\,(n-1)}\right]\\
&-(1-\alpha)\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{1^{[n-1,-\varrho]}x^{[k,-\varrho]}}{(1+x)^{[n+k-1,-\varrho]}}\left[\frac{k}{n^4\,(n-1)}\right]\\
&+\alpha\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{1^{[n,-\varrho]}x^{[k,-\varrho]}}{(1+x)^{[n+k,-\varrho]}}\left[\frac{k^4}{n^4}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(1-\alpha)\,(n-5)\,(n-1)^3}{n^4}\mathrm{I}_{n-1}^{\varrho}\left(t^4;x\right)-\frac{6\,(1-\alpha)\,(n-1)^2}{n^4}\mathrm{I}_{n-1}^{\varrho}\left(t^3;x\right)\\
&-\frac{4\,(1-\alpha)\,(n-1)}{n^4}\mathrm{I}_{n-1}^{\varrho}\left(t^2;x\right)-\frac{1}{n^4}\mathrm{I}_{n-1}^{\varrho}\,(t;x)+\alpha\mathrm{I}_{n}^{\varrho}\left(t^4;x\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(8\alpha n(n+1)(n+2)+(n-5)n(n+1)(n+2))}{(1-\varrho)(1-2\varrho)(1-3\varrho)(1-4\varrho)n^4}x^4\\
&+\frac{(48\alpha n(n+1)(\varrho(n-2)+1)+6n(n+1)(\varrho((n-7)n+14)+n-6))}{(1-\varrho)(1-2\varrho)(1-3\varrho)(1-4\varrho)n^4}x^3\\
&+\frac{1}{(1-\varrho)(1-2\varrho)(1-3\varrho)(1-4\varrho)n^4}\left[\begin{array}{c}8\alpha n(\varrho(\varrho(n(11n-39)+46)\\+18n-38)+8)+n(\varrho(\varrho(n(n\\(11n-94)+301)-362)\\+nn(18n-139)+291)\\+7n-57)\end{array}\right]x^2\\
&+\frac{1}{(1-\varrho)(1-2\varrho)(1-3\varrho)(1-4\varrho)\hat{m}^4}\left[\begin{array}{c}(16\alpha(\varrho(n-2)+1)(\varrho(3\varrho((m-3\\n+4)+3n-7)+1))\\+\varrho(6\varrho^3((n-5)n+8)^2\\+\varrho^2n(n(12n-109)+359)-416\varrho\\+\varrho(n-6)(7n-24))+n-16\end{array}\right]x.
\end{aligned}
$$

Lemma 4.2.3 *We establish the following limits of central moments by using Lemma 4.2.2 and,* $\lim_{n\to\infty} n\varrho = l$*:*

(i) $\lim\limits_{\mathfrak{z}\to\infty} \mathfrak{z}\overrightharpoonup{Q}_{\mathfrak{z}}^{(\alpha,\hat{\varrho})}(\vartheta;x) = (l+1)\,x\,(1+x)\,,$

(ii) $\lim\limits_{\mathfrak{z}\to\infty} \mathfrak{z}^2\overrightharpoonup{Q}_{\mathfrak{z}}^{(\alpha,\hat{\varrho})}(\vartheta;x) = 3x^2(1+l)^2(1+x)^2,$

where $\vartheta(\mathrm{v}) = (\mathrm{v}-x)^i$ *and,* $i = 2, 4$.

4.2.3 Direct Results

The renowned Bohman-Korovkin-Popoviciu theorem is used to obtain the uniform convergence of the α-Pólya-Baskakov operator (4.27).

Theorem 4.2.4 *For a non-negative parameter* α, *which may depend only* $\mathfrak{z}\in\mathbb{N}$, *let* $h\in C[0,\infty)$. *If* $\hat{\varrho}\to 0$ *when* $\mathfrak{z}\to\infty$, *then we have*

$$\lim_{n\to\infty}\overrightharpoonup{Q}_{\mathfrak{z}}^{(\alpha,\hat{\varrho})}(h;x) = h(x),$$

uniformly on each compact subset of $[0,\infty)$, *where* $C[0,\infty)$ *is the space of all real-valued functions continuous on* $[0,\infty)$.

Proof: *Taking Lemma (4.2.2) into consideration, it follows that:*

$$\lim_{\mathfrak{z}\to\infty}\overrightharpoonup{Q}_{\mathfrak{z}}^{(\alpha,\hat{\varrho})}\left(\mathrm{v}^{\mathrm{i}};x\right) = x^i,\quad i = 0,1,2$$

uniformly on each compact subset of $[0,\infty)$. *Thus, we arrive at the desired conclusion by applying the well-known Bohman-Korovkin-Popoviciu theorem.* α*-Pólya-Baskakov operator* (4.27) *asymptotic behaviour is now presented.*

Theorem 4.2.5 *Let* h *be a bounded and integrable function on* $[0,\infty)$. *for the first and second derivatives of* h *at a fixed point* $x\in[0,\infty)$, *then the function could be written as follows:*

$$\lim_{\mathfrak{z}\to\infty}\mathfrak{z}\left(\overrightharpoonup{Q}_{\mathfrak{z}}^{(\alpha,\hat{\varrho})}(\mathrm{h};x) - \mathrm{h}\,(x)\right) = (l+1)\,x\,(1+x)\,\mathrm{h}'\,(x) + 3x^2(1+l)^2(1+x)^2\mathrm{h}''(x).$$

Proof: *Applying Taylor's expansion, we can express the function h by writing,*

$$\mathrm{h}(\mathrm{v}) = \mathrm{h}(x) + (\mathrm{v}-x)\mathrm{h}'(x) + \frac{1}{2!}(\mathrm{v}-x)^2\mathrm{h}''(x) + \varepsilon(\mathrm{v},x)(\mathrm{v}-x)^2,$$

where, $\lim\limits_{\mathrm{v}\to x}\varepsilon(\mathrm{v},x) = 0$ *and* $\varepsilon(\mathrm{v},x)$ *is a bounded function. By linearity of* α*-Pólya-Baskakov operator* (4.27), *it follows*

$$\begin{aligned}\overrightharpoonup{Q}_{\mathfrak{z}}^{(\alpha,\hat{\varrho})}(\mathrm{h};x) - \mathrm{h}(x) =& \overrightharpoonup{Q}_{\mathfrak{z}}^{(\alpha,\hat{\varrho})}(\mathrm{v}-x;x)\,\mathrm{h}'(x) + \frac{1}{2}\overrightharpoonup{Q}_{\mathfrak{z}}^{(\alpha,\hat{\varrho})}\left((\mathrm{v}-x)^2;x\right)\mathrm{h}''(x) + \\ &+\overrightharpoonup{Q}_{\mathfrak{z}}^{(\alpha,\hat{\varrho})}\left(\varepsilon(\mathrm{v},x)\cdot(\mathrm{v}-x)^2;x\right).\end{aligned}$$

Taking Lemma 4.2.3 into account, we get

$$\lim_{\imath\to\infty} \imath\left(\overset{\rightharpoonup(\alpha,\hat{\varrho})}{Q_{\imath}}(\mathrm{h};x)-\mathrm{h}(x)\right)=(-2+l)\,x\mathrm{h}'(x)+(l+1)\,x(1+x)\,\mathrm{h}''(x)$$
$$+\lim_{\imath\to\infty} \imath\left(\overset{\rightharpoonup(\alpha,\hat{\varrho})}{Q_{\imath}}\left(\varepsilon(\mathrm{v},x)\cdot(\mathrm{v}-x)^2;x\right)\right). \tag{4.28}$$

As a result of the Cauchy-Schwarz inequality,

$$\imath\overset{\rightharpoonup(\alpha,\hat{\varrho})}{Q_{\imath}}\left(\varepsilon(\mathrm{v},x)(\mathrm{v}-x)^2;x\right)\leq\sqrt{\overset{\rightharpoonup(\alpha,\hat{\varrho})}{Q_{\imath}}(\varepsilon^2(\mathrm{v},x);x)}\sqrt{\imath^2\overset{\rightharpoonup(\alpha,\hat{\varrho})}{Q_{\imath}}\left((\mathrm{v}-x)^4;x\right)}. \tag{4.29}$$

Since $\varepsilon^2(\cdot,x)\in C[0,\infty)$ *and* $\varepsilon^2(x,x)=0$*, applying uniform convergence and from Theorem (4.2.4), we obtain*

$$\lim_{\imath\to\infty}\overset{\rightharpoonup(\alpha,\hat{\varrho})}{Q_{\imath}}\left(\varepsilon^2(\mathrm{v},x);x\right)=\varepsilon^2(x,x)=0. \tag{4.30}$$

Therefore, from Lemma 4.2.3 and (4.30) *yields*

$$\lim_{\imath\to\infty}\imath\overset{\rightharpoonup(\alpha,\hat{\varrho})}{Q_{\imath}}\left(\varepsilon(\mathrm{v},x)\cdot(\mathrm{v}-x)^2;x\right)=0,$$

and using (4.28)*, we obtain the asymptotic behavior of the α-Pólya-Baskakov operator* (4.27).

Lemma 4.2.6 *The following inequality is true for positive linear operators* (4.27).

$$\left|\overset{\rightharpoonup(\alpha,\hat{\varrho})}{Q_{\imath}}(\mathrm{h};x)\right|\leqslant\|\mathrm{h}\|.$$

Proof: From operators (4.27), we obtain

$$\left|\overset{\rightharpoonup(\alpha,\hat{\varrho})}{Q_{\imath}}(\mathrm{h};x)\right|=\left|\sum_{\jmath=0}^{\infty}q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\mathrm{h}\left(\frac{\jmath}{\imath}\right)\right|\leqslant\sum_{\jmath=0}^{\infty}q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\left|\mathrm{h}\left(\frac{\jmath}{\imath}\right)\right|$$
$$\leqslant\sum_{\jmath=0}^{\infty}q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\sup|\mathrm{h}(x)|=\|\mathrm{h}\|.$$

Let $C_B[0,\infty)$ denotes the space containing the real-valued continuous and bounded functions $\mathrm{h}\in C_B[0,\infty)$, equipped with the norm

$$\|\mathrm{h}\|=\sup_{x\in[0,\infty)}|\mathrm{h}(x)|.$$

The Peetre's K-functional for h has been characterized as follows:

$$K_2(h;\delta):=\inf_{x\in C_B^2[0,\infty)}\{\|\mathrm{h}-f\|+\delta\|f''\|\},\ \delta>0,$$

where $C_B^2[0,\infty) = \{f \in C_B[0,\infty) : f', f'' \in C_B[0,\infty)\}$. Now, according to Lorentz and DeVore ([49], p.177. Theorem 2.4) $\exists\, C > 0$, such that

$$K_2(\mathrm{h};\delta) \leqslant C\omega_2\left(\mathrm{h};\delta^{1/2}\right), \tag{4.31}$$

The modulus of continuity of second order $\omega_2\left(\mathrm{h};\delta^{1/2}\right)$ can be specified by

$$\omega_2\left(\mathrm{h};\delta^{1/2}\right) = \sup_{0<d\leqslant\delta^{1/2}}\sup_{x\in I}\left|\mathrm{h}(x+2d) - 2\mathrm{h}(x+d) + \mathrm{h}(x)\right|.$$

Furthermore, the modulus of smoothness of first order can be provided by

$$\omega\left(\mathrm{h};\delta^{1/2}\right) = \sup_{0<d\leqslant\delta^{1/2}}\sup_{x\in I}\left|\mathrm{h}(x+d) - \mathrm{h}(x)\right|.$$

Theorem 4.2.7 *For* $\mathrm{h} \in C_B[0,\infty)$*, we conclude*

$$\left|\overset{\rightharpoonup}{Q}_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{h};x) - \mathrm{h}(x)\right| \leqslant \omega\left(\mathrm{h}, \frac{(2\alpha+\hat{\varrho}\,\imath-2)}{(1-\hat{\varrho})\,\imath}x\right) + C\omega_2\left(\mathrm{h}, \frac{\sqrt{\psi_{\imath,\lambda}^{(\alpha)}(x)}}{2}\right),$$

where $C > 0$ is a constant and

$$\psi_{\imath,\lambda}^{(\alpha)}(x) = \overset{\rightharpoonup}{Q}_{\imath}^{(\alpha,\hat{\varrho})}\left((\mathrm{v}-x)^2;x\right) + \left\{\frac{(2\alpha+\hat{\varrho}\,\imath-2)}{(1-\hat{\varrho})\,\imath}x\right\}^2.$$

Proof: *We proceed with auxiliary operators*

$$Q_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{h};x) = \overset{\rightharpoonup}{Q}_{\imath}^{(\alpha,\hat{\varrho})}(h;x) + \mathrm{h}(x) - \mathrm{h}\left(\frac{2\alpha+\,\imath-2}{(1-\hat{\varrho})\,\imath}x\right). \tag{4.32}$$

We found that $\overset{\rightharpoonup}{Q}_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{h};x)$ *are linear for all* $x \in [0,\infty)$ *, hence*

$$Q_{\imath}^{(\alpha,\hat{\varrho})}(1;x) = 1 \;\; \text{and} \;\; Q_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{v};x) = x,$$

i.e., $Q_{\imath}^{(\alpha,\hat{\varrho})}$ *preserves linear functions intact as a result*

$$Q_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{v}-x;x) = 0. \tag{4.33}$$

For $\mathrm{v}, x \in [0,\infty)$ *and function* $g \in C_B^2[0,\infty)$*, according to Taylor's theorem we can imply that,*

$$g(v) = g(x) + (\mathrm{v}-x)g'(x) + \int_x^{\mathrm{v}}(\mathrm{v}-\varpi)g''(\varpi)d\varpi.$$

Applying the operator $Q_{\imath}^{(\alpha,\hat{\varrho})}$ *to both sides of above inequality, we can write*

$$\begin{aligned}
Q_{\imath}^{(\alpha,\hat{\varrho})}(g;x) - g(x) &= g'(x)Q_{\imath}^{(\alpha,\hat{\varrho})}((\mathrm{v}-x);x) + Q_{\imath}^{(\alpha,\hat{\varrho})}\left(\int_x^{\mathrm{v}}(\mathrm{v}-\varpi)g''(\varpi)d\varpi;x\right)\\
&= Q_{\imath}^{(\alpha,\hat{\varrho})}\left(\int_x^{\mathrm{v}}(\mathrm{v}-\varpi)g''(\varpi)d\varpi;x\right)\\
&= \overset{\rightharpoonup}{Q}_{\imath}^{(\alpha,\hat{\varrho})}\left(\int_x^{\mathrm{v}}(\mathrm{v}-\varpi)g''(\varpi)d\varpi;x\right)\\
&\quad - \int_x^{\frac{2\alpha+\imath-2}{(1-\hat{\varrho})\imath}x}\left(\frac{2\alpha+\,\imath-2}{(1-\hat{\varrho})\,\imath}x-\varpi\right)g''(\varpi)d\varpi.
\end{aligned}$$

Furthermore, we acquire

$$\left|Q_{\imath}^{(\alpha,\hat{\varrho})}(g;x) - g(x)\right| \leqslant \overset{\rightharpoonup}{Q}_{\imath}^{(\alpha,\hat{\varrho})}\left(\left|\int_x^{\mathrm{v}}(\mathrm{v}-\varpi)g''(\varpi)d\varpi\right|;x\right)$$
$$+\left|\int_x^{\frac{2\alpha+\imath-2}{(1-\hat{\varrho})\imath}x}\left(\frac{2\alpha+\imath-2}{(1-\hat{\varrho})\imath}x-\varpi\right)g''(\varpi)d\varpi\right|. \qquad (4.34)$$

Since $\left|\int_x^t(\mathrm{v}-\varpi)g''(\varpi)d\varpi\right| \leqslant (\mathrm{v}-x)^2\,\|g''\|$ *and*

$$\left|\int_x^{\frac{2\alpha+\imath-2}{(1-\hat{\varrho})\imath}x}\left(\frac{2\alpha+\imath-2}{(1-\hat{\varrho})\imath}x-\varpi\right)g''(\varpi)d\varpi\right| \leqslant \left\{\frac{2\alpha+\imath-2}{(1-\hat{\varrho})\imath}x-x\right\}^2\|g''\|.$$

As a equation, (4.34) *indicates*

$$\left|Q_{\imath}^{(\alpha,\hat{\varrho})}(g;x) - g(x)\right| \leqslant \left[\overset{\rightharpoonup}{Q}_{\imath}^{(\alpha,\hat{\varrho})}\left((\mathrm{v}-x)^2;x\right)+\left\{\frac{2\alpha+\imath-2}{(1-\hat{\varrho})\imath}x-x\right\}^2\right]\|g''\|$$
$$\leqslant \left[\overset{\rightharpoonup}{Q}_{\imath}^{(\alpha,\hat{\varrho})}\left((\mathrm{v}-x)^2;x\right)+\left\{\frac{(2\alpha+\hat{\varrho}\imath-2)}{(1-\hat{\varrho})\imath}x\right\}^2\right]\|g''\|$$
$$=\psi_{\imath,\lambda}^{(\alpha)}(x)\,\|g''\|, \qquad (4.35)$$

In accordance with Lemma 4.2.6 and auxiliary operators, we have

$$\left|\overset{\rightharpoonup}{Q}_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{h};x) - \mathrm{h}(x)\right| \leqslant \left|Q_{\imath}^{(\alpha,\hat{\varrho})}((\mathrm{h}-g);x)\right| + |g(x)-\mathrm{h}(x)| + \left|Q_{\imath}^{(\alpha,\hat{\varrho})}(g;x)-g(x)\right|$$
$$+\left|\mathrm{h}\left(\frac{2\alpha+\imath-2}{(1-\hat{\varrho})\imath}x\right)-\mathrm{h}(x)\right|$$
$$\leqslant 4\,\|\mathrm{h}-g\| + \psi_{\imath,\lambda}^{(\alpha)}(x)\,\|g''\| + \omega\left(\mathrm{h};\frac{(2\alpha+\hat{\varrho}\imath-2)}{(1-\hat{\varrho})\imath}x\right).$$

taking infimum on both side over $g \in C_B^2[0,\infty)$, *we get*

$$\left|\overset{\rightharpoonup}{Q}_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{h};x) - \mathrm{h}(x)\right| \leqslant 4K_2\left(\mathrm{h};\frac{\psi_{\imath,\lambda}^{(\alpha)}(x)}{4}\right)+\omega\left(\mathrm{h};\frac{(2\alpha+\hat{\varrho}\imath-2)}{(1-\hat{\varrho})\imath}x\right).$$

using equation (4.31), *we obtain*

$$\left|\overset{\rightharpoonup}{Q}_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{h};x) - \mathrm{h}(x)\right| \leqslant C\omega_2\left(\mathrm{h};\frac{\sqrt{\psi_{\imath,\lambda}^{(\alpha)}(x)}}{2}\right)+\omega\left(\mathrm{h};\frac{(2\alpha+\hat{\varrho}\imath-2)}{(1-\hat{\varrho})\imath}x\right).$$

For $\beta \in (0,1]$, *the Lipschitz-type space is defined as: [123]*

$$Lip_{\mathcal{D}}^*(\beta) := \left\{\mathrm{h}\in C_B[0,\infty) : |\mathrm{h}(\upsilon)-\mathrm{h}(x)| \leqslant \mathcal{D}\frac{|\upsilon-x|^\beta}{(x+\upsilon)^{\beta/2}}; x,\upsilon\in[0,\infty)\right\}.$$

Here, $\beta \in (0,1]$ *and* $\mathcal{D} > 0$ *is constant.*

Theorem 4.2.8 *Let* h $\in Lip^*_{\mathcal{D}}(\beta)$ *where* $0<\beta\in(0,1]$. *Then* $\forall x\in[0,\infty)$, *we get*

$$\left|\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{h};x)-\mathrm{h}(x)\right|\mathcal{D}\left(\frac{\Psi_{\imath}^{(\alpha,\hat{\varrho})}(x)}{x}\right)^{\beta/2}, \tag{4.36}$$

where $\Psi_{\imath}^{(\alpha,\hat{\varrho})}(x)=\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\hat{\varrho})}\left((\mathrm{v}-x)^2;x\right)$.

Proof: *Let us suppose that* $\beta=1$. *Then, for* h $\in Lip^*_{\mathcal{D}}(1)$ *and* $x\in[0,\infty)$, *we conclude*

$$\begin{aligned}\left|\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{h};x)-\mathrm{h}(x)\right| &\leqslant \left|\sum_{\jmath=0}^{\infty}q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\mathrm{h}\left(\frac{\jmath}{\imath}\right)-\mathrm{h}(x)\right|\\ &\leqslant \sum_{\jmath=0}^{\infty}q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\left|\mathrm{h}\left(\frac{\jmath}{\imath}\right)-f(x)\right|\leqslant\mathcal{D}\sum_{\jmath=0}^{\infty}q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\frac{\left|\frac{\jmath}{\imath}-x\right|}{\left(\frac{\jmath}{\imath}+x\right)^{1/2}}.\end{aligned}$$

Applying the Cauchy-Schwarz inequality for sum and $\frac{1}{\sqrt{\frac{k}{\imath}+x}}\leqslant\frac{1}{\sqrt{x}}$, *we obtain*

$$\begin{aligned}\left|\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{h};x)-\mathrm{h}(x)\right| &\leqslant \frac{\mathcal{D}}{\sqrt{x}}\sum_{\jmath=0}^{\infty}q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\left\{\left(\frac{\jmath}{\imath}-x\right)^2\right\}^{1/2}\\ &\leqslant \frac{\mathcal{D}}{\sqrt{x}}\left\{\sum_{\jmath=0}^{\infty}q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\right\}^{1/2}\left\{\sum_{\jmath=0}^{\infty}q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\left(\frac{\jmath}{\imath}-x\right)^2\right\}^{1/2}\\ &\leqslant \frac{\mathcal{D}}{\sqrt{x}}\left\{\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\hat{\varrho})}(1;x)\right\}^{1/2}\left\{\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\hat{\varrho})}\left((\mathrm{v}-x)^2;x\right)\right\}^{1/2}=\mathcal{D}\left(\frac{\Psi_{\imath}^{(\alpha,\hat{\varrho})}(x)}{x}\right)^{1/2}.\end{aligned}$$

As a result, the outcome for $\beta=1$ *is correct.*

As we continue, we will demonstrate the necessary outcome for $0<\beta<1$ *and consider* h $\in Lip^*_{\mathcal{D}}(\beta)$, *we get*

$$\left|\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{h};x)-\mathrm{h}(x)\right|\leqslant\sum_{\jmath=0}^{\infty}q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\left|\mathrm{h}\left(\frac{\jmath}{\imath}\right)-\mathrm{h}(x)\right|\leqslant\mathcal{D}\sum_{\jmath=0}^{\infty}q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\frac{\left|\frac{\jmath}{\imath}-x\right|^{\beta}}{\left(\frac{\jmath}{\imath}+x\right)^{\beta/2}}.$$

making use of inequality $\frac{1}{\sqrt{\frac{\jmath}{\imath}+x}}\leqslant\frac{1}{\sqrt{x}}$ *and using Holder's inequality in the above summation with* $p=\frac{2}{\beta}, q=\frac{2}{2-\beta}$. *As a result of applying the Holder's inequality to the sum with* $p=2/\beta, q=2/(2-\beta)$ *and inequality* $\frac{1}{\sqrt{\frac{\jmath}{\imath}+x}}\leqslant\frac{1}{\sqrt{x}}$, *we obtain*

$$\begin{aligned}\left|\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\hat{\varrho})}(\mathrm{h};x)-\mathrm{h}(x)\right| &\leqslant \frac{\mathcal{D}}{x^{\beta/2}}\sum_{\jmath=0}^{\infty}q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\left\{\left(\frac{\jmath}{\imath}-x\right)^2\right\}^{\beta/2}\\ &\leqslant \frac{\mathcal{D}}{x^{\beta/2}}\left\{\sum_{\jmath=0}^{\infty}q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\left(\frac{\jmath}{\imath}-x\right)^2\right\}^{\beta/2}\left\{\sum_{\jmath=0}^{\infty}q_{\imath,\jmath}^{(\alpha,\hat{\varrho})}(x)\right\}^{\frac{2-\beta}{2}}\\ &\leqslant \mathcal{D}\left\{\frac{\overrightarrow{\mathcal{Q}}_{\imath}^{(\alpha,\hat{\varrho})}\left((\mathrm{v}-x)^2;x\right)}{x}\right\}^{\beta/2}=\mathcal{D}\left(\frac{\Psi_{\imath}^{(\alpha,\hat{\varrho})}(x)}{x}\right)^{\beta/2}.\end{aligned}$$

4.2.4 Weighted Approximation

Gadjiev [60; 61] investigated the weight spaces $C_\Psi[0,\infty)$ and $B_\Psi[0,\infty)$ of real-valued functions defined on $[0,\infty)$ with $\Psi(x)=1+x^2$, in order to demonstrate Korovkin's theorem, which is not commonly true in all of these spaces.

$$B_\Psi[0,\infty) := \{\mathrm{h} : |\mathrm{h}(x)| \leqslant \mathcal{D}_{\mathrm{h}}\Psi(x)\},$$

with

$$\|\mathrm{h}\|_\Psi = \sup_{x\in[0,\infty)} \frac{|\mathrm{h}(x)|}{\Psi(x)},$$

and

$$C_\Psi[0,\infty) := \{\mathrm{h} : \mathrm{h} \in B_\Psi[0,\infty)\},$$

i.e. $C_\Psi[0,\infty) = C[0,\infty) \cap B_\Psi[0,\infty)$ is the subspace of $B_\Psi[0,\infty)$ containing continuous functions and

$$C^*_\Psi[0,\infty) := \left\{\mathrm{h} \in C_\Psi[0,\infty) : \lim_{x\to\infty} \frac{|\mathrm{h}(x)|}{\Psi(x)} < \infty\right\}.$$

The Korovkin's theorem, however, is valid throughout the range $C^*_\Psi[0,\infty)$.

The following is the definition of the usual modulus of continuity of h on $[0,b]$: The classical modulus of continuity of h on $[0,b]$ is defined as follows:

$$\omega_b(\mathrm{h};\delta) = \sup_{|\mathrm{v}-x|\leqslant\delta}\ \sup_{x,\mathrm{v}\in[0,b]} |\mathrm{h}(\mathrm{v}) - \mathrm{h}(x)|.$$

Theorem 4.2.9 *Let* $\mathrm{h} \in C^*_\Psi[0,\infty)$ *then for operators* $\overset{\rightharpoonup}{\mathcal{Q}}{}_{\imath}^{(\alpha,\hat\varrho)}(\mathrm{h};x)$ *we conculde*

$$\left\|\overset{\rightharpoonup}{\mathcal{Q}}{}_{\imath}^{(\alpha,\hat\varrho)}(\mathrm{h};x) - \mathrm{h}(x)\right\|_{C[a,b]} \leqslant 4\mathcal{D}_{\mathrm{h}}\left(1+b^2\right)\Psi_{\imath}^{(\alpha,\hat\varrho)}(x) + 2\omega\left(\mathrm{h}, \sqrt{\Psi_{\imath}^{(\alpha,\hat\varrho)}(x)}\right), \tag{4.37}$$

where $\Psi_{\imath}^{(\alpha,\hat\varrho)}(x) = \overset{\rightharpoonup}{\mathcal{Q}}{}_{\imath}^{(\alpha,\hat\varrho)}\left((\mathrm{v}-x)^2;x\right)$.

Proof: *Suppose* $\mathrm{v}-x>1$ *for* $\mathrm{v}\in(b+1,\infty)$ *and* $x\in[0,b]$, *then, we conclude*

$$\begin{aligned}|\mathrm{h}(\mathrm{v}) - \mathrm{h}(x)| &\leqslant \mathcal{D}_{\mathrm{h}}\Psi(\mathrm{v}-x) = \mathcal{D}_{\mathrm{h}}\left\{1+(\mathrm{v}-x)^2\right\} = \mathcal{D}_{\mathrm{h}}\left(\mathrm{v}^2 - 2x\mathrm{v} + x^2 + 1\right)\\ &\leqslant \mathcal{D}_{\mathrm{h}}\left(\mathrm{v}^2+x^2+2\right) = \mathcal{D}_{\mathrm{h}}\left\{(\mathrm{v}-x)^2 + 2x(\mathrm{v}-x) + 2 + 2x^2\right\}\\ &\leqslant \mathcal{D}_{\mathrm{h}}(\mathrm{v}-x)^2\left\{2x^2+2x+3\right\} \leqslant 4\mathcal{D}_{\mathrm{h}}(\mathrm{v}-x)^2\left(1+x^2\right)\\ &\leqslant 4\mathcal{D}_{\mathrm{h}}(\mathrm{v}-x)^2\left(1+b^2\right).\end{aligned} \tag{4.38}$$

For $x\in[0,b]$ *and* $\mathrm{v}\in[0,b+1]$, *we conclude*

$$|\mathrm{h}(\mathrm{v}) - \mathrm{h}(x)| \leqslant \omega_{b+1}(|\mathrm{v}-x|) \leqslant \left(1 + \frac{|\mathrm{v}-x|}{\delta}\right)\omega_{b+1}(\mathrm{h},\delta), \delta > 0. \tag{4.39}$$

We find by solving (5.7) *and* (4.39)

$$|\mathrm{h}(\mathrm{v}) - \mathrm{h}(x)| \leqslant 4\mathcal{D}_{\mathrm{h}}\left(1 + b^2\right)(\mathrm{v} - x)^2 + \left(1 + \frac{|\mathrm{v} - x|}{\delta}\right)\omega_{b+1}(\mathrm{h}, \delta),$$

making use of Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\overset{\rightharpoonup}{Q}_{\ell}^{(\alpha,\hat{\varrho})}\mathrm{h}(\mathrm{v};x) - \mathrm{h}(x)\right| &\leqslant 4\mathcal{D}_{\mathrm{h}}\left(1 + b^2\right)\overset{\rightharpoonup}{Q}_{\ell}^{(\alpha,\hat{\varrho})}\left((\mathrm{v} - x)^2; x\right) \\ &+ \left(1 + \frac{1}{\delta}\overset{\rightharpoonup}{Q}_{\ell}^{(\alpha,\hat{\varrho})}(|\mathrm{v} - x|\,; x)\right)\omega_{b+1}(\mathrm{h}, \delta) \\ &\leqslant 4\mathcal{D}_{\mathrm{h}}\left(1 + b^2\right)\overset{\rightharpoonup}{Q}_{\ell}^{(\alpha,\hat{\varrho})}\left((\mathrm{v} - x)^2; x\right) \\ &+ \left[1 + \frac{1}{\delta}\left\{\overset{\rightharpoonup}{Q}_{\ell}^{(\alpha,\hat{\varrho})}\left((\mathrm{v} - x)^2; x\right)\right\}^{1/2}\right]\omega_{b+1}(\mathrm{h}, \delta) \\ &\leqslant 4\mathcal{D}_{\mathrm{h}}\left(1 + b^2\right)\Psi_{\ell}^{(\alpha,\hat{\varrho})}(x) + \left[1 + \frac{1}{\delta}\sqrt{\Psi_{\ell}^{(\alpha,\hat{\varrho})}(x)}\right]\omega_{b+1}(\delta),\end{aligned}$$

choosing $\delta = \sqrt{\Psi_{\ell}^{(\alpha,\hat{\varrho})}(x)}$, *we receive the desired outcome.*

Theorem 4.2.10 *Suppose* $\mathrm{h} \in C_{\Psi}^{*}[0, \infty)$ *then, we obtain*

$$\lim_{\ell\to\infty}\left\|\overset{\rightharpoonup}{Q}_{\ell}^{(\alpha,\hat{\varrho})}(\mathrm{h}; x) - \mathrm{h}(x)\right\|_{\Psi} = 0. \tag{4.40}$$

Proof: *From [61], this is sufficient to validate the three equations shown below.*

$$\lim_{\ell\to\infty}\left\|\overset{\rightharpoonup}{Q}_{\ell}^{(\alpha,\hat{\varrho})}(\mathrm{v}^r; x) - \mathrm{v}^r\right\|_{\Psi} = 0,\ \ r = 0, 1, 2. \tag{4.41}$$

Clearly equation (4.41) *holds for* $r = 0$ *as* $\overset{\rightharpoonup}{Q}_{\ell}^{(\alpha,\hat{\varrho})}(1; x) = 1$.

From lemma 4.2.2 we can say that,

$$\begin{aligned}\left\|\overset{\rightharpoonup}{Q}_{\ell}^{(\alpha,\hat{\varrho})}(\mathrm{v}; x) - x\right\|_{\Psi} &= \sup_{x\in[0,\infty)}\frac{1}{\Psi(x)}\left|\frac{2\alpha + \ell - 2}{(1 - \hat{\varrho})\ell}x - x\right| \\ &= \sup_{x\in[0,\infty)}\frac{x}{x^2 + 1}\left|\frac{(2\alpha + \hat{\varrho}\ell - 2)}{(1 - \hat{\varrho})\ell}\right| \leqslant \frac{(2\alpha + \hat{\varrho}\ell - 2)}{(1 - \hat{\varrho})\ell}.\end{aligned}$$

This implies that $\lim_{\ell\to\infty}\left\|\overset{\rightharpoonup}{Q}_{\ell}^{(\alpha,\hat{\varrho})}(\mathrm{v}; x) - x\right\|_{\Psi} = 0.$ *similarly, we have*

$$\left\|\overset{\rightharpoonup}{Q}_{\ell}^{(\alpha,\hat{\varrho})}\left(\mathrm{v}^2; x\right) - x^2\right\|_{\Psi} = \sup_{x\in[0,\infty)}\frac{1}{\Psi(x)}\left|\begin{array}{c}\frac{(4\alpha\ell+(\ell-3)n)}{(1-\hat{\varrho})(1-2\hat{\varrho})\ell^2}x^2 \\ +\frac{4\alpha\hat{\varrho}(\ell-2)+4\alpha+\hat{\varrho}((\ell-5)\ell+8)+\ell-4}{(1-\hat{\varrho})(1-2\hat{\varrho})\ell^2}x - x^2\end{array}\right|$$

$$\leqslant \sup_{x\in[0,\infty)}\frac{x^2}{\Psi(x)}\left|\frac{(4\alpha\ell + (\ell - 3)\ell)}{(1 - \hat{\varrho})(1 - 2\hat{\varrho})\ell^2} - 1\right|$$

$$+ \sup_{x\in[0,\infty)}\frac{x}{\Psi(x)}\left|\frac{x(4\alpha\hat{\varrho}(\ell - 2) + 4\alpha + \hat{\varrho}((\ell - 5)\ell + 8) + \ell - 4)}{(1 - \hat{\varrho})(1 - 2\hat{\varrho})\ell^2}\right|.$$

Thus

$$\lim_{\wr\to\infty}\left\|\overrightarrow{Q}_{\wr}^{(\alpha,\hat{\varrho})}\left(\mathrm{v}^2;x\right)-x^2\right\|_{\Psi}=0.$$

The anticipated outcome is attained.

Chapter 5

Convergence and Difference Estimates between Mastroianni and Gupta Operators

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In this chapter, we are concerned about investigating the difference of operators. Gupta operators are a modified form of Srivastava-Gupta operators. We estimate the difference between Mastroianni operators with Gupta operators in terms of the modulus of continuity of first order. We also study the weighted approximation of functions and obtain the rate of convergence with the help of the moduli of continuity as well as Peetre's K-functional of Gupta operators.

5.1 Introduction

Acu-Rasa [10], Aral et al. [26] and Gupta [69] studied some fascinating results for the difference of operators in a general sense. Several results on this topic are compiled in the recent book of Gupta et al. [79]. We study here the Mastroianni operators [103] are mentioned below:

$$\mathcal{M}_{n,c}(f;x) \;=\; \sum_{i=0}^{\infty} v_{n,i}(x,c)\mathcal{F}_{n,i}(f), \tag{5.1}$$

where

$$v_{n,i}(x,c) = \frac{(-x)^i}{i!}\tau_{n,c}^{(i)}(x), \mathcal{F}_{n,i}(f) = f\left(\frac{i}{n}\right),$$

with particular cases considered as:

(i) If $\tau_{n,0}(x) = exp(-nx)$, then $v_{n,i}(x,0) = exp(-nx)\frac{(nx)^i}{i!}$, and the operators $\mathcal{M}_{n,0}$ becomes Szász operators.

(ii) If $c \in \mathbb{N}$ and $\tau_{n,c}(x) = \frac{1}{(1+cx)^{n/c}}$, then we have $v_{n,i}(x,c) = \frac{(n/c)_i}{i!}\frac{(cx)^i}{(1+cx)^{\frac{n}{c}+i}}$, and we obtain classical Baskakov operators.

(iii) If $\tau_{n,-1}(x) = (1-x)^n$, then $v_{n,i}(x,-1) = \binom{n}{i}x^i(1-x)^{n-i}$, and the operators (5.1) reduce to Bernstein polynomials,

where $\mathcal{F}_{n,i} : \mathcal{S} \to \mathbb{R}$ is a functional (linear and positive) defined on $\mathcal{S}$ and $\mathcal{S} \subset C[0,\infty)$. Case ($iii$) has not been considered here, we will continue with this case in our next upcoming research.

Srivastava-Gupta operator (see [39], [139]) reproduces only constant functions. Recently Gupta in [68] studied a few examples of genuine operators (operators preserving linear functions), we consider here the following operators

$$\mathcal{G}_{n;c}(f;x) \;=\; \sum_{i=0}^{\infty} v_{n,i}(x,c)\mathcal{H}_{n,i}(f), \tag{5.2}$$

where $v_{n,i}(x,c)$ is defined in (5.1) and

$$\mathcal{H}_{n,i}(f) = (n+c)\int_0^{\infty} v_{n+2c,i-1}(t,c)f(t)dt, \quad 1 \le i < \infty, \quad \mathcal{H}_{n,0}(f) = f(0).$$

Remark 5.1.1 *For operators (5.1), we have $\mathcal{F}_{n,i}(f) = f\left(\frac{i}{n}\right)$ such that*

$$\mathcal{F}_{n,i}(e_0) = 1, \text{ and } b^{\mathcal{F}_{n,i}} = \mathcal{F}_{n,i}(e_1).$$

If we denote $T_r^{\mathcal{F}_{n,i}} = \mathcal{F}_{n,i}(e_1 - b^{\mathcal{F}_{n,i}}e_0)^r$, $r \in \mathbb{N}$, then by simple computation, we have

$$T_r^{\mathcal{F}_{n,i}} = \mathcal{F}_{n,i}(e_1 - b^{\mathcal{F}_{n,i}}e_0)^r = 0, \; r = 2,4.$$

5.2 Preliminaries

Remark 5.2.1 *For the Gupta type operators (5.2), by simple computation, we have*

$$\mathcal{H}_{n,i}(e_r) \;=\; \frac{(i+r-1)!}{(i-1)!}\frac{\Gamma\left(\frac{n}{c}-r+1\right)}{c^r.\Gamma\left(\frac{n}{c}+1\right)},$$

where $\mathcal{H}_{n,i}(e_0) = 1$, $b^{\mathcal{H}_{n,i}} := \mathcal{H}_{n,i}(e_1) = \frac{i}{n}$. If we denote $T_r^{\mathcal{H}_{n,i}} = \mathcal{H}_{n,i}(e_1 - b^{\mathcal{H}_{n,i}}e_0)^r$, $r \in \mathbb{N}$, then after simple computation, we have

$$T_2^{\mathcal{H}_{n,i}} \;:=\; \mathcal{H}_{n,i}(e_1 - b^{\mathcal{H}_{n,i}}e_0)^2 = \frac{ci^2+ni}{n^2(n-c)},$$

and

$$\begin{aligned}
T_4^{\mathcal{H}_{n,i}} &:= \mathcal{H}_{n,i}(e_1 - b^{\mathcal{H}_{n,i}} e_0)^4 \\
&= \mathcal{H}_{n,i}(e_4, x) - 4\mathcal{H}_{n,i}(e_3, x)\left(\frac{i}{n}\right) + 6\mathcal{H}_{n,i}(e_2, x)\left(\frac{i}{n}\right)^2 \\
&\quad -4\mathcal{H}_{n,i}(e_1, x)\left(\frac{i}{n}\right)^3 + \mathcal{H}_{n,i}(e_0, x)\left(\frac{i}{n}\right)^4 \\
&= \frac{(i+3)(i+2)(i+1)i}{n(n-c)(n-2c)(n-3c)} - 4\frac{(i+2)(i+1)i^2}{n^2(n-c)(n-2c)} + 6\frac{(i+1)i^3}{n^3(n-c)} - \frac{3i^4}{n^4}.
\end{aligned}$$

Lemma 5.2.2 *Few moments of Mastroianni operators are given by*

$$\begin{aligned}
\mathcal{M}_n(e_0; x) &= 1; \\
\mathcal{M}_n(e_1; x) &= x; \\
\mathcal{M}_n(e_2; x) &= \frac{x}{n}[x(n+c)+1]; \\
\mathcal{M}_n(e_3; x) &= \frac{x}{n^2}[x^2(n+c)(n+2c) + 3x(n+c) + 1]; \\
\mathcal{M}_n(e_4; x) &= \frac{x}{n^3}[x^3(n+c)(n+2c)(n+3c) + 6x^2(n+c)(n+2c) + 7x(n+c) + 1]; \\
\mathcal{M}_n(e_5; x) &= \frac{x}{n^4}[x^4(n+c)(n+2c)(n+3c)(n+4c) + 10x^3(n+c)(n+2c)(n+3c) \\
&\quad +25x^2(n+c)(n+2c) + 15x(n+c) + 1]; \\
\mathcal{M}_n(e_6; x) &= \frac{x}{n^5}[x^5(n+c)(n+2c)(n+3c)(n+4c)(n+5c) + 15x^4(n+c)(n+2c) \\
&\quad (n+3c)(n+4c) + 65x^3(n+c)(n+2c)(n+3c) + 90x^2(n+c)(n+2c) \\
&\quad +31x(n+c) + 1].
\end{aligned}$$

Lemma 5.2.3 *Let* $f(t) = e_i,\ i = 0, 1, 2, 3, 4$ *and c is the element of the set* $\{0, 1, 2, \}$, *then we have*

$$\begin{aligned}
\mathcal{G}_{n,c}(e_0; x) &= 1; \\
\mathcal{G}_{n,c}(e_1; x) &= x; \\
\mathcal{G}_{n,c}(e_2; x) &= \frac{(n+c)}{(n-c)}x^2 + \frac{2}{(n-c)}x,\ n > c; \\
\mathcal{G}_{n,c}(e_3; x) &= \frac{(n+c)(n+2c)}{(n-c)(n-2c)}x^3 + \frac{6(n+c)}{(n-c)(n-2c)}x^2 + \frac{6}{(n-c)(n-2c)}x,\ n > 2c; \\
\mathcal{G}_{n,c}(e_4; x) &= \frac{(n+c)(n+2c)(n+3c)}{(n-c)(n-2c)(n-3c)}x^4 + \frac{12(n+c)(n+2c)}{(n-c)(n-2c)(n-3c)}x^3 \\
&\quad + \frac{36(n+c)}{(n-c)(n-2c)(n-3c)}x^2 + \frac{24}{(n-c)(n-2c)(n-3c)}x,\ n > 3c.
\end{aligned}$$

Consequently,

$$\mathcal{G}_{n,c}\left((e_1 - x); x\right) = 0;$$
$$\mathcal{G}_{n,c}\left((e_1 - x)^2; x\right) = \frac{2x(1+cx)}{n-c}, \ n > c;$$
$$\mathcal{G}_{n,c}\left((e_1 - x)^4; x\right) = \frac{12c^2(n+7c)}{(n-c)(n-2c)(n-3c)}x^4 + \frac{24c^2(13n+c)}{(n-c)(n-2c)(n-3c)}x^3 + \frac{12c^2(n+9c)}{(n-c)(n-2c)(n-3c)}x^2 + \frac{24}{(n-c)(n-2c)(n-3c)}x, \ n > 3c.$$

Very recently, Pratap and Deo [131] considered genuine Gupta-Srivastava operators and studied fundamental properties, the rate of convergence, Voronovskaya type estimates, convergence estimates and weighted approximation. In the year 2018, Garg et al. [63] studied the weighted approximation properties for Stancu generalized Baskakov operators. In the same year, Acu et al. [8] also studied the order of approximation for Srivastava-Gupta operators via Peetre's $K-$functional and weighted approximation properties and some numerical considerations regarding the approximation properties, were considered. Several researchers studied approximation operators and their variants, and they were given some impressive results on the asymptotic formula, Voronovskaya-type formula, rate of convergence and bounded variation (See [1], [8], [26], [28], [29], [37], [40], [41], [64], [72], [89], [111], [120], [122]).

5.3 Difference of Operators

Let $C_B[0,\infty)$ be the class of bounded continuous functions defined on the interval $[0,\infty)$ equipped with the norm $\|.\| = \sup_{x\in[0,\infty)} |f(x)| < \infty$.

Theorem A[67; 69] Let $f^{(s)} \in C_B[0,\infty)$, s is a member of set $\{0,1,2\}$ and x belongs to $[0,\infty)$, then for all natural numbers n, we get

$$|(\mathcal{G}_{n,c} - \mathcal{M}_{n,c})(f,x)| \ \le \ \|f''\|\alpha(x) + \omega(f'',\delta_1)(1+\alpha(x)) + 2\omega(f,\delta_2(x)),$$

where

$$\alpha(x) = \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)(T_2^{\mathcal{F}_{n,i}} + T_2^{\mathcal{H}_{n,i}}),$$

and

$$\delta_1^2 = \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)(T_4^{\mathcal{F}_{n,i}} + T_4^{\mathcal{H}_{n,i}}), \quad \delta_2^2 = \sum_{i=0}^{\infty} v_{n,i}(x,c)(b^{\mathcal{F}_{n,i}} - b^{\mathcal{H}_{n,i}})^2.$$

We give the quantitative estimate for the difference of Mastroianni and Gupta type operators as an application of Theorem A:

Theorem 5.3.1 *Let $f^{(j)} \in C_B[0,\infty)$, j is a member of set $\{0,1,2\}$ and x belongs to $[0,\infty)$, then for all natural numbers n, we get*

$$|(\mathcal{G}_{n,c} - \mathcal{M}_{n,c})(f;x)| \leq \|f''\|\beta(x) + \omega(f'',\delta_1)(1+\beta(x)),$$

where

$$\beta(x) = \frac{cx[x(n+c)+1]}{2n(n-c)} + \frac{nx}{2n(n-c)},$$

and

$$\begin{aligned}\delta_1^2 &= \frac{1}{2n^4(n-c)(n-2c)(n-3c)}\Big[\Big\{3c^2(n+c)(n+2c)(n+3c)(n+6c)\Big\}x^4 \\ &\quad +6c(n+c)(n+2c)\{3c(n+6c)+2n(n+2c)\}x^3 \\ &\quad +(n+c)\Big\{21c^2(n+6c)+36nc(n+2c)+n^2(3n+c)\Big\}x^2 \\ &\quad +\Big\{3c^2(n+6c)+12nc(n+2c)+n^2(3n+c)+6n^3\Big\}x\Big].\end{aligned}$$

Proof: *First using Remark (5.1.1), Remark (5.2.1) and applying Lemma (5.2.2), we get*

$$\begin{aligned}\beta(x) &= \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)(T_2^{\mathcal{F}_{n,i}} + T_2^{\mathcal{H}_{n,i}}) \\ &= \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)\frac{ci^2+ni}{n^2(n-c)} \\ &= \frac{c}{2(n-c)}\mathcal{M}_n(e_2,x) + \frac{n}{2n(n-c)}\mathcal{M}_n(e_1,x) \\ &= \frac{cx[x(n+c)+1]}{2n(n-c)} + \frac{nx}{2n(n-c)}.\end{aligned}$$

$$\begin{aligned}\delta_1^2 &= \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)(T_4^{\mathcal{F}_{n,i}} + T_4^{\mathcal{H}_{n,i}}) \\ &= \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)T_4^{\mathcal{H}_{n,i}} \\ &= \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)\Big[\frac{(i+3)(i+2)(i+1)i}{n(n-c)(n-2c)(n-3c)} - 4\frac{(i+2)(i+1)i^2}{n^2(n-c)(n-2c)} + 6\frac{(i+1)i^3}{n^3(n-c)} - \frac{3i^4}{n^4}\Big] \\ &= \frac{1}{2}\sum_{i=0}^{\infty} \frac{v_{n,i}(x,c)}{n^4(n-c)(n-2c)(n-3c)}\Big[\Big(i^4+6i^3+11i^2+6i\Big)n^3 \\ &\quad -4\Big(i^4+3i^3+2i^2\Big)n^2(n-3c) + 6(i^4+i^3)n(n-2c)(n-3c) \\ &\quad -3i^4(n-c)(n-2c)(n-3c)\Big] \\ &= \frac{1}{2}\sum_{i=0}^{\infty} \frac{v_{n,i}(x,c)}{n^4(n-c)(n-2c)(n-3c)}\Big[i^4\Big\{n^3 - 4n^2(n-3c) + 6n(n-2c)(n-3c) \\ &\quad -3(n-c)(n-2c)(n-3c)\} + i^3\Big\{6n^3 - 12n^2(n-3c) + 6n(n-2c)(n-3c)\Big\} \\ &\quad + i^2\Big\{11n^3 - 8n^2(n-3c)\Big\} + 6in^3\Big]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=0}^{\infty}\frac{v_{n,i}(x,c)}{n^4(n-c)(n-2c)(n-3c)}\left[3i^4c^2(n+6c)+12i^3nc(n+2c)+i^2n^2(3n+c)+6in^3\right]\\
&= \frac{1}{2n^4(n-c)(n-2c)(n-3c)}\left[3n^4c^2(n+6c)\mathcal{M}_n(e_4,x)+12n^4c(n+2c)\mathcal{M}_n(e_3,x)\right.\\
&\quad\left.+n^4(3n+c)\mathcal{M}_n(e_2,x)+6n^4\mathcal{M}_n(e_1,x)\right]\\
&= \frac{1}{2}\left[\frac{3xc^2(n+6c)\left\{x^3(n+c)(n+2c)(n+3c)+6x^2(n+c)(n+2c)+7x(n+c)+1\right\}}{n^4(n-c)(n-2c)(n-3c)}\right.\\
&\quad+\frac{12nc(n+2c)x\left\{x^2(n+c)(n+2c)+3x(n+c)+1\right\}}{n^4(n-c)(n-2c)(n-3c)}\\
&\quad\left.+\frac{n^2(3n+c)x\{x(n+c)+1\}}{n^4(n-c)(n-2c)(n-3c)}+\frac{6n^3x}{n^4(n-c)(n-2c)(n-3c)}\right]\\
&= \frac{1}{2n^4(n-c)(n-2c)(n-3c)}\left[\left\{3c^2(n+c)(n+2c)(n+3c)(n+6c)\right\}x^4\right.\\
&\quad+6c(n+c)(n+2c)\{3c(n+6c)+2n(n+2c)\}x^3\\
&\quad+(n+c)\left\{21c^2(n+6c)+36nc(n+2c)+n^2(3n+c)\right\}x^2\\
&\quad\left.+\left\{3c^2(n+6c)+12nc(n+2c)+n^2(3n+c)+6n^3\right\}x\right],
\end{aligned}
$$

and

$$\delta_2^2 = \sum_{i=0}^{\infty} v_{n,i}(x,c)(b^{\mathcal{F}_{n,i}}-b^{\mathcal{H}_{n,i}})^2 = 0.$$

5.4 Weighted Approximation

The usual first order of modulus of continuity of f on bounded interval $[0,b]$ is defined as:

$$\omega_b(f;\delta) = \sup_{0<|t-x|\leq\delta}\ \sup_{t,x\in[0,b]} |f(t)-f(x)|.$$

Let

$$B_2[0,\infty) := \left\{f:[0,\infty)\to\mathbb{R}\,|f(x)|\leq M_f\left(1+x^2\right)\right\},$$

where M_f is a constant dependant on f, with the norm

$$\|f\|_2 = \sup_{x\geq 0}\frac{|f(x)|}{1+x^2}.$$

Let

$$C_2[0,\infty) = C[0,\infty)\cap B_2[0,\infty).$$

In [84], Ispir acquainted the weighted modulus of continuity $\Omega(f;\delta)$ as:

$$\Omega(f;\delta) = \sup_{0\leq|k|<\delta,x\geq 0}\frac{|f(x+k)-f(x)|}{(1+k^2)(1+x^2)},\quad f\in C_2[0,\infty). \tag{5.3}$$

Let

$$C_2'[0,\infty) = \left\{ f \in C_2[0,\infty) : \lim_{t\to\infty} \frac{|f(x)|}{1+t^2} < \infty \right\}.$$

From [84; 86], if $f \in C_2'[0,\infty)$, then $\lim_{\delta\to 0} \Omega(f;\delta) = 0$ and

$$\Omega(f;p\delta) \le 2(1+p)\left(1+\delta^2\right)\Omega(f;\delta), \quad p > 0. \tag{5.4}$$

From (5.3) and (5.4) and for $f \in C_2'[0,\infty)$, we have

$$\begin{aligned} |f(t) - f(x)| &\le \left(1+(t-x)^2\right)\left(1+x^2\right)\Omega(f;|t-x|) \\ &\le 2\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\Omega(f;\delta)\left(1+(t-x)^2\right)\left(1+x^2\right). \end{aligned}$$

Now we give the rate of approximation of unbounded functions in the theorem of first order of modulus of continuity.

Theorem 5.4.1 *Let $f \in C_2[0,\infty)$, then we get*

$$\left|\mathcal{G}_{n,c}(f,x) - f(x)\right| \le 4M_f\left(1+b^2\right)\delta_n^2(x) + 2\omega_{b+1}(f;\delta),$$

where $\delta = \delta_n(x) = \sqrt{\mathcal{G}_{n,c}\left((t-x)^2, x\right)}$.
Proof: *For $x \in [0,b]$ and $t \ge 0$, we have*

$$|f(t) - f(x)| \le 4M_f\left(1+b^2\right)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f;\delta), \quad \delta > 0.$$

Applying operator $\mathcal{G}_{n,c}$ and using Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left|\mathcal{G}_{n,c}(f;x) - f(x)\right| &\le 4M_f\left(1+b^2\right)\mathcal{G}_{n,c}\left((t-x)^2, x\right) \\ &+ \left(1+\frac{\mathcal{G}_{n,c}(|t-x|, x)}{\delta}\right)\omega_{b+1}(f,\delta) \\ &\le 4M_f\left(1+b^2\right)\mathcal{G}_{n,c}\left((t-x)^2, x\right) \\ &+ \left(1+\frac{1}{\delta}\sqrt{\mathcal{G}_{n,c}\left((t-x)^2, x\right)}\right)\omega_{b+1}(f,\delta). \end{aligned}$$

After choosing $\delta = \sqrt{\mathcal{G}_{n,c}\left((t-x)^2, x\right)}$, we obtain the required result.

Theorem 5.4.2 *Let $f \in C_2'[0,\infty)$, then we have*

$$\lim_{n\to\infty} \left\|\mathcal{G}_{n,c}(f) - f\right\|_2 = 0.$$

Proof: *From [83], it is sufficient to verify the following by the well-known Bohman-Korovkin theorem:*

$$\lim_{n\to\infty} \left\|\mathcal{G}_{n,c}\left(t^i;x\right) - x^i\right\|_2 = 0, \quad i = 0,1,2. \tag{5.5}$$

From Lemma (5.2.3), the result is true for $i = 0, 1$. *Again using Lemma (5.2.3), we get*

$$\left\|\mathcal{G}_{n,c}\left(t^2;x\right)-x^2\right\|_2=\sup_{x\geq 0}\left|\frac{(n+c)}{(n-c)}x^2+\frac{2}{(n-c)}x-x^2\right|.$$

Finally, we have

$$\lim_{n\to\infty}\left\|\mathcal{G}_{n,c}\left(t^2;x\right)-x^2\right\|_2=0.$$

Thus we get the desired result.

Theorem 5.4.3 *Let* $g \in C_2'[0,\infty)$ *and* $\eta > 0$, *we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|\mathcal{G}_{n,c}(g;x)-g(x)\right|}{(1+x^2)^{1+\eta}}=0,\quad x_0\in(0,\infty].$$

Proof: *Let* $x_0 > 0$ *be any arbitrary fixed value and* $x_0 \in (0,\infty]$ *then, we have*

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{\left|\mathcal{G}_{n,c}(g;x)-g(x)\right|}{(1+x^2)^{1+\eta}} &\leq \sup_{x\leq x_0}\frac{\left|\mathcal{G}_{n,c}(g;x)-g(x)\right|}{(1+x^2)^{1+\eta}}+\sup_{x>x_0}\frac{\left|\mathcal{G}_{n,c}(g;x)-g(x)\right|}{(1+x^2)^{1+\eta}}\\ &\leq \left|\mathcal{G}_{n,c}(g)-g\right|_{C[0,x_0]}+\|g\|_2\sup_{x>x_0}\frac{\left|\mathcal{G}_{n,c}\left(1+t^2;x\right)\right|}{(1+x^2)^{1+\eta}}\\ &\quad+\sup_{x>x_0}\frac{|g(x)|}{\left(1+x_0^2\right)^{1+\eta}}.\end{aligned}\tag{5.6}$$

From Theorem (5.4.2), the first term of the above inequality tends to zero.
Since $|g(x)| \leq \|g\|_2\left(1+x^2\right)$, *we have*

$$\sup_{x>x_0}\frac{|g(x)|}{(1+x^2)^{1+\eta}}\leq\frac{\|g\|_2}{\left(1+x_0^2\right)^{\eta}}.$$

Let $\varepsilon > 0$ *be arbitrary and if we choose* x_0 *very big then*

$$\frac{\|g\|_2}{\left(1+x_0^2\right)^{\eta}}<\frac{\varepsilon}{2},\tag{5.7}$$

Since $\lim_{n\to\infty}\sup_{x>x_0}\frac{\mathcal{G}_{n,c}\left(1+t^2;x\right)}{1+x^2}=1$, *we have*

$$\sup_{x>x_0}\frac{\mathcal{G}_{n,c}\left(1+t^2;x\right)}{1+x^2}\leq\frac{\left(1+x_0^2\right)^{\eta}}{\|g\|_2}\frac{\varepsilon}{2}+1,\quad as\ \ n\to\infty.$$

Therefore

$$\|g\|_2\sup_{x>x_0}\frac{\mathcal{G}_{n,c}\left(1+t^2;x\right)}{(1+x^2)^{1+\eta}}\leq\frac{\|g\|_2}{(1+x_0{}^2)^{\eta}}\sup_{x>x_0}\frac{\mathcal{G}_{n,c}\left(1+t^2;x\right)}{(1+x^2)}\leq\frac{\varepsilon}{2}+\frac{\|g\|_2}{(1+x^2)^{\eta}}.\tag{5.8}$$

From Theorem (5.4.1), and for sufficient large n, we have

$$\left\|\mathcal{G}_{n,c}(g)-g\right\|_{C[0,x_0]}<\varepsilon.\tag{5.9}$$

Estimates from (5.7) to (5.9), the Theorem is proved.

Theorem 5.4.4 *Let $f \in C_2' [0,\infty)$. For sufficient large n, we have*

$$\sup_{x\in[0,\infty)} \frac{\left|\mathcal{G}_{n,c}(f;x) - f(x)\right|}{(1+x^2)^{5/2}} \leq \hat{C}\Omega\left(f; n^{-1/2}\right),$$

where $\hat{C} > 0$ is constant.

Proof: *For x is a point of interval $\in [0,\infty)$ and δ is a positive number and by using the definition of the weighted modulus of continuity and Lemma (5.2.3), we obtain*

$$\begin{aligned} |f(t) - f(x)| &\leq \left(1 + (x + |t-x|)^2\right)\Omega\left(f; |t-x|\right) \\ &\leq 2\left(1+x^2\right)\left(1+(t-x)^2\right)\left(1 + \frac{|t-x|}{\delta}\right)\Omega\left(f;\delta\right). \end{aligned}$$

Applying operator $\mathcal{G}_{n,c}$ both sides, we get

$$\begin{aligned} \left|\mathcal{G}_{n,c}(f;x) - f(x)\right| \leq 2\left(1+x^2\right)\Omega\left(f;\delta\right)\Big\{1 + \mathcal{G}_{n,c}\left((t-x)^2; x\right) \\ + \mathcal{G}_{n,c}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta}; x\right)\Big\}. \end{aligned}$$

Applying Cauchy-Schwarz inequality, Lemma (5.2.3) and choosing $\delta = \frac{1}{\sqrt{n}}$, we obtain the required result.

Chapter 6

Iterative Combinations of Generalised Approximation Operators

The purpose of this chapter is to consider the generalised form of iterative combinations of positive linear operators with well-known Bernstein and Baskakov operators as its particular case. We have estimated the r^{th} moment of the iterative operator and found a recurrence relation between the central moments and their derivatives. We deduce the Voronovskaya type asymptotic formula and the relation between the error of continuous function and its norm with restrictions on its higher derivatives.

6.1 Introduction

In the year 1973, Micchelli [116] had given several results on saturation class and studied properties of semigroups of operators of the Bernstein operator. In the last section of his Ph. D. thesis, he defined iterative operator as:

$$\mathrm{T}_{n,k}\left(f;x\right) = \left[I - (I - B_n)^k\right](f;x) = -\sum_{v=0}^{k}(-1)^v\binom{k}{v}B_n^v\left(f;x\right), \tag{6.1}$$

where B_n is the Bernstein operator and B_n^r is the r^{th} iterate of the operator B_n with $k = 1, 2, ...$ and established the following result

$$\left|T_{n,k}(f;x) - f(x)\right| \leqslant \frac{3}{2}\left(2^k - 1\right)\omega\left(f;\delta\right),$$

where $\omega(f;\delta)$ is the modulus of continuity. The operators (6.1) proved to be a better approach to finding the order of approximation for the Bernstein operator.

Inspired by Micchelli [97; 116], Agarwal [18] et al. gave more intellectual and sharpened results for the new Micchelli type linear operators such as Voronovskaya type asymptotic approximation of sufficiently smooth function for the Bernstein operators.

In 1998, Agarwal [18] extended his work on the Micchelli combination of Bernstein operators and gave simultaneous approximation results related to the inverse theorem. Deo [37; 38] had studied Beta as well as Baskakov operators and estimated results for Baskakov-type operators based on Micchelli [116]. Some interesting approximation results are studied by several mathematicians [1; 5; 85; 96].

We now consider the sequence with the weighted function

$$q_{n.k}(x) = \frac{(-x)^k\,\phi_{n,c}^{(k)}(x)}{k!},$$

where

$$\phi_{n,c}(x) = \begin{cases} (1+cx)^{-n/c}\ ;\ c=-1,\ x\in[0,1] \\ (1+cx)^{-n/c}\ ;\ c>0,\ \ x\in[0,\infty). \end{cases}$$

The generalised form of linear positive operators $L_{n,c} : C_2(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined as:

$$L_{n,c}(f;x) = \sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{k}{n}\right), \tag{6.2}$$

where $C_2(\mathbb{R}_+) = \left\{ f \in C(\mathbb{R}_+) : \lim\limits_{x\to\infty} \frac{|f(x)|}{1+x^2} < 0 \right\}$. The space $C_2(\mathbb{R}_+)$ endowed with the norm

$$||f|| = \sup\left\{ \frac{|f(x)|}{1+x^2}; x \geq 0 \right\},$$

such that $C_2(\mathbb{R}_+)$ is a Banach space. For $c = -1$, operators (6.2) represent Bernstein operators and for $c > 0$, operators (6.2) represent Baskakov operators.

Now we consider Micchelli-type iterative combinations of generalized positive linear operators as:

$$T_{n,k} :\ C_2(\mathbb{R}_+) \to C(\mathbb{R}_+),$$

and

$$T_{n,k}(f;x) = \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} L_{n,c}^{r}(f;x). \tag{6.3}$$

Very recently, Deo et al. [44] studied generalised positive linear operators based on Pólya-Eggenberger distribution (PED) as well as inverse Pólya-Eggenberger distribution (IPED), which are actually generalised form of classical Bernstein and Baskakov operators and authors established direct results for these operators and variant form of these generalised operators were studied by Dhamija et al. [50]. In year 2011, Deo et al. [41; 50] established a generalised form of Bernstein and Baskakov operators and gave a better approximation with the help of King's [98] idea.

6.2 Auxiliary Results

This section of the paper consists of some basic properties and definitions that will be used further to prove the main theorems.

Definition 6.2.1 *(Exponential operators) Let $S_\lambda(f;t)$ be positive operators of the form*

$$S_\lambda(f;t) = \int_{-\infty}^{\infty} W(\lambda,t,x)f(x)dx,$$

such that $S_\lambda(f;t)$ satisfies the following homogenous partial differential equation:

$$\frac{\partial}{\partial t}W(\lambda,t,u) = \frac{\lambda}{p(t)}W(\lambda,t,u)(u-t), \tag{6.4}$$

where $p(t)$ is analytic and positive for $t \in (A,B)$ for some $A,B,-\infty \leqslant A < B \leqslant +\infty$, and the normalization condition

$$S_\lambda(1;t) = \int_{-\infty}^{\infty} W(\lambda,t,u)du = 1.$$

Then operators $S_\lambda(f;t)$ are referred to as exponential operators.

Let $m \in \mathbb{N}^\circ = \mathbb{N}\bigcup\{0\}$ (the set of all non-negative integers), $p \in \mathbb{N}$ (set of Natural numbers) and λ denote the integral part of λ.

Let the m^{th} order moment $a^{\{p\}}_{n,m}(x)$ be defined as:

$$a^{\{p\}}_{n,m}(x) = L^{\{p\}}_{n,c}\left((t-x)^n;x\right).$$

We shall use $a_{n,m}(x) = a^{\{1\}}_{n,m}(x)$, $a^{(1)}_{n,m}(x)$ for the derivative of $a_{n,m}(x)$ with respect to x and $R(j,k;x)$ be the coefficient of $1/n^k$ in the $T_{n,k}\Big((t\ -\ x)^j;x\Big)$.

Lemma 6.2.2 *For the operator $L^r_{n,c}(f;x)$*

(i) $L^r_{n,c}(1;x) = 1;$

(ii) $L^r_{n,c}(t;x) = x;$

(iii) $L^r_{n,c}(t^2;x) = x^2\left(1+\frac{c}{n}\right)^r + \frac{x}{c}\left[\left(1+\frac{c}{n}\right)^r - 1\right].$

Proof: *This result can be easily proved by mathematical induction.*

Lemma 6.2.3 *There is a recurrence holds for the function $a_{n,m}(x)$ as ,*

$$n\, a_{n,m+1}(x) = x(1+cx)\left[a^{(1)}_{n,m}(x) + m\, a_{n,m-1}(x)\right];$$

with $a_{n,0}(x) = 1$, $a_{n,1}(x) = 0$, and $a_{n,2}(x) = \frac{x(1+cx)}{n}$.
Proof: *The values corresponding to $a_{n,0}(x), a_{n,1}(x)$ and $a_{n,2}(x)$ follows immediately from the definition. This proof will proceed by proving*

$$x(1+cx)\, a^{(1)}_{n,m}(x) = n\, a_{n,m+1}(x) - x(1+cx)m\, a_{n,m-1}(x).$$

The rest is basic computation which is left to the reader.

6.3 Direct Results

This section is dedicated to introducing approximation theorems and determining the asymptotic Voronovskaya-type formula to discuss the convergence properties.

Theorem 6.3.1 *For every $f \in C_2(\mathbb{R}_+)$ with $k \in \mathbb{N}^\circ$ and $n \in \mathbb{N}$,*

$$|\, T_{n,k}(f;x) - f(x)| \le \omega(f;\delta)\left[\left(2^k - 1\right) + \frac{1}{\delta^2}\left\{\left(2+\frac{c}{n}\right)^k - 2^k\right\}\left(x^2 + \frac{x}{c}\right)\right].$$

Proof: *We consider,*

$$T_{n,k}\,(f;x) - f(x) = \sum_{r=1}^{k} (-\,1)^{r+1}\binom{k}{r} L^r_{n,c}\,(f(t) - f(x);x)\,.$$

Therefore,

$$\left|T_{n,k}(f;x) - f(x)\right| \leqslant \sum_{r=1}^{k}\binom{k}{r} L^r_{n,c}\,(|f(t) - f(x)|;x)\,. \tag{6.5}$$

Mond and et al. [119] given the result that for all $t, x \in [0,\infty]$ and $\delta > 0$,

$$|f(t) - f(x)| \le \left(1 + \frac{(t-x)^2}{\delta^2}\right)\omega(f;\delta). \tag{6.6}$$

From (6.5) *and* (6.6) *and using lemma (6.2.2), we have*

$$\begin{aligned}|\, T_{n,k}(f;x) - f(x)| &\leqslant \sum_{r=1}^{k}\binom{k}{r} L^r_{n,c}\left(\left(1 + \frac{(t\,-\,x)^2}{\delta^2}\right);x\right)\omega(f;\delta)\\ &\leqslant \omega(f;\delta)\left[\sum_{r=1}^{k}\binom{k}{r} L^r_{n,c}(1;x) + \frac{1}{\delta^2}\sum_{r=1}^{k} L^r_{n,c}\left((t-x)^2;x\right)\right],\end{aligned}$$

and with the help of the central moment

$$L^r_{n,c}((t-x)^2;x) = \left(x^2+\frac{x}{c}\right)\left[\left(1+\frac{c}{n}\right)^r-1\right],$$

we get

$$\begin{aligned}|\,T_{n,k}(f;x)-f(x)| &\leq \omega(f;\delta)\left[\left(2^k-1\right)+\frac{1}{\delta^2}\left\{\left(2+\frac{c}{n}\right)^k-2^k\right\}\left(x^2+\frac{x}{c}\right)\right]\\ &= \omega(f;\delta)\left(2^k-1\right)\left[1+\frac{1}{\delta^2}\left\{\frac{\left(2+\frac{c}{n}\right)^k-2^k}{2^k-1}\right\}\left(x^2+\frac{x}{c}\right)\right].\end{aligned}$$

Choosing $\delta = n^{-1/2}$ *in above, we finally get*

$$\|T_{n,k}(f)-f(x)\| \leq \omega(f;1/\sqrt{n})(2^k-1)\left[1+n\left\{\frac{\left(2+\frac{c}{n}\right)^k-2^k}{2^k-1}\right\}x\left(x+\frac{1}{c}\right)\right],$$

which is the desired answer.

Remark 6.3.2 *It is easy to verify that for* $c>0$,

$$\frac{\left(2+\frac{c}{n}\right)^k-2^k}{2^k-1} \leqslant c^k\frac{k}{n} = \frac{c_1}{n}, \quad \textit{where } c_1 = c^k k.$$

So we get

$$\|\,T_{n,k}(f)-f\| \leqslant \omega\left(f;1/\sqrt{n}\right)\left(2^k-1\right)\left[1+k\,c^k x\left(x+\frac{1}{c}\right)\right], \quad x\in[0,\infty),$$

and for $c=-1$, *we have*

$$\|\,T_{n,k}(f)-f\| \leqslant \omega\left(f;1/\sqrt{n}\right)\left(2^k-1\right)\left(1+\frac{k}{4}\right), \quad x\in[0,1].$$

Lemma 6.3.3 *[119].* $a_{n,m}(x)$ *is a polynomial in x and* $1/n$ *with degree of* $a_{n,m}(x)$ *in both is less than equal to m with* $a_{n,m}(x) = o\left(n^{-[(m+1)/2]}\right)$. *Also, the coefficient of* $(1/n)^m$ *in* $a_{n,2m}(x)$ *is* $(2m-1)!!\phi(x)^m$, *where a!!= semi factorial of a and* $\phi(x) = x(1+cx)$ *and the coefficient of* $(1/n)^m$ *in* $a_{n,2m+1}(x)$ $(2m+1)!!\,\phi^m(x)\phi'(x)\left(\frac{m}{3}\right)$.

Proof: *By the definition of exponential operators, (6.2) satisfies the partial differential equation (6.4). Also, Bernstein polynomials, Szàsz, Post-Widder and Baskakov all are exponential types. So, the proof follows from [112] (Prop.3.2).*

Lemma 6.3.4 *[18]. There is the recurrence relation:*

$$a^{\{p+1\}}_{n,m}(x) = \sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{m-j}\frac{1}{i!}\frac{d\left(a^{\{p\}}_{n,m}(x)\right)}{dx}\,a^{\{p\}}_{n,i+j}(x),$$

where $\frac{dy}{dx}$ *denotes the derivative of y with respect to x.*

Lemma 6.3.5 *[18]. We have* $a_{n,m}^{\{p\}}(x) = o\left(n^{-[(m+1)/2]}\right)$.

Lemma 6.3.6 *[18]. For* $l-th$ *moment (* $l \in \mathbb{N}$ *) of* $T_{n,k}$*, we have*

$$T_{n,k}\left((t\ -\ x)^l; x\right) = o\left(\ n^{-k}\right).$$

Definition 6.3.7

$$U_{n,s}(x) = \sum_{k=0}^{\infty} \frac{(-x)^k \phi_{n,c}^{(k)}(x)}{k!}\,(k-nx)^s.$$

The inequality

$$0 \le U_{n,s}(x) \le K\,n^{[s/2]};\quad 0 \le x < \infty,$$

follows from lemma (6.3.3).

Theorem 6.3.8 *Let* $f \in C_2(\mathbb{R}_+)$ *and* $f^{(2k)}$ *exists at a fixed point* $x \in [0,\infty)$*, then*

$$\lim_{n\to\infty} n^k\left[T_{n,k}(f;x) - f(x)\right] = \sum_{j=2}^{2k} \frac{f^{(j)}(x)}{j!} R(j,k;x). \tag{6.7}$$

Also, if $f^{(2k-1)} \in A.C.[0,c]$ *with* $f^{(2k)} \in L_\infty[0,c]$*, then for any proper interval* $[a,b]$ *of* $[0,c]$*, we have*

$$\left\|T_{n,k}(f;) - f\right\|_{C[a,b]} \leqslant \frac{C}{n^k}\left\{\|f\|_{C[a,b]} + \left\|f^{(2k)}\right\|_{L_\infty[0,c]}\right\}. \tag{6.8}$$

Proof: *Since* f *is continuous in* $[0,\infty)$*, so it has a Taylors series expansion at* $t = x$*,*

$$f(t) = f(x) + \sum_{j=1}^{2k} \frac{f^{(j)}(x)\,(t-x)^j}{j!} + \frac{f^{2k+1}(\xi)\,(t-x)^{2k+1}}{(2k+1)!},\ 0 < \xi < \infty. \tag{6.9}$$

Now we consider

$$\begin{aligned} n^k\left[T_{n,k}(f;x) - f(t)\right] &= n^k \sum_{j=1}^{2k} \frac{f^j(x)}{j!}\, T_{n,k}[(t-x)^j;x] \\ &\quad + n^k \sum_{r=1}^{k} (-\,1)^{r+1}\binom{k}{r} L_{n,c}^r\left(\epsilon(t;x)(t\ -\ x)^{2k};x\right) \\ &= G_1 + G_2, \end{aligned}$$

where $\epsilon(t,x) \to 0$ *as* $t \to x$*.*

So for a given $\varepsilon > 0$*, there exist a* $\delta(\varepsilon) > 0$ *such that*

$$|\epsilon(t,x)| < \varepsilon \quad \textit{whenever} \quad 0 < |t-x| < \delta.$$

First we evaluate G_1*,*

$$G_1 = n^k \sum_{j=2}^{2k} \frac{f^{(j)}(x)}{j!}\, T_{n,k}\left((t\ -\ x)^j;x\right) +\ n^k\left[f'(x)\,T_{n,k}(t-x;x)\right].$$

From lemma (6.3.6), we get

$$T_{n,k}((t-x)^j;x) = o(n^{-k}).$$

We can directly compute that,

$$G_1 = \sum_{j=2}^{2k} \frac{f^j(x)}{j!} R(j,k;x) + o(1). \tag{6.10}$$

Now, we estimate G_2,

Let $\Theta_\delta(t)$ be the characteristic function in $(t-\delta, t+\delta)$, we have

$$\begin{aligned} |G_2| &\leqslant n^k \sum_{r=1}^{k} \binom{k}{r} L_{n,c}^r \left(|\varepsilon(t,x)||t-x|^{2k} \Theta_\delta(t); x \right) \\ &+ n^k \sum_{r=1}^{k} \binom{k}{r} L_{n,c}^r \left(|\varepsilon(t,x)||t-x|)^{2k} \left(1-\Theta_\delta(t)\right); x \right) \\ &= G_{21} + G_{22}. \end{aligned}$$

Therefore,

$$\begin{aligned} G_{21} &\leqslant \sup_{|t-x|<\delta} |\varepsilon(t,x)| n^k \sum_{r=1}^{k} \binom{k}{r} L_{n,c}^r \left((t-x)^{2k}; x \right) \\ &\leqslant \varepsilon n^k \sum_{r=1}^{k} a_{n,2k}^{\{r\}} \binom{k}{r} = \varepsilon C_1. \end{aligned}$$

Now

$$|G_{22}| \leqslant n^k \sum_{r=1}^{k} \binom{k}{r} L_{n,c}^r \left(|\varepsilon(t,x)|(t-x)^{2k} \left(1-\Theta_\delta(t)\right); x \right).$$

For an arbitrary $p > k$ and using lemma (6.3.5), we have

$$L_{n,c}^r(|\varepsilon(t,x)|\,|t-x|^{2k};x) \leq \frac{M_2}{\delta^{2(p-k)}} L_{n,c}^r((t-x)^{2p};x)$$

and

$$G_{22} \leqslant \frac{M_3}{n^{p-k}} = o(1).$$

Since $\varepsilon > 0$ is an arbitrary, $G_2 \to 0$ as $n \to \infty$. From G_1, G_2 and (6.7) follows immediately

$$\begin{aligned} T_{n,k}(f(t);x) - f(x) &= T_{n,k}\left(\varphi(t)\left(f(t)-f(x)\right);x\right) + T_{n,k}\left(\left(1-\varphi(t)\right)\left(f(t)-f(x)\right);x\right) \\ &= G_3 + G_4, \end{aligned}$$

where $\varphi(x)$ is the characteristics function of the closed interval on $[0,c]$.

For all $t \in [0,c]$ and $x \in [a,b]$, we have

$$f(t) - f(x) = \sum_{l=0}^{2k-1} \frac{f^{(l)}(x)}{l!}(t-x)^l + \frac{1}{(2k-1)!}\int_x^t (t-s)^{2k-1} f^{(2k)}(s)ds.$$

And

$$\begin{aligned} G_3 &= \sum_{l=0}^{2k-1} \frac{f^{(l)}(x)}{l!} T_{n,k}\left(\varphi(t)(t-x)^l; x\right) + \frac{1}{(2k-1)!} T_{n,k}\left(\int_x^t \varphi(t)(t-s)^{2k} f^{(2k)}(s)ds\right) \\ &= \sum_{l=0}^{2k-1} \frac{f^{(l)}(x)}{l!}\left[T_{n,k}\left((t-x)^l; x\right) + T_{n,k}\left((\varphi(t)-1)(t-x)^l; x\right)\right] \\ &\quad + \frac{1}{(2k-1)!} T_{n,k}\left(\int_x^t \varphi(t)(t-s)^{2k} f^{(2k)}(s)ds\right) \\ &= \sum_{l=0}^{2k-1} \frac{f^{(l)}(x)}{l!}(G_{31}+G_{32}) + G_{33}. \end{aligned}$$

Clearly, $G_{31} = o\left(\frac{1}{n^k}\right)$ by lemma (6.3.6). Similarly by definition of characteristics function, $G_{31} = o\left(\frac{1}{n^k}\right)$. Convergence of G_{31} and G_{32} is uniform for all $x \in [a,b]$ & $t \in [0,b]$. Also,

$$G_{33} \leqslant \frac{\left\|f^{(2k)}\right\|_{L_\infty[0,c]}}{(2k-1)!} \leqslant \frac{M_{11}\left\|f^{(2k)}\right\|_{L_\infty[0,c]}}{n^k}, \text{ where } M_{11} \text{ is a constant.}$$

Combining the above results and the interpolation property of norms introduced by Goldberg and Meir [65], we have

$$G_3 \leqslant M_{12}\left\{\|f\|_{C[a,b]} + \left\|f^{(2k)}\right\|_{L_\infty[0.c]}\right\}. \tag{6.11}$$

For G_4, we proceed on the same lines as we have done for (6.7) and for G_3.

$$G_4 \leqslant M_{21}\left\{\sum_{l=0}^{2k-1}\left\|f^{(l)}\right\|_{C[a,b]} + \left\|f^{(2k)}\right\|_{L_\infty[0,c]}\right\}. \tag{6.12}$$

Hence from (6.11) and (6.12), we obtain the required result (6.8).

Theorem 6.3.9 *Suppose that $k \in \mathbb{N}^\circ$. If the function $f, f', f''.....f^{(2k+1)}$ are in the class of $C[0,\infty)$ and $f^{(2k+1)} \in Lip_M\, 1$ on $[0,\infty)$, then*

$$|\, T_{n.k}(f;x) - f(x)| = o\left(\frac{1}{n^{k+1}}\right),$$

uniformly as $n \to \infty$ on $[0,\infty)$.

Proof: *Let us define*

$$\|f\|_k = \max_{0 \le j \le 2k}\left\{\|\, f^j\, \|, M\right\}.$$

It will be sufficient to prove that

$$|T_{n,k+1}(f;x) - f(x)| \leq \frac{A_k \, \|f\|_k}{n^{k+1}},$$

where A_k is a constant independent of f and n. We will prove it by mathematical induction. For the case $k = 0$,

$$|\,T_{n,0}(f;x) - f(x)| = |\,L_{n,c}(f;x) - f(x)|.$$

By Taylor series expansion, we obtain

$$|\,L_{n,c}(f(t) - f(x);x)| \leq \frac{x(1+cx)}{2n} f''(x).$$

Suppose the theorem is true for $j < k$ and f satisfies the hypothesis of the theorem then we have

$$f(x) - \sum_{i=0}^{2k+1} \frac{f^{(i)}(x)}{i!}(x-x_0)^i = \frac{1}{(2k+1)!}\int_{x_0}^{x}(x-\eta)^{2k+1} f^{(2k+1)}(\eta)d\eta$$

$$\leq \frac{1}{(2k+1)!}\frac{\left[(x-\eta)^{2k+2}\right]_{x_0}^{x}\|f\|_k}{(2k+2)}.$$

Since $L_{n,c}$ are positive linear operators,

$$\left|L_{n,c}f(x) - \sum_{i=0}^{2k+1}\frac{f^{(i)}(x)}{n^i i!}\,U_{n,i}(x)\right| \leqslant \frac{1}{(2k+1)!\,n^{2k+2}}U_{n,2k+2}(x)\|f\|_k,$$

we can rewrite this inequality as:

$$(L_{n,c} - I)(f;x) = \sum_{i=0}^{2k+1}\frac{f^{(i)}(x)}{n^i\,i!}\,U_{n,i}(x) + e_n(x),$$

where

$$|\,e_n(x)| \leq \frac{\|f\|_k\, a_k}{n^{k+1}}.$$

Now, consider the function

$$J_{n,l}(x) = \frac{f^{(l)}(x)\,U_{n,l}(x)}{l!\,n^{[l/2]}}, \quad l = 2, 3, ...2k.$$

This satisfies the hypothesis of the theorem for the integers $k_l = \left[k - \frac{l}{2}\right]$ as $k_l < k$.

By the definition of $J_{n,l}$ applying induction

$$\left|(L_{n,c} - I)^{k_i+1}(J_{n,i};x)\right| \leqslant \frac{b_{\,k_i}}{n^{k_i}}\|J_{n,i}\|_{k_i}.$$

Also $\frac{J_{n,i}}{i!\,n^{[i/2]}}$ is a polynomial of degree i which is uniformly bounded in n.

$$0 \leq \frac{J_{n,i}}{i!\,n^{[i/2]}} \leq \frac{C}{i!};$$

and

$$\|J_{n,i}\|_{k_i} = \| f^{(i)} \| \le \|f\|_k \, .$$

We can easily estimate,

$$|(L_{n,c} - I)^k (J_{n,i}; x)| \le \frac{b_k}{n^{k+1}} \|f\|_k, \quad i = 2, 3...2k.$$

Moreover, $\| (L_{n,c} - I)^k \| \le 2^k m_k$, *where* m_k *is independent of* f *and* n. *Therefore we have,*

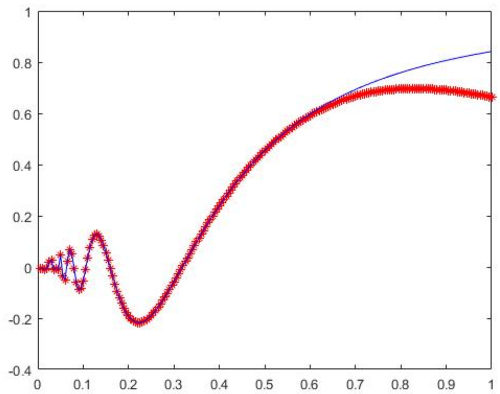
$$\begin{aligned} |T_{n,k+1}(f;x) - f(x)| &= \left| (L_{n,c} - I)^k \left(\sum_{i=2}^{2k} \frac{J_{n,i}\, n^{[i/2]}}{n^i} + e_n(x); x \right) \right| \\ &\le \left\{ A_k \sum_{i=2}^{2k} \frac{n^{[i/2]}}{n^{k_i+1}\, n^i} + \frac{2^k a_k m_k}{n^{k+1}} \right\} \|f\|_k \\ &= \frac{2^k a_k m_k + b_k(2k-1)}{n^{k+1}} \|f\|_k \, . \end{aligned}$$

So, we obtain the required result by mathematical induction.

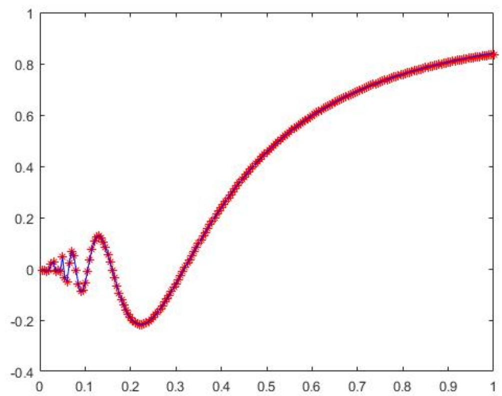
Example 6.3.10 *We have approximated the rate of convergence of the operators* $T_{n,k}(f)$ *to the function* $f(x) = xsin(1/x)$ *for different values of* k *while keeping* $c > 0$. *As the conclusion comes from the table and using the graphical technique, for the value of* $k = 10$ *the error estimation is less than* 0.1 *and for* $k = 30$ *the error is less than 0.001.*

x	$\|T_{n,10}(f;x) - f(x)\|$	$\|T_{n,30}(f;x) - f(x)\|$
0.895	0.11650	0.00160
0.9050	0.12240	0.00180
0.9150	0.12830	0.00210
0.9300	0.13440	0.00240
0.940	0.14040	0.00280
0.950	0.14660	0.00310
0.960	0.15280	0.00360
0.970	0.15900	0.00400
0.980	0.16530	0.00450
0.990	0.17160	0.00510
1.00	0.17800	0.00560

Table 6.1: Comparing between error of operators for different values of k towards function f(x)=$xsin(1/x)$ with $c > 0$.

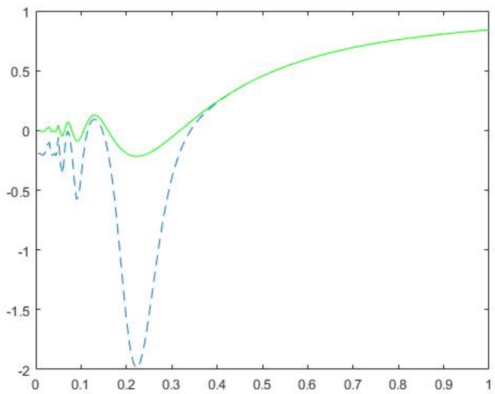


(a) k=10, c =23.6

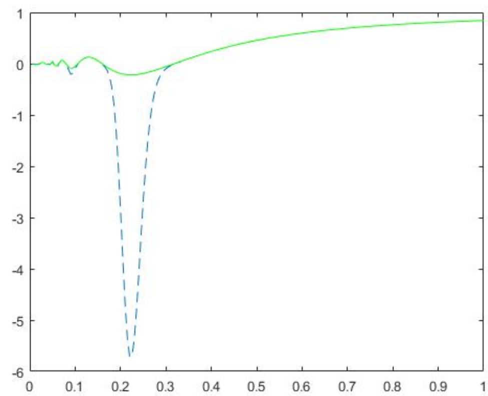


(b) k=30, c =23.6

Figure 6.1: Convergence of $T_{n,k}(f;x)$$* * **$, for the function $f(x) = xsin(1/x)$$- - --$, for $n = 100$.



(a) k=10, c = -1



(b) k=30, c = -1

Figure 6.2: Convergence of $T_{n,k}(f;x)$− − −, for the function $f(x) = xsin(1/x)$− − − − −, for $n = 100$.

Conclusion and Future scope

Conclusion

This chapter aims to present concluding remarks to our thesis and illustrate some of the prospects that define our current and future endeavours in scientific research. This thesis is mainly a study of convergence estimates of various approximation operators and their variants. The introductory chapter consists of definitions and a literature survey of concepts used throughout this thesis.

The second chapter presents a Durrmeyer-type construction involving a class of orthogonal polynomials called Apostol-Genocchi polynomials. The main goal of the first section is to construct Durrmeyer-type operators of Apostol-Genocchi operators (2.18) based on the Jain operators (2.9) with real parameters α, β, and λ. In the second section, we present Beta operators associated with the Apostal-Genocchi polynomials and studied the approximation properties of these Durrmeyer operators. We give a direct approximation theorem using first and second-order modulus of continuity, local approximation results for Lipschitz class functions and a direct theorem for the usual modulus of continuity.

Inspired by King's approach, the next chapter deals with the modification of the so-known Lupaş-Kantrovich that preserves constant functions and exponential function e^{-x}. Followed by some useful lemmas, we determine the rate of convergence of the proposed operators in terms of the usual modulus of continuity and Peetre's K- functional. Further, the degree of approximation is also established for the function of bounded variation. We also illustrate convergence and absolute error via figures and tables.

In the fourth chapter, we discuss approximation operators due to the development of the theory of inverse Pólya-Eggenberger distribution. The first section of this chapter presents the summation-integral type operators involving inverse Pólya-Eggenberger distribution [142] and Pǎltǎnea operators [126]. The vital target of this chapter is to contemplate the approximation properties of operators (4.5) including K-functional and second-order modulus of smoothness. Lastly, we set up the rate of convergence for functions with derivative of bounded variation. The main goal of the second section is to construct α-Pólya-Baskakov operator based on inverse Pólya-Eggenberger distribution (4.23), where α being a non-negative parameter, which may depend only on the natural number $\mathscr{r}$, with $\alpha \to 0$ when $\mathscr{r} \to \infty$, $\mathscr{r} \geq 1$, $x \in [0, \infty)$. As a result of this study, we can obtain some approximation results, including the Voronovskaya type asymptotic formula, error estimate in terms of modulus of continuity and the sense of k-functional, and weighted approximation.

In the fifth chapter, we discuss the approximation properties of Gupta operators. We find an estimate for the difference between Mastroianni operators and Gupta operators in terms of modulus of continuity of first order. We give the rate of convergence with the help of the moduli of continuity and Peetre's $K-$functional and the weighted approximation of functions is studied.

The purpose of the sixth chapter is to give generalised results for the operators (6.3) and obtain better approximation results. We study auxiliary results and derive the value of the iterative operators (6.3) at the basic test functions $1, t, t^2$. Also, we obtain Voronovskaya-type results, direct estimates in terms of modulus of continuity and provide a computational approximation which relates the rate of convergence graphically with the error estimation done.

Academic future plans

The objective of my work is to share my research results with the mathematical community and to continue carrying further additional studies in the area of approximation by linear positive operators. Furthermore, I shall intend to put forward parametric generalisations of existing operators which permits us to approximate an additional array of functions. I intend to pursue research and contribute to the development of new operators in this field in addition to contributing modifications to those currently exist.

We have examined the Mastroianni and Gupta operators in this thesis. Bede et al. [31] have been associated with introducing the max-product version of families of linear approximation operators. In general, a family of nonlinear (or more accurately sub-linear) operators with improved approximation properties of their original version is the max-product version of a sequence or net of linear operators; in many instances, the order of convergence is faster than that of linear operators. My goal is to expand on this research in the future by establishing the max-product form of Mastroianni and Gupta operators.

Another appealing problem related to approximation theory is to identify which operators provide the best approximation. There are several operators with high convergence rates, some of which are investigated in this thesis. As part of my continuing study, I am interested in comparing the existing operators employing the difference of operators examined to find out which ones generate the best approximation.

The corresponding semi-exponential operators for Bernstein, Baskakov and Ismail-May operators were obtained by Abel et al. [3]. Since the area of semi-exponential operators is not much explored, I intend to study the approximation properties of these operators such as the complete asymptotic expansion, and the behaviour of their derivatives through simultaneous approximation and other convergence estimates.

References

[1] Abel U., Gupta V. and Ivan M., Asymptotic approximation of functions and their derivatives by generalized Baskakov-Szász-Durrmeyer operators, Anal. Theory Appl., 21(1)(2005), 15-26.

[2] Abel U., Gupta V. and Ivan M., The complete asymptotic expansion for Baskakov-Szász-Durrmeyer operators, Ann. Tiberiu Popoviciu Semin. Funct. Eqn. Approx. Convexity, (1)(2003), 3-15.

[3] Abel U., Gupta V. and Sisodia M., Some new semi-exponential operators. RACSAM., 2(116)(2022), 87.

[4] Abramowitz M. and Stegun I. A. (eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series 55, 4th Printing, Washington, 1965.

[5] Acu A. M., Acar T., Muraru C. V. and Radu V. A., Some approximation properties by a class of bivariate operators, Math. Methods Appl. Sci., 42 (2019), 1-15.

[6] Acu A.M. and Gonska H., Classical Kantorovich operators revisited, Ukrainian Math. J., 71(6)(2019), 843-852.

[7] Acu A. M. and Gupta V., Direct results for certain summation-integral type Baskakov-Szász operators, Results. Math., 72(2017), 1161-1180.

[8] Acu A. M. and Muraru C. V., Certain approximation properties of Srivastava-Gupta operators, J. Math. Inequal., 12(2)(2018), 583-595.

[9] Acu A. M. and Radu V. A., Approximation by Certain Operators Linking the α–Bernstein and the Genuine α-Bernstein-Durrmeyer Operators, Mathematical Analysis I: Approximation Try: Springer Proceedings in Mathematics & Statistics, eds. N. Deo et al., 306(2020), 77-88.

[10] Acu A. M. and Rasa I., New estimates for the differences of positive linear operators, umer. Algorithms, 73(2016), 775-789.

[11] Acar T., Acu A.M. and Manav N., Approximation of functions by genuine Bernstein-Durrmeyer type operators. J. Math. Inequalities, 12(4)(2018), 975-987.

[12] Acar T., Aral A., Cárdenas-Morales, D. and Garrancho P., Szász–Mirakyan type operators which fix exponentials, Results Math., 72(3)(2017), 1393-1404.

[13] Acar T., Aral A. and Gonska H., On Szász–Mirakyan operators preserving $e^{2ax}, a > 0$, Mediterr. J. Math., 14(1)(2017), 1660-5454.

[14] Acar T., Aral A. and Mohiuddine S. A., Approximation by bivariate (p, q)-Bernstein-Kantorovich operators, Iran J Sci Technol Trans Sci., 42(2)(2018), 655-662.

[15] Acar T., Aral A. and Raşa I., The new forms of Voronovskayas theorem in weighted spaces, Positivity, 20(1)(2016), 25-40.

[16] Adell J. A. and De la Cal J., On a Bernstein-type operator associated with the inverse Pólya-Eggenberger distribution, Rend. Circolo Matem. Palermo, Ser. II, 33(1993), 143-154.

[17] Agrawal P.N., Ispir N. and Kajla A., Approximation properties of Lupaş-Kantorovich operators based on Pólya distribution, Rend. Circ. Mat. Palermo, Ser.II, 65(2016), 185-208.

[18] Agarwal P. N. and Kasana H. S., On the iterative combinations of Bernstein polynomials, Demonstr. Math, 16(1984), 777-783.

[19] Agratini O., On an approximation process of integral type, Appl. Math. Comput., 236(1)(2014), 195-201.

[20] Agratini O., Approximation properties of a class of linear operators, Math. Meth. Appl. Sci., 36(2013), 2353-2358.

[21] Altomare F. and Campiti M., Korovkin-type Approximation Theory and its Application, de Gruyter studies in Mathematics, 17, Walter de Gruyter & Co., Berlin, (1994).

[22] Anastassiou G.A. and Gal S., Approximation Theory, Moduli of Continuity and Global Smoothness Preservation, Birkhauser, Boston, (2000).

[23] Apostol T. M., On the Lerch zeta function, Pacific J. Math., 1(1951), 161-167.

[24] Araci S., Novel identities involving Genocchi numbers and polynomials arising from applications of umbral calculus. Appl. Math. Comput., 233(2014), 599-607.

[25] Aral A., Inoan D. and Rasa I., Approximation properties of Szász-Mirakyan operators preserving exponential functions, Positivity, 23(1)(2019), 233-246.

[26] Aral A., Inoan D. and Rasa I., On differences of linear positive operators, Anal. Math. Phys., 9(2019), 1227-1239.

[27] Aral A. and Hasan E., Parametric generalization of Baskakov operator, Math.Commun., (24)(2019), 119-131.

[28] Barbosu D. and Deo N., Some Bernstein-Kantorovich operators, Automation Computers Applied Mathematics, 22(1)(2013), 15-21.

[29] Barbosu D. and MiclăuşD., On the Voronovskaja-type formula for the Bleimann, Butzer and Hahn bivariate operators, Carpathian J. Math., 33(1)(2017), 35-42.

[30] Baskakov V. A., An instance of a sequence of linear positive operators in the space of continuous functions, Dokl. Akad. Nauk., (113)(1957), 249-251.

[31] Bede, Barnabás and Coroianu, Lucian and Gal, Sorin G, Approximation by max-product type operators, Springer, 2016.

[32] Bernstein S. N., Démonstration du théorém de Weierstrass fondée sur la calcul des probabilitiés, Comm. Soc. Math. Charkow Sér.2 t., 13(1912), 1-2.

[33] Bernstein S. N., Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné, mimoires acad. Académie Royale de Belgique. Classe des Sciences. Mémoires, 4(1912).

[34] Bohman H., On approximation of continuous and analytic functions, Arkiv för Matematik, 2(1)(1952), 43-56.

[35] Boyanov B. D. and Veselinov V. M., A note on the approximation of functions in an infinite interval by linear positive operators, Bull. Math. Soc. Sci. Math. Roum., 14(62)(1970), 9-13.

[36] Cardenas-Morales D., Garrancho P. and Munos-Delgado F. J., Shape-preserving approximation by Bernstein-type operators which fix polynomials, Appl. Math. Comput., 182(2006), 1615-1622.

[37] Deo N., A note on equivalent theorem for Beta operators, Mediterr. J. Math., 4(2)(2007), 245-250.

[38] Deo N., On the iterative combinations of Baskakov operator, Gen. Math., 15(2007), 51-58.

[39] Deo N., Faster rate of convergence on Srivastava-Gupta operators, Appl. Math. Comput., 218(21)(2012), 10486-10491.

[40] Deo N., Voronovskaya type asymptotic formula for Lupaş-Durrmeyer operators, Rev. Un. Mat. Argentina., 48(1)(2007), 47-54.

[41] Deo N. and Bhardwaj N., Some approximation results for Durrmeyer operators, Appl. Math. Comput., 217(12)(2011), 5531-5536.

[42] Deo N. and Bhardwaj N., Direct and inverse theorems for beta–Durrmeyer operators. In Modern Mathematical Methods and High Performance Computing in Science and Technology, Springer, Singapore (2016), 179-191.

[43] Deo N., Bhardwaj N. and Singh S. P., Simultaneous approximation on generalized BernsteinDurrmeyer operators, Afr. Mat., 24(1)(2013), 77-82.

[44] Deo N. and Dhamija M., Generalized positive linear operators based on PED and IPED, Iranian J. Sci. Technol. Trans. Sci., 43(2019), 507-513.

[45] Deo N. and Dhamija M., Miclăuş D., Stancu-Kantorovich operators based on inverse Pólya-Eggenberger distribution, Appl. Math. Comput., 273(2016), 281–289.

[46] Deo N., Dhamija M. and Miclăuş D., New modified Baskakov operators based on the inverse Pólya-Eggenberger distribution, Filomat, 33(11)(2019), 3537-3550.

[47] Deo N., Ozarslan M.A. and Bhardwaj N., Statistical convergence for General Beta Operators, Korean J. Math., 22(4)(2014), 671-681.

[48] Derriennic M. M., Sur 1 Approximation des Fonctions Intégrables sur $[0, 1]$ par des Polynômes de Bernstein Modifiés, J. Approx. Theory, 31(1981), 325-343.

[49] DeVore R. A. and Lorentz G. G., Constructive Approximation, Springer, Berlin, 303(1993).

[50] Dhamija M. and Deo N., Approximation by generalized positive linear-Kantorovich operators, Filomat, 31(14)(2017), 4353-4368.

[51] Dhamija M. and Deo N., Better approximation results by Bernstein-Kantorovich operators, Lobachevskii J. Math., 38(1)(2017), 94-100.

[52] Dhamija M. and Deo N., Jain-Durrmeyer operators associated with the inverse Pólya-Eggenberger distribution, Appl. Math. Comput., 286(2016), 15-22.

[53] Dhamija M., Pratap R. and Deo N., Approximation by Kantorovich form of modified Szász-Mirakyan operators, Appl. Math. Comput., 317(2018), 109-120.

[54] Ditzian Z. and Totik V., Moduli of Smoothness, Springer, New York, 19(1988).

[55] Durrmeyer J.L., Une formule dinversion de la transforme de Laplace, Application la thorie des moments. PhD thesis Facult des Sciences de Universit de Paris, (1967).

[56] Duman O. and Özarslan M. A., Szász–Mirakjan type operators providing a better error estimation. Appl. Math. Lett., 20(12)(2007), 1184-1188.

[57] Farcaş A., An asymptotic formula for Jain's operators, Stud. Univ. Babeş-Bolyai Math., 57(4)(2012), 511-517.

[58] Favard J., Sur les meilleurs procédés d'approximation de certaines classes de fonctions par des polynômes trigonométriques. Bull. sci. math, 61(1937)(209-224), 243-256.

[59] Finta Z., On approximation properties of stancus operators, Stud. Univ. Babeş-Bolyai Math., XLVII(4)(2002), 47-55.

[60] Gadjiev A. D., The convergence problem for a sequence of positive linear operators on unbounded sets and theorems analogous to that of P. P. Korovkin, Dokl. Akad. Nauk SSSR, 218(1974), 1001-1004. Also in Soviet Math Dokl., 15(1974), 1433-1436.

[61] Gadjiev A. D., Theorems of the type of P. P. Korovkin type theorems, Mat. Zametki, 20(5)(1976), 781-786.

[62] Garg T., Acu A. M. and Agrawal P. N., Further results concerning some general Durrmeyer type operators, RACSAM, 113(3)(2019), 2373-2390.

[63] Garg T., Agrawal P. N. and Kajla A., Jain-Durrmeyer operators involving inverse Polya-Eggenberger distribution, Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci., 89(2019), 547-557.

[64] Gavrea I. and Ivan M., Asymptotic behaviour of the iterates of positive linear operators, Abstr. Appl. Anal., 20(11)(2011), 11.

[65] Goldberg S. and Meir V., Minimum moduli of ordinary differential operators, Proc. London Math Soc, 23(3)(1971), 1-15.

[66] Gonska H., Heilmann M. and Raşa I., Kantorovich operators of order k, Numer. Funct. Anal. Optim., 32(7)(2011), 717-738.

[67] Gupta V., On difference of operators with applications to Szász type operators, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas volume, 113(3)(2019), 2059-2071.

[68] Gupta V., Some examples of Genuine approximation operators, Gen. Math., 26(1-2)(2018), 3-9.

[69] Gupta V., Difference of operators of Lupas type, Constructive Math. Anal., 1(1)(2018).

[70] Gupta V., A note on modified Baskakov type operators, Approx. Theory Appl.,10(1994), 74-78.

[71] Gupta V. and Acu A.M., On Baskakov–Szász–Mirakyan–type operators preserving exponential type functions, Positivity, 22(3) (2018), 919-929.

[72] Gupta V. and Agarwal R. P., Convergence Estimates in Approximation Theory, Springer, (2014), Switzerland AG.

[73] Gupta V. and Deo N., On the rate of convergence for bivariate Beta operators, Gen. Math., 13(3)(2005), 109-116.

[74] Gupta V. and Gancho T., A modified post widder operators preserving e^{Ax} Stud. Univ. Babeş-Bolyai Math, 67 (2022), 599-606.

[75] Gupta V. and Greubel G. C., Moment Estimations of new Szász-Mirakyan-Durrmeyer operators, Appl. Math. Comput., 271(15)(2015), 540-547.

[76] Gupta V. and López-Moreno A. J., Phillips operators preserving arbitrary exponential functions, e^{at}, e^{bt}, Filomat, 32(14)(2018), 5071-5082.

[77] Gupta V., Malik N. and Rassias T.M., Moment Generating Functions and Moments of Linear Positive Operators. In: Daras, N., Rassias, T. (eds) Modern Discrete

Mathematics and Analysis, Springer Optimization and Its Applications, 131(2018), Springer, Cham.

[78] Gupta V. and Rassias T. M., Lupaş-Durrmeyer operators based on Pólya distribution, Banach J. Math. Anal., 8(2)(2014), 145-155.

[79] Gupta V., Rassias T. M., Agrawal. P. N. and Acu A. M., Recent Advances in Constructive Approximation Theory, Springer Optimization and Its Applications, Springer, Cham, 138(2018).

[80] Gupta V. and Srivastava G. S., Simultaneous approximation by Baskakov-Szász type operators, Bull. Math. Sci. Math. Roumaine (N. S.), 37(85)(1993), 73-85.

[81] Gupta V., Vasishtha V. and Gupta M K., Rate of convergence of summation-integral type operators with derivatives of bounded variation, J. Inequal. Pure Appl. Math., 4(2)(2003), 1-8.

[82] Holhoş A. The rate of approximation of functions in an infinite interval by positive linear operator. Stud. Univ. Babeş-Bolyai Math., 55(2)(2010), 133-142.

[83] Ibikli E. and Gadjieva E. A., The order of approximation of some unbounded function by the sequences of positive linear operators, Turk J. Math., 19(3)(1995), 331-337.

[84] Ispir N., On modified Baskakov operators on weighted spaces, Turk J. Math., 25(3)(2001), 355-365.

[85] Ispir N., Aral A. and Dogru O., On Kantorovich process of a sequence of the generalized linear positive operators, Numer. Funct. Anal. Optim., 29(5-6)(2008), 574-589.

[86] Ispir N. and Atakut C., Approximation by modified Szász-Mirakjan operators on weighted spaces, Proc Indian Acad. Sci. (Math Sci), 112(4)(2002), 571-578.

[87] Jackson D., On approximation by trigonometric sums and polynomials. Trans. Amer. Math. Soc., 13(4)(1912), 449-515.

[88] Jain G. C., Approximation of functions by a new class of linear operators, J. Austral. Math. Soc., 13(3)(1972), 271-276.

[89] Jain S. and Gangwar R. K., Approximation degree for generalized integral operators, Rev. Un. Mat. Argentina., 50(1)(2009), 61-68.

[90] Johnson N. L. and Kotz S., Discrete Distributions, Houghton-Mifflin, Boston, 1969.

[91] Jolany H., Sharifi H. and Alikelaye R. E., Some results for the Apostol-Genocchi polynomials of higher order. Bull. Malays. Math. Sci. Soc., 36(2)(2013), 465-479

[92] Kantorovich L. V., Sur la convergence de la suite de polynomes de S. Bernstein en dehors de linterval fundamental, Bull. Acad. Sci. USSR, (1931), 1103-1115.

[93] Kanat K. and Sofyalioglu M., On Stancu type Szász-Mirakyan-Durrmeyer Operators Preserving $e^{2ax}, a > 0$, GUJS, (2021), 196-209.

[94] Kajla A., Acu A. M. and Aggarwal P N., Baskakov-Szász type operators on inverse Pólya-Eggenberger distribution, Ann. Funct. Anal, 8(2017), 106-123.

[95] Kajla A., Acu A M. and Agrawal P. N., Baskakov-Szász type operators based on inverse Pólya-Eggenberger distribution, Ann. Funct. Anal., 8(1)(2017), 106-123.

[96] Kajla. A and Agrawal P N., Approximation properties of Szász type operators based on Charlier polynomials, Turk J. Math., 39(6), (2015), 990-1003.

[97] Karlin S. and Zeigler Z., Iteration of positive approximation operators, J. Approx. Theory, 3(1970), 310-319.

[98] King J. P., Positive linear operators which preserve x^2, Acta Math. Hungar, 99(3)(2003), 203-208.

[99] Korovkin P. P., Convergence of linear positive operators in the spaces of continuous functions(Russian), Doklady Akad. Nauk. SSSR(N. N.), 90(1953), 961-964.

[100] Korovkin P. P., Linear Operators and Approximation Theory, Hindustan Publication Co., Delhi (1960).

[101] Lenze B., On Lipschitz-type maximal functions and their smoothness spaces, Nederl. Akad. Wetensch. Indag. Math., 50(1)(1988), 53-63.

[102] Lipi Km. and Deo N., General family of exponential operators, Filomat, 34(12)(2020), 4043-4061.

[103] A -J. López-Moreno and J -M. Latorre-Palacios, Localization results for generalized Baskakov/Mastroianni and composite operators, J. Math. Anal appl., 380(2)(2011), 425-439.

[104] Luo Q -M., Fourier expansions and integral representations for the Genocchi polynomials, J. Integer Seq., 12(2009), 1-9.

[105] Luo Q -M., q-Extensions for the Apostol-Genocchi Polynomials, Gen. Math., 17(2)(2009), 113-125.

[106] Luo Q -M., Extensions of the Genocchi polynomials and their Fourier expansions and integral representations, Osaka J. Math., 48(2011), 291-309.

[107] Luo Q -M., Apostol-Euler polynomials of higher order and Gaussian hypergeometric functions, Taiwanese J. Math., 10(2006), 917-925.

[108] Luo Q -M., Srivastava H. M., Some generalizations of the Genocchi polynomials and the Stirling numbers of the second kind, Appl. Math. Comput., 217(2011), 5702-5728.

[109] Lupas A., The approximation by means of some linear positive operators, Muller M.W., Felten M. and Mache D. H. (eds.) Approximation Theory (Proceedings of the International Dortmund Meeting IDoMAT 95, Held in Witten, March 13-17, 1995), Mathematical Research, vol. 86, 201-229. Akademie Verlag, Berlin (1995)

[110] Magnus W., Oberhettinger F. and Soni R. P., 1966, Formulas and Theorems for the Special Functions of Mathematical Physics, 3*rd* Enlarged ed., Springer-Verlag, New York.

[111] Malik N., Some approximation properties for generalized Srivastava-Gupta operators, Appl. Math. Comput., 269(2015), 747-758.

[112] May C. P., Saturation and inverse theorems for combinations of a class of exponential type operators, Canad. J. Math., 28(1976), 1224-1250.

[113] Mastroianni G., Su un operatore lineare e positivo, Rend. Acc. Sc. Fis. Mat., Napoli, 4(46)(1979), 161-176.

[114] Mastroianni G., Su una classe di operatori e positivi, Rend. Acc. Sc. Fis. Mat., Napoli, 4(48)(1980), 217-235.

[115] Mazhar S. M., Totik V., Approximation by modified Szász operators, Acta Sci. Math., 49(1985), 257-269.

[116] Micchelli C. A., Saturation classes and iterates of operators, Ph.D. Thesis, Stanford University, 100(1969).

[117] Mihesan V., Uniform approximation with positive linear operators generated by generalized Baskakov method, Automat. Comput. Appl. Math., 7(1998), 34-37.

[118] Mishra N. S. and Deo N., Kantorovich variant of Ismail-May Operators, Iran. J. Sci. Technol. Trans. A Sci., (44)3(2020), 739-748.

[119] Mond B., On the degree of approximation by linear positive operators, J. Approx. Theory, 18(1976), 304-306.

[120] Neer T., Ispir N. and Agrawal P. N., Bezier variant of modified Srivastava-Gupta operators, Rev. Un. Mat. Argentina., 58(2)(2017), 199-214.

[121] Nörlund N. E., Vorlesungen über Differentzenrechnung, Springer-Verlag, Berlin, 1924, Reprinted by Chelsea Publishing Company, Bronx, New York,1954.

[122] Özarslan M. A. and Aktuglu H., Local approximation for certain King-type operators, Filomat, 27(2013), 173-181.

[123] Özarslan M. A. and Duman O., Local approximation behaviour of modified SMK operators, Miskolc Math. Notes, 11(1)(2010), 87-99.

[124] Ozden H. and Simsek Y., Modification and unification of the Apostol-type numbers and polynomials and their applications, Appl. Math. Comput., 235(2014), 338-351.

[125] Ozsarac F. and Acar T., Reconstruction of Baskakov operators preserving some exponential functions. Math. Methods Appl. Sci., 42(16)(2019), 5124-5132.

[126] Păltănea R., Modified Szász-Mirakjan operators of integral form, Carpathian J. Math., 24(3)(2008), 378-385.

[127] Pethe S., On the Baskakov operator, Indian J. Math., 26(1984), 43-48.

[128] Pop Ovidiu T., Bărbosu D. and Miclăuş D., The Voronovskaja type theorem for an extension of Sász-Mirakjan operators, Demonstr. Math., 45(1)(2012), 107-115.

[129] Prakash C., Verma D. K. and Deo N., Approximation by a new sequence of operators involving Apostol-Genocchi polynomials, Math. Slovaca, 5(71)(2021), 1179-1188.

[130] Pratap R. and Deo N., Rate of convergence of Gupta-Srivastava operators with certain parameters, J. Class. Anal., 14(2),(2019), 137-153.

[131] Pratap R. and Deo N., Approximation by genuine Gupta-Srivastava operators, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, 113(3)(2019), 2495-2505.

[132] Razi Q., Approximation of a function by Kantorovich type operators, Mat. Vesnic., 41(1989), 183-192.

[133] Sahai A. and Prasad G., On simultaneous approximation by Modified Lupas operators, J. Approx. Theory, 45(1985), 122-128.

[134] Sándor J. and Crstici B., Handbook of Number Theory, Kluwer Academic Publishers, Dordrecht, Boston, London., 2(2004).

[135] Schumaker L. L., Spline Functions, Basic Theory, New-York, John Wiley & Sons, (1981).

[136] Sofyalioğlu M. and Kanat K., Approximation properties of generalized Baskakov–Schurer–Szász–Stancu operators preserving $e^{-2ax}, a > 0$, Inequal. Appl. , 112(2019), 1-16.

[137] Srivastava H. M., Some formulas for the Bernoulli and Euler polynomials at rational arguments, Math. Proc. Cambridge Philos. Soc., 129(2000), 77-84.

[138] Srivastava H. M. and Choi J., Series Associated with the Zeta and Related Funcions, Kluwer Academic Publishers, Dordrecht, Boston, London(2001).

[139] Srivastava H. M. and Gupta V., A certain family of summation-integral type operators, Math. Comput. Modelling, 37(2003), 1307-1315.

[140] Srivastava H. M., Özarslan M. A. and Kaanoğlu C., Some generalized Lagrange-based Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Russ. J. Math. Phys., 20(1)(2013), 110-120.

[141] Srivastava H. M. and Pintér Á., Remarks on some relationships between the Bernoulli and Euler polynomials, Appl. Math. Lett., 17(2004), 375-380.

[142] Stancu D. D., Two classes of positive linear operators, Anal. Univ. Timişoara, Ser. Ştin. Matem., 8(1970), 213-220.

[143] Szász O., Generalization of S. Bernstein's polynomials to the infinite interval. J. Research Nat. Bur. Standards 45(1950), 239-245.

[144] Tachev G., On two modified Phillips operators, Stud. Univ. Babeş-Bolyai Math., 64(3)(2019), 305-312.

[145] Tarabie S., On Jain-Beta linear operators, Appl. Math. Inf. Sci., 6(2)(2012), 213-216.

[146] Verma D. K., Gupta V. and Agrawal P. N., Some approximation properties of Baskakov-Durrmeyer-Stancu operators, Appl. Math. Comput., 218(11)(2012), 6549-6556.

[147] Weierstrass K., Über die analytische Darstellbarkeit sogenannter willkürlicher Functioneneiner reellen Veränderlichen, Sitzungsberichteder der Koniglich preussischen Akademie der Wissenschcaften zu Berlin., (1885), 789-805.

[148] Yilmaz Ö. G., Bodur M. and Aral A., On approximation properties of Baskakov–Schurer–Szász operators preserving exponential functions, Filomat, 32(15)(2018), 5433-5440.

[149] Yilmaz Ö. G., Gupta V. and Aral A., A note on Baskakov–Kantorovich type operators preserving e^{-x}, Math. Methods Appl. Sci., (2018), 1-7.

List of Publications

1. Neha and Naokant Deo. Integral Modification of Apostol-Genocchi Operators. *Filomat* 2021: 35(8): 2533-2544.
   SCIE, Impact Factor-0.988

2. Neha and Naokant Deo. On Approximation of Functions with Exponential Growth by using Modified Lupaş-Kantrovich Operators. *Numerical Functional Analysis and Optimization* 2023: 44(14): 1510-1522.
   SCIE, Impact Factor-1.30

3. Neha and Naokant Deo. Generalization of parametric Baskakov Operators based on the I-P-E Distribution, *Filomat*.
   SCIE, Impact Factor-0.988, **Accepted.**

4. Neha and Naokant Deo. Integral Modification of Beta-type Apostol-Genocchi Operators. *Mathematical Foundations of Computing* 2023: 6(3): 474-483.
   ESCI and Scopus

5. Neha and Naokant Deo. Convergence and Difference Estimates between Mastroianni and Gupta Operators. *Kragujevac Journal of Mathematics* 2023: 47(2): 259-269.
   ESCI and Scopus

6. Neha, Naokant Deo and Ram Pratap. Bézier Variant of Summation-Integral type Operators. *Rendiconti del Circolo Matematico di Palermo Series 2* 2022: 72: 889-900.
   ESCI and Scopus

7. Neha and Naokant Deo. Iterative Combinations of Generalised Operators, *Matematicki Vesnik*.
   ESCI and Scopus, Accepted

Papers Presented in International Conferences

1. Jain-Durrmeyer Operators Involving Apostol-Genocchi Polynomial in the *International Conference on Emerging Trends in Pure and Applied Mathematics* held on 12-13 in blended mode organized by the Department of Applied Sciences, School of Engineering in association with Department of Mathematical Sciences, School of Science, Tezpur University.

2. On the Approximation of Functions with Exponential Growth by Using Modified Lupaş-Kantrovich Operators in the *International Conference on Mathematical Analysis and Applications* held on December 15-17, 2022. in blended mode organized by the Department of Mathematics, National Institute of Technology Tiruchirappalli-620015, Tamil Nadu, India.