ROLE OF MULTIPARTITE ENTANGLED STATES IN QUANTUM TELEPORTATION

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in

MATHEMATICS

By

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DECLARATION

I, the undersigned, hereby declare that the research work reported in this thesis entitled "**Role of Multipartite Entangled States in Quantum Teleportation**" for the award of the degree of *Doctor of Philosophy in Mathematics* has been carried out by me under the supervision of *Dr. Satyabrata Adhikari*, Department of Applied Mathematics, Delhi Technological University, Delhi, India.

The research work embodied in this thesis, except where otherwise indicated, is my original research. This thesis has not been submitted by me earlier in part or full to any other University or Institute for the award of any degree or diploma. This thesis does not contain other person's data, graphs, or other information unless specifically acknowledged.

Anuma

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CERTIFICATE

On the basis of a declaration submitted by **Ms. Anuma Garg**, Ph.D. scholar, I hereby certify that the thesis titled **"Role of Multipartite Entangled States in Quantum Teleportation"** submitted to the Department of Applied Mathematics, Delhi Technological University, Delhi, India for the award of the degree of *Doctor of Philosophy in Mathematics*, is a record of bonafide research work carried out by her under my supervision.

I have read this thesis and, in my opinion, it is fully adequate in scope and quality as a thesis for the degree of Doctor of Philosophy.

To the best of my knowledge, the work reported in this thesis is original and has not been submitted to any other Institution or University in any form for the award of any Degree or Diploma.

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Dedicated to My Parents

(Manisha Garg and Ravindra Kumar Garg)

For their love, guidance, selfless sacrifices, and constant support

Contents

Preface

Quantum information science is a rapidly expanding field. Entanglement can be considered as heart of the quantum information theory. Although much research has been done on two-qubit entanglement, very little is known about multipartite entanglement. A better understanding of multipartite entanglement aids our understanding of the many-body system. There are many unanswered questions in the theory of multipartite systems. This is due to the complex structure of the multipartite system. The complexity of a multi-qubit system grows in proportion to the number of qubits and the dimension of the system. Entanglement is a quantum mechanical property that can be used as a resource in computational and communication tasks. It is essential in many information processing protocols, including quantum cryptography, quantum superdense coding, and quantum teleportation. Although entangled states are useful in various quantum information processing tasks, the practical use of an entangled resource is restricted to the successful experimental realization of the resource. Non-locality is another quantum mechanical phenomenon which is not same as the entanglement. Although non-locality and quantum entanglement go hand in hand and they correspond to quantum correlation present in quantum states conceptually, they are very much distinct. At the level of a two-qubit entangled state and also for higher dimensional bipartite and multipartite entangled quantum states, it is possible to obtain more non-locality with less entanglement. Thus, we may expect that non-local states with less entanglement may be more useful as resource states. Many novel applications of non-locality have been developed for quantum computation and communication, including communication complexity, and quantum cryptography. Quantum teleportation is an important topic to study in quantum information science. It plays a vital role in the development of quantum information theory and quantum technologies. Bennett et. al. have developed the first protocol of quantum teleportation for two-qubit system. By using quantum teleportation protocol, we can send information encoded in a quantum state in a more efficient way than the existed classical protocols. The efficiency of the quantum teleportation protocol can be measured through the fidelity of a state known as teleportation fidelity. Another form of quantum teleportation is known as controlled quantum teleportation (CQT). CQT works perfectly for a three-qubit state shared between Alice, Bob and, Charlie where the third party Charlie acts as a controller. The importance of the CQT lies in the fact that in the CQT protocol, the controller has the power to enhance the teleportation fidelity of the two-qubit state possessed by Alice and Bob by performing measurements on his qubit. Here, we have derived a different form of criteria, which is based on the maximum eigenvalue, for the detection of entangled state useful in quantum teleportation. The developed criterion may also be implementable in an experiment. Then, we have extensively studied the non-locality of the two-qubit state by defining a quantity that measures the strength of the non-locality. Later, we considered the three-qubit state (pure/mixed) and studied the non-locality of its reduced two-qubit state with the power of the controller of the three-qubit state in controlled quantum teleportation. Also, we have connected the non-locality of the three-qubit state with the non-locality of the two-qubit state by deriving the upper bound and lower bound of the Svetlichny operator. The derived state dependent bounds may be used to detect the genuine non-locality of any general three-qubit quantum state. Thus, the detection of genuine non-locality guarantees that the three-qubit state is genuinely entangled. At the end of the thesis, we studied the controlled quantum teleportation protocol using a three-qubit state and derived the lower bound of the controller's power in terms of the introduced witness operator. Thus, our study may help to estimate the power of the controller in an experiment.

Chapter 1 is introductory in nature. Chapters 2-5 are based on the research work published/communicated in the form of research papers in reputed journals. Finally, we conclude the thesis with future scope and references. Each chapter begins with a brief outline of the work carried out in that chapter.

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List of symbols

V	Vector space
$\mathbb{F}$	Field of complex or real numbers
D	Linear operator
$M_{m,n}$	Matrix of order $m \times n$
$\|\|.\|\|$	Norm
$\langle .,. \rangle$	Inner product
$\mathscr{H}$	Hilbert space
T	Transpose
$\dagger$	Conjugate transpose
$\otimes$	Tensor product
PT	Partial transposition
T_B	Partial transpose with respect to a subsystem B
$\|.\rangle$	Ket vector
$\langle .\|$	Bra vector
$\sigma_x,\ \sigma_y,\ \sigma_z$	Pauli matrices
F	Singlet fraction
f	Teleportation fidelity
N	Negativity
C	Concurrence
f_{NC}	Non-Conditioned fidelity
f_C	Conditioned fidelity
$P/P_{CT}^{(k)}$	Controller's power
S_{NL}	Strength of non-locality
S_v	Svetlichny operator

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Chapter 1

General Introduction

"As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality."

- Albert Einstein

The introduction gives an overview of the basic definitions, and concepts of linear algebra and a few recapitulations of quantum mechanics. We then study the theory of entanglement, in particular, bipartite and tripartite entanglement. We recapitulate the theory of non-locality, in which the bipartite and tripartite non-locality is discussed. In bipartite non-locality, we discussed the concept of non-locality using inequality. We have considered the Mermin's inequality and Svetlichny inequality to discuss the non-locality in tripartite system. A brief review of quantum communication protocols, in particular, quantum teleportation and controlled quantum teleportation is discussed. In the last section, we emphasized on the non-locality in quantum communication.

1.1 Basics of linear algebra

Linear algebra is the study of vector spaces and operations defined on it. In this section, we recall a few fundamental terms of linear algebra which would be needed for a better understanding of quantum information theory [1–3].

Vector space

A non-empty set V with two binary operations "addition" and "multiplication" denoted by "+" and "." respectively is said to be a vector space over a field $\mathbb{F}$ if it satisfies the following axioms:

1. **Closed under addition:** $u+v \in V$, $\forall\, u,v \in V$
2. **Addition is associative:** $(u+v)+w = u+(v+w)$, $\forall\, u,v,w \in V$
3. **Additive identity:** There exist a zero element $0 \in V$ such that $0+u = u$, $\forall\, u \in V$
4. **Additive inverse:** For every vector $u \in V$, there exist a vector $-u \in V$ such that $u+(-u) = 0$
5. **Addition is commutative:** $u+v = v+u$, $\forall\, u,v \in V$
6. **Closed under multiplication:** $\alpha.u \in V$, $\forall\, \alpha \in \mathbb{F}$ and $u \in V$
7. **Multiplicative Identity:** There exist a unity element 1 $\in V$ such that $1.u = u$, $\forall$ $u \in V$
8. $\alpha.(u+v) = \alpha.u+\alpha.v$, $\forall\, \alpha \in \mathbb{F}$ and $\forall\, u,v \in V$
9. $(\alpha+\beta).u = \alpha.u+\beta.u$, $\forall\, \alpha,\beta \in \mathbb{F}$ and $\forall\, u \in V$
10. $(\alpha.\beta).u = \alpha.(\beta.u)$, $\forall\, \alpha,\beta \in \mathbb{F}$ and $\forall\, u \in V$

Linear dependence and linear independence

A set of finite non-zero vectors v_1, v_2,...,v_n of a vector space V over a field $\mathbb{F}$ is said to be linearly independent if there exist scalars $\alpha_1,\alpha_2,\dots,\alpha_n \in \mathbb{F}$ such that $\alpha_1 v_1+\alpha_2 v_2+\dots+\alpha_n v_n = 0$ implies $\alpha_i = 0,\ \forall\ i$. Otherwise, the set is linearly dependent.

Basis and dimension

Let V be a vector space over a field $\mathbb{F}$.

(a) **Basis:** A set $S = \{v_1, v_2, \dots, v_n\} \subseteq V$ is said to be a basis if

(i) $v_1, v_2, ..., v_n$ are linearly independent vectors.

(ii) S spans the vector space V over field $\mathbb{F}$.

(b) **Dimension:** The number of elements in the basis set is known as the dimension of the vector space V.

Linear operator

A linear operator D is a function defined as $D : V \to V$, where V is a vector space over a field $\mathbb{F}$ such that

$$D(\alpha u + \beta v) = \alpha D(u) + \beta D(v), \forall\ u, v \in V \text{ and } \forall\ \alpha, \beta \in \mathbb{F} \tag{1.1.1}$$

Matrix representation of an operator

A linear operator D may be represented in the form of a matrix. Let D be a linear operator from V_1 into V_2, where V_1 and V_2 are two vector spaces of dimension n and m respectively. Let $v_j (j = 1, \ldots, n)$ and $w_i (i = 1, \ldots, m)$ be the basis elements of vector spaces V_1 and V_2 respectively. Then a linear operator D can be represented by $(m \times n)$ matrix as

$$D(v_j) = a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{mj}w_m = \sum_i^m a_{ij}w_i, \quad j \in \{1, 2, \ldots, n\} \tag{1.1.2}$$

where a_{ij} are the elements of matrix representation of operator D.

Transpose and conjugate transpose of a matrix

For a given matrix $A = \sum_i \sum_j a_{ij}$, the interchanging of rows of the matrix into columns and the columns into rows is known as transpose of a matrix and is denoted by $A^T = \sum_j \sum_i a_{ji}$. The conjugate transpose of a given matrix $A = \sum_i \sum_j a_{ij}$ is defined by applying the complex conjugate of every element of the matrix and then interchanging rows with columns of a matrix or vice versa gives the conjugate transpose of a matrix and is denoted as $A^\dagger = \sum_j \sum_i a^*_{ji}$.

Eigenvalue and eigenvector

Eigenvalue: A linear operator D defined on a vector space V over the field $\mathbb{F}$. A scalar $\lambda \in \mathbb{F}$ is called eigenvalue of D if there is a non-zero vector $v \in V$ such that

$$Dv = \lambda v \tag{1.1.3}$$

The equation (1.1.3) is called eigenvalue equation.

Eigenvector: If λ is an eigenvalue of D, then any non-zero vector corresponding to it is called the eigenvector of D.

Remark: The lower and upper bound of the maximal eigenvalue of a Hermitian operator has been obtained by using various matrix norms. Here, we provide a result known as Dembo's bound [4, 5] for obtaining the lower and upper bound of the maximal eigenvalues of any positive-semidefinite Hermitian operator.

Dembo's bound: For any $n \otimes n$ Hermitian positive semi-definite operator R_n with eigenvalues $\lambda_1 \leq \lambda_2 \leq ..... \leq \lambda_n$, Dembo's bound [4, 5] is given by

$$\frac{c+\eta_1}{2} + \sqrt{\frac{(c-\eta_1)^2}{4} + (b^*)^T b} \leq \lambda_n(R_n) \leq \frac{c+\eta_{n-1}}{2} + \sqrt{\frac{(c-\eta_{n-1})^2}{2} + (b^*)^T b} \quad (1.1.4)$$

where $R_n = \begin{pmatrix} R_{n-1} & b \\ (b^*)^T & c \end{pmatrix}$, η_1 is the lower bound on minimal eigenvalue of R_{n-1}, η_{n-1} is the upper bound on maximal eigenvalue of R_{n-1} and b is a vector of dimension $n-1$.

Inner product space

Let *V* be a vector space over a field $\mathbb{F}$. A function which assigns each ordered pair of vectors to an element of the field, defined as

$$\langle u, v \rangle : V \times V \to \mathbb{F}, \quad u, v \in V \quad (1.1.5)$$

is said to be an inner product if it follows the following conditions

1. **Linearity**: $\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle, \ \forall \ u, v, w \in$ *V* and $\forall \ \alpha, \beta \in \mathbb{F}$.
2. **Conjugate symmetry**: $\langle u, v \rangle = \overline{\langle v, u \rangle}$, $\forall \ u, v \in$ *V*
3. $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ if and only if $u = 0$, for any $u \in$ *V*

Any vector space *V* over field $\mathbb{F}$ associated with an inner product is known as an inner product space.

Orthogonal vectors

Let u and v be two vectors of a vector space *V* over the field $\mathbb{F}$. The vector u is said to be orthogonal to v, if the inner product of u and v is zero, i.e., $\langle u, v \rangle = 0$.

Norm

A norm is a real-valued function defined on a vector space *V* over a field $\mathbb{F}$ as

$$||.|| : V \to \mathbb{F} \quad (1.1.6)$$

$||.||$ satisfies the following properties:

1. **Triangular Property**: $\|u+v\| \leq \|u\| + \|v\|, \forall\ u,v \in V$

2. **Non-Negativity**: $\|v\| \geq 0, \forall\ v \in V$. $\|v\| = 0$ if and only if $v = 0$.

3. $\|\alpha v\| = |\alpha|\|v\|, \forall\ v \in V$ and $\forall\ \alpha \in \mathbb{F}$

Unit vector

If the norm of a vector is unity, then the vector is known as a unit vector.

Orthonormal vectors

A set of vectors is known as orthonormal vectors if the inner product of any vector u with itself is 1, i.e., $\langle u,u \rangle = 1$ and the inner product of any vector u with any other distinct vector v is 0, i.e., $\langle u,v \rangle = 0$.

Norm and inner product of two operators

(a) **Inner product of two operators:** For any two finite dimensional linear operators D_1 and D_2, the inner product of D_1 and D_2 is defined as

$$\langle D_1, D_2 \rangle = Tr[D_1^\dagger D_2] \tag{1.1.7}$$

(b) **Norm of a operator:** Norm of a linear operator D is defined as

$$\|D\| = \sqrt{\langle D, D \rangle} \tag{1.1.8}$$

Normed linear space

A vector space V is called normed linear space if for every element $v \in V$, there is a unique real number associated with it, which may be the norm of v.

Banach space

A complete normed linear space is known as Banach space.

Hilbert space

A complete inner product space is called Hilbert space.

Remark: A Banach space is a Hilbert space if and only if the parallelogram law holds.

Tensor product

Let V_1 and V_2 be two vector spaces over a field $\mathbb{F}$ of dimension m and n respectively. The tensor product of V_1 and V_2 is denoted by $V_1 \otimes V_2$ and it is of dimension mn.

Suppose A be a matrix of order $m \times n$ and B be a matrix of order $p \times q$. Therefore, the matrix A and B may be represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \; B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pq} \end{bmatrix} \tag{1.1.9}$$

The matrix representation of $A \otimes B$ is given as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \tag{1.1.10}$$

It is now clear from (1.1.10) that $A \otimes B$ represent a matrix of order $mp \times nq$.

Properties of tensor product:

If $A, B, C, D \in M_{m,n}$ and $\alpha \in \mathbb{F}$ then

1. $(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha (A \otimes B)$
2. $(A \otimes B)^{\dagger} = A^{\dagger} \otimes B^{\dagger}$
3. $(A + B) \otimes C = A \otimes C + B \otimes C$
4. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
5. $(A \otimes B)(C \otimes D) = AC \otimes BD$
6. $Tr(A \otimes B) = Tr(B \otimes A)$
7. $det(A \otimes B) = det(B \otimes A)$
8. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$
9. $I_m \otimes I_n = I_{mn}$, where I_m and I_n are identity matrix of order m and n respectively.

Partial transpose of a matrix

If a matrix X of order $d^2 \times d^2$ is subdivided into d^2 blocks of order $d \times d$ then the partial transposition of a matrix X is obtained by performing the transposition operation on every block of order d of the matrix X.

To illustrate it, let us consider a 4×4 matrix X, which is given in the block matrix form as

$$X = \left(\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right)$$

The partial transposition of the matrix X may be given as

$$X^{PT} = \left(\begin{array}{cc|cc} a_{11} & a_{21} & a_{13} & a_{23} \\ a_{12} & a_{22} & a_{14} & a_{24} \\ \hline a_{31} & a_{41} & a_{33} & a_{43} \\ a_{32} & a_{42} & a_{34} & a_{44} \end{array}\right)$$

Remark:
(i) If X^{PT} has no negative eigenvalue, i.e., if it is positive semidefinite, then the matrix X is said to be **positive partial transpose**
(ii) If X^{PT} has atleast one negative eigenvalue then the matrix X is said to have **negative partial transpose**.

1.1.1 A few results in linear algebra

Result 1.1: For any two Hermitian $d \times d$ matrices A and B, we have [6]

$$\lambda_{min}(A)Tr(B) \leq Tr(AB) \leq \lambda_{max}(A)Tr(B) \tag{1.1.11}$$

where the eigenvalues of A are arranged as $\lambda_{min} = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq .... \leq \lambda_d = \lambda_{max}$.
Result 1.2: If M be any $n \times n$ complex matrix and N be any $n \times n$ Hermitian matrix, then we have [2, 6]

$$\lambda_{min}(\overline{M})Tr(N) \leq R(Tr(MN)) \leq \lambda_{max}(\overline{M})Tr(N) \tag{1.1.12}$$

where $\overline{M} = \frac{M+M^{\dagger}}{2}$ and $R(x)$ denotes the real part of x.
Proof:- Let us assume that the eigenvalues of $\overline{M}$ may be arranged in an ascending order as $\lambda_{min} = \lambda_1 \leq \lambda_2 \leq ...... \leq \lambda_n = \lambda_{max}$. To prove Result 1.2, let us recall the lower

and upper bound of $R(Tr(MN))$, which is given in [6],

$$\sum_{i=1}^{n} \lambda_i(\overline{M})\lambda_{n-i+1}(N) \leq R(Tr(MN)) \leq \sum_{i=1}^{n} \lambda_i(\overline{M})\lambda_i(N) \tag{1.1.13}$$

In L.H.S., Replacing all the eigenvalues of $\overline{M}$ by its minimum eigenvalues and in R.H.S. if we replace all the eigenvalues of $\overline{M}$ by its maximum eigenvalue, we get the desired result given in (1.1.12).

Result 1.3: If M be any $n \times n$ complex matrix and N be any $n \times n$ Hermitian matrix, then we have

$$Tr(\overline{M})\lambda_{min}(N) \leq R(Tr(MN)) \leq Tr(\overline{M})\lambda_{max}(N) \tag{1.1.14}$$

where $\overline{M} = \frac{M+M^\dagger}{2}$ and $R(x)$ denotes the real part of x.

Result 1.4: If M be any $n \times n$ complex matrix and N be any $n \times n$ Hermitian matrix, then we have

$$Tr(\overline{M})\lambda_k(N) \leq R(Tr(MN)) \tag{1.1.15}$$

where $\lambda_k(N)$ denote the first non-zero eigenvalue of N.

1.2 Origin of quantum mechanics

Until the end of the 19th century, classical mechanics appeared to be sufficient to explain all the physical phenomena, but it fails to account for thermodynamic equilibrium between matter and radiation, which may be considered as the root of classical mechanics. In the late nineteen century and the first quarter of the twentieth century, it was found that the amount of energy radiated from black body radiation calculated experimentally does not match with the theoretical result obtained by using the theory of classical mechanics. Thus, we require a new concept to explain the difference obtained in the theoretical and experimental results. Many attempts were made in different ways to explain the difference but all failed until a revolutionary theory called quantum mechanics was introduced. The theory of quantum mechanics was developed between the years 1925 and 1930. Quantum mechanics provides a consistent description of matter on the microscopic scale and can be considered one of the greatest intellectual achievements of the twentieth century. Two equivalent approaches to the theory were proposed at nearly the same time. The first is known as matrix mechanics developed by W. Heisenberg, M. Born, and P. Jordan, and the second is known as wave mechanics which was proposed by E. Schrodinger [7]. Now, we shall present the principles of quantum mechanics in a more general way, as a set of postulates.

1.2.1 Postulates of quantum mechanics

The postulates of quantum mechanics map a mathematical theory to physical systems. Hence, they are the result of years of discoveries made by experimenters as well as theoreticians in the field of quantum mechanics. However, this also means that the postulates may still undergo more or less minor changes over time. Currently, the postulates may be formulated in the following form [1].

1. **State space:** Any isolated physical quantum system is associated with a Hilbert space, which may be called *state space* in quantum mechanics. In state space, the physical system is completely described by a vector known as *state vector*. A quantum state is a column vector of a Hilbert space $\mathscr{H}$ which is denoted as $|v\rangle$ and $\langle v|$ denote its dual vector. If $\mathscr{H}$ denotes a two dimensional Hilbert space spanned by $\{|0\rangle, |1\rangle\}$ vectors then the basis vectors $\{|0\rangle, |1\rangle\}$ can be represented as

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.2.1}$$

In general, a quantum state in a d-dimensional Hilbert space can be represented as

$$|\psi\rangle = \sum_{i=1}^{d} b_i |k_i\rangle \tag{1.2.2}$$

where k_i denote the basis of the Hilbert space $\mathscr{H}$ and b_i are the scalars belonging to field $\mathbb{F}$.

Since every vector in a Hilbert space is represented as a column vector, a d-dimensional quantum state can also be represented as a column vector given by

$$|\psi\rangle = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

2. **Evolution:** If $|\psi\rangle$ represents a state in a closed quantum system, then its evolution is described by the unitary operator. A state $|\psi\rangle$ at time t_1 is related to state $|\psi^{'}\rangle$ at time t_2 by a unitary operator $U(t_1,t_2)$, which can be expressed as

$$|\psi^{'}(t_2)\rangle = U(t_1,t_2)|\psi(t_1)\rangle \tag{1.2.3}$$

3. **Measurements:** Measurements on a quantum system in Hilbert space $\mathscr{H}$ are described by a collection of measurement operators from set $\{M_i\}$ with completeness relation

$$\Sigma_{i=1} M_i^{\dagger} M_i = I \tag{1.2.4}$$

   These operators $\{M_i\}$ act on the state space that is being measured, and the index i refers to the measurement outcome that occurs after measurement.

4. **Composite system:** The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. If we consider n quantum systems $|\psi_1\rangle, |\psi_2\rangle, ..., |\psi_n\rangle$, then the joint state of the composite system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$.

1.2.2 Formalism of quantum mechanics

The postulates of quantum mechanics are the basic principles that help to develop the mathematical formalism of it. They play an important role in understanding the behavior of a physical system.

1.2.2.1 Observables

In quantum mechanics, observables are dynamical variables like energy, momentum, position, and angular momentum, which can be measured. Every dynamical variable is represented by Hermitian operator. The requirement of the Hermitian operator is due to the fact that it has real eigenvalues.

Let us consider the eigenvalue equation [8]

$$A|\phi\rangle = \lambda|\phi\rangle \tag{1.2.5}$$

where Hermitian operator A represents a physical system, λ denote the real eigenvalue and $|\phi\rangle$ is the corresponding eigenvector. In general, if we assume that $|a_i\rangle$ denote the eigenstate of the Hermitian operator A then the eigenvalue equation is given by

$$A|a_i\rangle = a_i|a_i\rangle \tag{1.2.6}$$

If the system is assumed to be the superposition of eigenstates of A, then an arbitrary normalized state $|\alpha\rangle$ can be represented as

$$|\alpha\rangle = \sum_i c_i|a_i\rangle, \quad \sum_i |c_i|^2 = 1 \tag{1.2.7}$$

When a measurement is performed on a physical system $|\alpha\rangle$ then it collapses to one of the eigenstates $|a_i\rangle$ and its probability is given by

$$|c_i|^2 = |\langle a_i|\alpha\rangle|^2 \tag{1.2.8}$$

1.2.2.2 Expectation value of an observable in quantum mechanics

The expectation value of a Hermitian operator A of an arbitrary state $|\alpha\rangle$ is denoted by $\langle A\rangle$, and is defined as $\langle\alpha|A|\alpha\rangle$. If $|a_i\rangle' s$ represent the orthonormal eigenstates of A corresponding to the eigenvalues $\lambda_i' s,\ i = 1, 2, \ldots$ then we have

$$\langle a_i|A|a_j\rangle = \lambda_j\langle a_i|a_j\rangle = \lambda_i\delta_{ij} \tag{1.2.9}$$

The expectation value of A can be expressed in terms of its eigenvalues as

$$\begin{aligned}
\langle A\rangle = \langle\alpha|A|\alpha\rangle &= \sum_i\sum_j \langle\alpha|a_i\rangle\langle a_i|A|a_j\rangle\langle a_j|\alpha\rangle \\
&= \sum_i\sum_j \langle\alpha|a_i\rangle\lambda_j\delta_{ij}\langle a_j|\alpha\rangle \\
&= \sum_j \lambda_j\langle\alpha|a_j\rangle\langle a_j|\alpha\rangle \\
&= \sum_j \lambda_j|\langle a_j|\alpha\rangle|^2
\end{aligned} \tag{1.2.10}$$

In the second step, we have used the completeness relation, which is given by

$$\sum_i |a_i\rangle\langle a_i| = 1 \tag{1.2.11}$$

1.2.2.3 Projection operator

Consider a Hilbert space H spanned by the complete orthonormal eigenvectors $|a_i\rangle' s$ of a Hermitian operator A. An arbitrary normalized state $|\psi\rangle \in H$ can be represented as

$$|\psi\rangle = \sum_i c_i |a_i\rangle, \quad \sum_i |c_i|^2 = 1 \tag{1.2.12}$$

If the operator $P_1 = |a_1\rangle\langle a_1|$ is applied on an arbitrary state $|\psi\rangle$, then the operator P_1 project $|\psi\rangle$ on one of its eigenstates $|a_1\rangle$. This type of operator is known as a projection operator.
In general, for the i^{th} eigenstate $|a_i\rangle$, we can have the projection operator P_i which is given by

$$P_i = |a_i\rangle\langle a_i|, \; i = 1, 2, \ldots \tag{1.2.13}$$

Completeness condition gives $\sum_i P_i = I$. Since $\langle a_i|a_j\rangle = \delta_{ij}$, then

$$P_i P_j = \delta_{ij} P_j, \text{ where } \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1.2.14}$$

Therefore, the projection operator can be considered as an idempotent operator and its eigenvalues are only 0 and 1 respectively.

1.2.2.4 Commuting observables

The commutator between two operators A and B can be defined as

$$[A, B] = AB - BA \tag{1.2.15}$$

If $AB = BA$, i.e., $[A, B] = 0$, we say A commutes with B. Two operators A and B are compatible if they commute, i.e., $[A, B] = 0$. In other words, two observables are compatible when their corresponding Hermitian operators commute. These two operators are called commuting sets of observables. If $[A,B] \neq 0$, then the operators are incompatible. If two operators A and B commute, then they are simultaneously measurable.

1.2.2.5 Pauli matrices

Pauli matrices is a set of unitary and Hermitian operators which are represented in the form of 2×2 complex matrices. The representation of Pauli matrices are as follows

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.2.16}$$

Properties of Pauli matrices:

1. $\sigma_x^\dagger = \sigma_x$, $\sigma_y^\dagger = \sigma_y$, $\sigma_z^\dagger = \sigma_z$.
2. Eigenvalues of all Pauli matrices belong to the set $S = \{\pm 1\}$.
3. The determinant of Pauli matrices is -1.
4. $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$.
5. $Tr(\sigma_x) = Tr(\sigma_y) = Tr(\sigma_z) = 0$.

1.2.2.6 Heisenberg uncertainty relation

The uncertainty principle is a fundamental concept in quantum mechanics that limits our ability to simultaneously determine certain pairs of properties such as the position and momentum of quantum particles. In other words, if we intend to measure more accurately one of these properties, then the accuracy of measuring the other property will become less. Let A and B be two Hermitian operators that are incompatible, i.e., $AB \neq BA$. By the Heisenberg uncertainty principle, the Hermitian operators A and B are not simultaneously measurable. So, we can find an error in our measurement. The difference between the product AB and BA of two incompatible operators A and B is given by

$$AB - BA = [A,B] = iC \tag{1.2.17}$$

where C is another Hermitian operator.

If the error occurred during the measurement is measured by the standard deviations $\triangle A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$ and $\triangle B = \sqrt{\langle B^2 \rangle - \langle B \rangle^2}$ of the incompatible operators A and B then the uncertainty principle may be expressed as

$$\triangle A \triangle B \geq \frac{\langle C \rangle}{2} \tag{1.2.18}$$

1.3 Information theory

Gaining knowledge about anything through some process is known as information. It is an essential component of the transmission of messages or knowledge and it was being used with or without awareness in our everyday life long before computers made their way into our life [9]. An event contains some information, if there is a non-zero probability (less than unity) of happening the event. Therefore, if the event is certain then it contains no information. Thus, information about any event may be measured by the amount of uncertainty of happening the event. Alternatively, the information in the random variable X can be expressed as the expectation value of $X's$ unexpectedness. The unexpectedness of the event x may be defined as $-log(p_x)$, where p_x is the probability of occurring an event x.
When information is entered into and stored in a computer, it is generally referred to as data. After processing the input data, we obtain the output data that again can be perceived as information. When information is compiled or used to better understand something or to do something, it becomes knowledge. Information theory deals with the study of data and storing and communicating data. In the present scenario, we can divide the information theory into two parts: (i) Classical information theory and (ii) Quantum information theory.

1.4 Quantum information theory

Quantum information theory is an interdisciplinary subject that deals with the transfer of information using the principles of quantum mechanics. It extends classical information theory to the quantum world, where quantum properties such as superposition and entanglement can be used. Therefore, quantum information theory may provide a way to overcome the limitations of classical information theory.

1.4.1 Basics of quantum information theory

1.4.1.1 Qubit

Quantum bit or qubit is the fundamental unit of quantum information. A qubit is nothing but it is a superposition of $|0\rangle$ and $|1\rangle$. Mathematically, a qubit can be expressed as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,\ |\alpha|^2 + |\beta^2| = 1 \tag{1.4.1}$$

where α and β are two complex numbers.

1.4.1.2 Density operator

An operator ρ which is associated with some ensemble $(p_i, |\psi_i\rangle)$ is a density operator if and only if it satisfies the following conditions:

1. Trace condition: $Tr(\rho) = 1$
2. Positive semi-definiteness: ρ is a positive semi-definite operator, i.e. $\rho \geq 0$. This implies that eigenvalues of ρ are either positive or zero.
3. Hermitianity: ρ must be a Hermitian operator, i.e. $\rho^\dagger = \rho$.

1.4.1.3 Pure and mixed state

Pure state: A pure state is described by the projector. A state described by the density operator ρ is said to be pure if it satisfies

$$Tr(\rho^2) = 1 \tag{1.4.2}$$

Mixed state: A state ρ is said to be mixed if it is expressed as a convex combination of pure states $|\psi_i\rangle,\ i = 1, 2, \ldots, n$. Mathematically, a mixed state ρ which is associated to some ensemble $(p_i, |\psi_i\rangle)$ can be expressed as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|,\ \sum_i p_i = 1,\ 0 \leq p_i \leq 1 \tag{1.4.3}$$

Remark 1: A state ρ is mixed if it satisfies tr(ρ^2)<1.

Remark 2: Generally, a single qubit may be described by the density operator

$$\rho = \frac{1}{2}(I_2 + \vec{r}.\vec{\sigma}) \tag{1.4.4}$$

where $\vec{r} = (r_x, r_y, r_z) \in R^3$, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and I_2 denote the identity matrix of order 2.

Remark 3: A qutrit system is described by the density operator [10]

$$\rho = \frac{1}{3}(I_3 + \vec{b}.\vec{\Gamma}) \tag{1.4.5}$$

where $\vec{b} = (b_1, b_2, \ldots, b_8) \in R^8$, $\vec{\Gamma} = (S_1, S_2, \ldots, S_8)$ and I_3 denote the identity matrix of order 3. $S_1, S_2, \ldots, S_8$ are known as Gell-Mann matrices and they are given by

$$S_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; S_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.4.6}$$

$$S_4 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; S_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \; S_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{1.4.7}$$

$$S_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; S_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix} \tag{1.4.8}$$

Here S_1, S_2 and S_3 are three symmetric matrices; S_4, S_5 and S_6 are three anti-symmetric matrices; S_7 and S_8 are two diagonal matrices.

1.4.1.4 Partial trace

Suppose the composite system of two physical systems A and B is described by a density operator ρ_{AB}. The reduced density operator of ρ_{AB} for subsystem A and B respectively then defined as

$$\rho_A = Tr_B(\rho_{AB}) \tag{1.4.9}$$

$$\rho_B = Tr_A(\rho_{AB}) \tag{1.4.10}$$

where Tr_B and Tr_A denote the partial trace over the system B and A respectively. For example, if the first and second system is represented by $|a_1\rangle$ and $|b_1\rangle$ respectively then the partial traces over the system A and B are given by

$$Tr_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2| \; Tr(|b_1\rangle\langle b_2|) \tag{1.4.11}$$

$$Tr_A(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |b_1\rangle\langle b_2| \; Tr(|a_1\rangle\langle a_2|) \tag{1.4.12}$$

Partial trace is a very common function of the composite system. It is not only viewed as a mathematical operation but also has operational meaning. Partial trace is a unique mapping from a composite system to a subsystem and it can be described as a density operator for the reduced system. The physical interpretation of partial trace has also been provided by two different approaches [11–13]. In [14], the numerical calculation of the partial trace function has been analyzed and have presented the number of operations required to calculate the partial trace function on classical computers in bipartite and the multipartite system as well. For Instance, for a bipartite system, $\mathscr{H}_A \otimes \mathscr{H}_{\mathscr{B}}$ with dimension d_1 and d_2 respectively, it has been observed that the number of operations for calculating the partial trace may also be optimized using Bloch's parametrization with generalized Gell Mann's matrices and the optimized number of operations required to calculate partial trace function are of the order $O(d_1^2 d_2)$ [14].

1.4.2 Quantum measurement

We again move on to the topic of quantum measurement as in this thesis, it needs special attention. Being a physical theory, quantum mechanics significantly depends on the results of experiments. Measurement is the process of obtaining the result of an experiment, and it is described by the measurement operators. Let $M = \{M_i\}_{i=1}^n$ be a set of linear operators on a Hilbert space $\mathscr{H}$ of dimension n then $M_i's$ may represent a measurement operator satisfying the inequality

$$\Sigma_{i=1}^k M_i^\dagger M_i \leq I, \quad k = 2,3,\ldots,n \tag{1.4.13}$$

where I is an identity operator.
If $k = n$ i.e. if the number of measurement operators is equal to the dimension of the Hilbert space then the measurement operators satisfying the relation

$$\Sigma_{i=1}^n M_i^\dagger M_i = I \tag{1.4.14}$$

The property (1.4.14) is called the completeness property.
If the state $|\phi\rangle \in \mathscr{H}$ is being measured then the probability to get the outcome i is given by

$$p(i) = \langle\phi|M_i^\dagger M_i|\phi\rangle \tag{1.4.15}$$

and the state $|\phi\rangle$ collapses onto the state $|\phi'\rangle$, which is given by

$$|\phi\rangle \rightarrow \frac{M_i|\phi\rangle}{\sqrt{\langle\phi|M_i^\dagger M_i|\phi\rangle}} \equiv |\phi'\rangle \tag{1.4.16}$$

Let us consider a quantum state described by the density operator ρ, which is given by

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| \tag{1.4.17}$$

If the initial state is $|\psi_i\rangle$, then the probability of getting result m is given by

$$p(m|i) = \langle\psi_i|M_m^\dagger M_m|\psi_i\rangle = tr(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|) \tag{1.4.18}$$

By the law of total probability, the probability of obtaining m is given by

$$\begin{aligned} p(m) &= \sum_i p(m|i)p_i \\ &= \sum_i (tr(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|))p_i \\ &= Tr(M_m^\dagger M_m \sum_i p_i|\psi_i\rangle\langle\psi_i|) \\ &= Tr(M_m^\dagger M_m\rho) \end{aligned} \tag{1.4.19}$$

Now, if the initial state is $|\psi_i\rangle$ then after the measurement performed by the measurement operator M_m, the post-measurement state described by the density operator

$$\begin{aligned} \rho_m &= \sum_i p(i|m)|\psi_i^m\rangle\langle\psi_i^m| \\ &= \sum_i p_i \frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{Tr(M_m^\dagger M_m\rho)} \\ &= \frac{M_m\rho M_m^\dagger}{Tr(M_m^\dagger M_m\rho)} \end{aligned} \tag{1.4.20}$$

where $|\psi_i^m\rangle = \frac{M_m|\psi_i\rangle}{\sqrt{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}}$.

The concept of measurement presented above, provides the most general way to define the measurement process. Hence, it is sometimes referred to as the general measurement. Different types of measurements are as follows:

1) **Projective measurement:** A projective measurement is a type of measurement on a quantum system that establishes the value of an observable corresponding to the

physical property such as position, momentum, or spin. It is described by a Hermitian operator M that can be expended using a spectral decomposition

$$M = \sum_m m P_m \tag{1.4.21}$$

where P_m is the projector onto the eigenspace of M with eigenvalue m.

Von Neumann measurement is a type of projective measurement that enables the determination of an observable without changing the quantum state. A single qubit Von Neumann measurement in the computational basis may be described as $\{\pi_k = |k\rangle\langle k|, k = 0,1\}$. In general, a single qubit measurement operator in an arbitrary basis can be described as

$$B_k = V \pi_k V^\dagger : k = 0,1 \tag{1.4.22}$$

where V denotes the single qubit unitary operator which may be expressed as [15]

$$V = tI + i\overrightarrow{y}.\overrightarrow{\sigma}, \;\; t^2 + y_1^2 + y_2^2 + y_3^2 = 1 \tag{1.4.23}$$

where $t \in \mathbb{R}$ and $\overrightarrow{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$.

2) **Positive operator valued measurements (POVM):** POVM are known as generalization of projective measurements. In POVM, the set of measurements is described by a set of positive semi-definite operators, and the sum of these operators is identity. Let us suppose that a measurement is performed on a state $|\psi\rangle$, which is described by the measurement operators $\{M_m\}$. Define $E_m \equiv M_m^\dagger M_m$. From the definition, it is clear that E_m is a positive operator and it follows the completeness relation $\sum_m E_m = I$. When the measurement operator E_m performing on the system $|psi\rangle$ then the probability of obtaining an outcome m is given by $p(m) = \langle\psi|E_m|\psi\rangle$. The operators E_m are known as POVM elements and the complete set of $\{E_m\}$ operators is known as POVM. POVM is especially helpful when the measurement results are not orthogonal or when the measuring tool is not precise or defined. Measurement errors and noise can also be handled by POVMs.

1.5 Quantum entanglement

Ever since Einstein, Podolsky, and Rosen [16] raised the issue of the lack of compatibility of quantum mechanics with the assumptions of local realism, quantum correlation has been the subject of ongoing debates and studies. The violation of Bell-type [17] inequalities confirms the fundamentally different nature of quantum correlations as compared to classical correlations. In order to describe such correlations, Schrodinger [18] first used the term entanglement and considered it to be the characteristic trait of quantum mechanics. Bohm [19] later explored entanglement in a simpler context, that of a pair of spins in the singlet state, which has since been central to the investigation of the foundations of quantum mechanics and quantum information. Following these developments, Bell greatly advanced the investigation of quantum entanglement by deriving, what is now known as Bell's inequality [17] that must be obeyed by systems which are correlated but whose interactions are local as against the systems whose correlations are spatially extended and cannot be explained by the assumption of locality. The potential offered by the efficient use of such entangled systems as resources for quantum information, communication, cryptography, and quantum computing has led to many interesting protocols.

The use of entangled resources to achieve efficient and optimal success in quantum information and communication, in comparison to classical resources, is based on quantum correlations existing between the particles. The existence of such long-range correlations in quantum systems with no classical analogues thus distinguishes the quantum world from its classical counterparts. Moreover, quantum correlations not only shed light on the complex nature of entanglement but also provide physical insights into quantum computing and quantum communication protocols like teleportation [20], quantum cryptography [21, 22], superdense coding [23], quantum repeaters [24] or measurement based quantum computation [25].

1.5.1 Bipartite entanglement

A bipartite system described by the Hilbert space $\mathscr{H}_{AB}$ is composed of two individual systems A and B described by the Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$ respectively. Therefore, the composite system $\mathscr{H}_{AB}$ is described by tensor product of $\mathscr{H}_A$ and $\mathscr{H}_B$ i.e. $\mathscr{H}_{AB} \equiv \mathscr{H}_A \otimes \mathscr{H}_B$. If $\rho^A \in \mathscr{H}_A$, $\rho^B \in \mathscr{H}_B$ and a bipartite state $\rho_{AB} \in \mathscr{H}_{AB}$ is expressed in the form $\rho_{AB} = \rho^A \otimes \rho^B$, then the state ρ_{AB} is said to be a product state. Moreover, if a state $\rho_{AB} \in \mathscr{H}_A \otimes \mathscr{H}_B$ is expressed as the convex combination of product states i.e. if ρ_{AB} is expressed in the form as

$$\rho_{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B, 0 \leq p_i \leq 1, \sum_i p_i = 1 \quad (1.5.1)$$

then ρ_{AB} is said to be a separable state. Otherwise, the state ρ_{AB} is said to be an entangled state.

A general form of pure two-qubit state $|\psi\rangle_{AB} \in \mathscr{H}_{AB}$ ($\equiv \mathscr{H}_A \otimes \mathscr{H}_B$) is given by

$$|\psi\rangle_{AB} = \alpha|00\rangle_{AB} + \beta|01\rangle_{AB} + \gamma|10\rangle_{AB} + \delta|11\rangle_{AB}, \ \ |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1 \quad (1.5.2)$$

(i) If $\gamma = \delta = 0$, then the state $|\psi\rangle_{AB}$ given in (1.5.2) reduces to

$$\begin{aligned} |\psi_1\rangle_{AB} &= \alpha|00\rangle_{AB} + \beta|01\rangle_{AB}, \ \ |\alpha|^2 + |\beta|^2 = 1 \\ &= |0\rangle_A \otimes (\alpha|0\rangle_B + \beta|1\rangle_B) \end{aligned} \quad (1.5.3)$$

Therefore, $|\psi_1\rangle_{AB}$ is a separable state.

(ii) If $\beta = \gamma = 0$, then the state $|\psi\rangle_{AB}$ given in (1.5.2) reduces to

$$|\psi_2\rangle_{AB} = \alpha|00\rangle_{AB} + \delta|11\rangle_{AB}, \ \ |\alpha|^2 + |\delta|^2 = 1 \quad (1.5.4)$$

Therefore, $|\psi_2\rangle_{AB}$ represents an entangled state.

Let us consider a two-qubit state described by the density operator as

$$\rho_{AB} = \frac{1}{4}[I \otimes I + \overrightarrow{a}.\overrightarrow{\sigma} \otimes I + I \otimes \overrightarrow{b}.\overrightarrow{\sigma} + \sum_{i,j} c_{ij} \sigma_i \otimes \sigma_j] \quad (1.5.5)$$

where $\vec{a} = (a_1, a_2, a_3) \in R^3$, $\vec{b} = (b_1, b_2, b_3) \in R^3$, $c_{ij} = Tr[\rho_{AB}(\sigma_i \otimes \sigma_j)]$ and $\sigma_i' s \ \ (i = x, y, z)$ denotes the Pauli matrices.

If we are given the general form of a two-qubit system given in (1.5.5) then it would be very difficult to say whether the state is entangled or not. This is due to the fact that the representation of a quantum state is not unique, as one basis can be obtained from the other basis just by using a unitary transformation. Thus, we have to check for the given quantum state whether it can be expressed in the form (1.5.1) for every possible basis.

1.5.2 Tripartite entanglement

Let us consider a tripartite state $\rho_{ABC} \in \mathscr{H}_{ABC}$, which is composed of three individual systems A, B and C described by the Hilbert spaces $\mathscr{H}_A$, $\mathscr{H}_B$ and $\mathscr{H}_C$ respectively. In terms of quantum correlation, a tripartite system can be classified into three stochastic local operation and classical communication (SLOCC) inequivalent classes: (i) fully separable state, (ii) biseparable state, and (iii) genuinely entangled state [26].

(i) **A fully separable state:** A pure tripartite state $|\psi\rangle_{ABC}$ is said to be a fully separable state if it can be expressed as

$$|\psi\rangle_{ABC} = |\psi\rangle_A \otimes |\psi\rangle_B \otimes |\psi\rangle_C \tag{1.5.6}$$

More generally, if a state ρ_{ABC} is expressed as the convex combination of product states then it is known as a fully separable state. Mathematically, it can be expressed as

$$\rho_{sep}^{ABC} = \sum_i p_i \rho_i^A \otimes \rho_i^B \otimes \rho_i^C, \;\; 0 \leq p_i \leq 1, \;\; \sum_i p_i = 1 \tag{1.5.7}$$

(ii) **Biseparable state:** If a pure three-qubit state is expressed as

$$|\phi\rangle_{bisep}^{A-BC} = |\phi^{(1)}\rangle_A \otimes |\phi^{(2)}\rangle_{BC} \tag{1.5.8}$$

where $|\phi^{(1)}\rangle_A$ represent a single-qubit state and $|\phi^{(2)}\rangle_{BC}$ denote a two-qubit entangled state in a composite system B and system C respectively.

A biseparable state in a tripartite system may be further classified into three sub-classes: (a) Biseparable state in $A-BC$ cut, (b) Biseparable state in $B-AC$ cut, and (c) Biseparable state in $C-AB$ cut. The definition given in (1.5.8) denotes a pure biseparable state in $A-BC$ cut. In a similar manner, the pure biseparable state in $B-AC$ cut and $C-AB$ cut may be defined as

$$|\phi\rangle_{bisep}^{B-AC} = |\phi^{(3)}\rangle_B \otimes |\phi^{(4)}\rangle_{AC}$$
$$|\phi\rangle_{bisep}^{C-AB} = |\phi^{(5)}\rangle_C \otimes |\phi^{(6)}\rangle_{AB} \tag{1.5.9}$$

where $|\phi^{(3)}\rangle_B$, $|\phi^{(5)}\rangle_C$ represent a single qubit pure state in system B and C respectively and $|\phi^{(4)}\rangle_{AC}$, $|\phi^{(6)}\rangle_{AB}$ denote the pure two-qubit entangled pure state.

If $|a_i\rangle, |b_i\rangle$ and $|c_i\rangle$ denote the basis vectors in the system A, B and C respectively and $|\phi_i\rangle_{AB}$, $|\phi_i\rangle_{BC}$ and $|\phi_i\rangle_{AC}$ represent the two-qubit entangled state in their respective composite system then a mixed biseparable state can be expressed as a convex combination of pure biseparable states, which may be expressed as

$$\rho_{ABC} = p_1\rho_{bisep}^{A-BC} + p_2\rho_{bisep}^{B-AC} + p_3\rho_{bisep}^{C-AB}, \; 0 \leq p_i \leq 1, \; p_1 + p_2 + p_3 = 1 \qquad (1.5.10)$$

where,

$$\begin{aligned} \rho_{bisep}^{A-BC} &= \sum_i |a_i\rangle_A\langle a_i| \otimes |\phi_i\rangle_{BC}\langle\phi_i| \\ \rho_{bisep}^{B-AC} &= \sum_i |b_i\rangle_B\langle b_i| \otimes |\phi_i\rangle_{AC}\langle\phi_i| \\ \rho_{bisep}^{C-AB} &= \sum_i |c_i\rangle_C\langle c_i| \otimes |\phi_i\rangle_{AB}\langle\phi_i| \end{aligned}$$

(iii) **Genuine entangled state:** If a tripartite state described by the density operator ρ_{ABC} is not expressed in any of the forms given in (1.5.6), (1.5.7), (1.5.8), (1.5.9), and (1.5.10), then the state ρ_{ABC} is called a genuinely entangled state. In other words, a three-qubit state is a genuine entangled state if there is a correlation between every pair of particles. There are two types of SLOCC inequivalent classes of three-qubit genuine entangled states known as GHZ and W class of state.
The general form of GHZ class of state can be expressed as

$$|\psi_G\rangle_{ABC} = \lambda_0|000\rangle_{ABC} + \lambda_1 e^{i\theta}|100\rangle_{ABC} + \lambda_2|101\rangle_{ABC} + \lambda_3|110\rangle_{ABC} + \lambda_4|111\rangle_{ABC}, \sum_{i=0}^{4}\lambda_i^2 = 1 \qquad (1.5.11)$$

where $\theta \in [0,\pi]$, $\lambda_i \geq 0$ for $i = 1,2,3$ and $\lambda_0, \lambda_4 > 0$.
The general form of W class of state can be expressed either as $|\psi_{W_1}\rangle_{ABC}$ or $|\psi_{W_2}\rangle_{ABC}$. The form of $|\psi_{W_1}\rangle_{ABC}$ and $|\psi_{W_2}\rangle_{ABC}$ can be given as

$$\begin{aligned} |\psi_{W_1}\rangle_{ABC} &= \lambda_0|000\rangle_{ABC} + \lambda_1 e^{i\theta}|100\rangle_{ABC} + \lambda_2|101\rangle_{ABC} + \lambda_3|110\rangle_{ABC}, \\ &\quad \theta \in [0,\pi]; \lambda_i \geq 0, i = 0,1,2,3; \; \sum_{i=0}^{3}\lambda_i^2 = 1 \qquad (1.5.12) \\ |\psi_{W_2}\rangle_{ABC} &= \lambda_1 e^{i\theta}|100\rangle_{ABC} + \lambda_2|101\rangle_{ABC} + \lambda_3|110\rangle_{ABC} + \lambda_4|111\rangle_{ABC}, \\ &\quad \theta \in [0,\pi]; \lambda_i \geq 0, i = 1,2,3,4; \; \sum_{i=1}^{4}\lambda_i^2 = 1 \qquad (1.5.13) \end{aligned}$$

1.5.3 Detection of entanglement

Detection of entanglement in either higher dimensional or multipartite system is not an easy task and thus it may be considered as an important problem in quantum information theory. Now, we present a few important criteria for the detection of the entanglement in a bipartite or multipartite system.

1a. PPT criteria for bipartite system: PPT criteria is the first entanglement detection criteria which has been introduced by A. Peres [27]. Let us consider a $M \otimes N$ dimensional bipartite state ρ_{AB} written in the form as

$$\rho_{AB} = \sum_{i,j}^{M} \sum_{k,l}^{N} \rho_{ij,kl} |i\rangle\langle j| \otimes |k\rangle\langle l| \tag{1.5.14}$$

The partial transposition of ρ_{AB} with respect to the subsystem B may be expressed as

$$\rho_{AB}^{T_B} = \sum_{i,j}^{M} \sum_{k,l}^{N} \rho_{ij,kl} |i\rangle\langle j| \otimes |l\rangle\langle k| \tag{1.5.15}$$

Partial transposition of of ρ_{AB} with respect to subsystem A may be expressed as

$$\rho_{AB}^{T_A} = \sum_{i,j}^{M} \sum_{k,l}^{N} \rho_{ij,kl} |j\rangle\langle i| \otimes |k\rangle\langle l| \tag{1.5.16}$$

PPT Criteria states that if a bipartite quantum state ρ_{AB} is separable, then the partial transpose of ρ_{AB} with respect to system A or system B has zero or positive eigenvalues, i.e, $\rho_{AB}^{T_A}$ (or $\rho_{AB}^{T_B}$) is a positive semidefinite operator. PPT Criterion is only a necessary condition but not sufficient for $M \otimes N$ system, where $MN > 6$. Later, Horodecki et. al. [28] proved PPT criterion to be a necessary and sufficient condition for $2 \otimes 2$, $2 \otimes 3$ and $3 \otimes 2$ systems. PPT criterion is applicable for bipartite systems only and since PPT criterion is computed with the help of partial transposition and it is a positive map but not a completely positive map, so this criterion may not be directly applicable in an experiment.

1b. PPT criteria for tripartite system: To develop the PPT criteria for tripartite system, let us first define the partial transposition operation on three-qubit system. To discuss this operation, let us first consider a three-qubit state described by a density operator ρ_{ABC} which is given by

$$\rho_{ABC} = \begin{pmatrix} P & Q & R & S \\ Q^* & T & U & V \\ R^* & U^* & W & X \\ S^* & V^* & X^* & Y \end{pmatrix} \tag{1.5.17}$$

where P,Q,R,S,T,U,V,W,X,Y denote the 2×2 block matrices and "$*$" denote the complex conjugation.

When the partial transposition operation acts on the first subsystem A of the state ρ_{ABC}, the state ρ_{ABC} transformed into $\rho_{ABC}^{T_A}$, which can be expressed as

$$\rho_{ABC}^{T_A} \equiv [T\otimes I\otimes I]\rho_{ABC} = \begin{pmatrix} P & Q & R^* & U^* \\ Q^* & T & S^* & V^* \\ R & S & W & X \\ U & V & X^* & Y \end{pmatrix} \tag{1.5.18}$$

When the partial transposition operation acts on the second subsystem B of the state ρ_{ABC}, the state ρ_{ABC} transformed into $\rho_{ABC}^{T_B}$, which can be expressed as

$$\rho_{ABC}^{T_B} \equiv [I\otimes T\otimes I]\rho_{ABC} = \begin{pmatrix} P & Q^* & R & U \\ Q & T & S & V \\ R^* & S^* & W & X^* \\ U^* & V^* & X & Y \end{pmatrix} \tag{1.5.19}$$

When the partial transposition operation acts on the third subsystem C of the state ρ_{ABC}, the state ρ_{ABC} transformed into $\rho_{ABC}^{T_C}$, which can be expressed as

$$\rho_{ABC}^{T_C} \equiv [I\otimes I\otimes T]\rho_{ABC} = \begin{pmatrix} P^* & Q^* & R^* & S^* \\ Q & T^* & U^* & V^* \\ R & U & W^* & X^* \\ S & V & X & Y^* \end{pmatrix} \tag{1.5.20}$$

If any one of the partial transpose of ρ_{ABC} with respect to subsystem A, subsystem B or subsystem C, i.e $\rho_{ABC}^{T_A}$ (or $\rho_{ABC}^{T_B}$ or $\rho_{ABC}^{T_C}$) have at least one negative eigenvalue, then the tripartite state ρ_{ABC} is said to be an entangled state. PPT criteria for a tripartite system is only a necessary condition for the detection of entanglement of a tripartite state. Further, PPT criteria is not a suitable criterion for the discrimination of the biseparable state and a genuine entangled three-qubit state.

2. Structural physical approximation of partial transposition (SPA-PT): Partial transposition operation is not directly accessible in an experiment as it is a positive but not a completely positive map [28]. So, to make partial transposition operation realizable in an experiment, we can approximate the partial transposition operation by using structural physical approximation (SPA) which has been introduced by P. Horodecki and A. Ekert [29]. Structural physical approximation generally refers to a method or technique that can be used to approximate a non-physical system in such a way so that it can be a physically realizable system. SPA map has been implemented on a single qubit system [30], two-qubit system [31] and qutrit systems [32] as well.

In SPA, a precise amount of white noise is added to a non-physical operator $\wedge$, such that it gets approximated into a completely positive operator $\widetilde{\wedge}$. For a d-dimensional system ρ, the approximate map $\widetilde{\wedge}$ can be written as

$$\widetilde{\wedge(\rho)} = (1-p)\wedge(\rho) + pD(\rho) \tag{1.5.21}$$

where $0 \leq p \leq 1$, $D(\rho) = \frac{I_d}{d}$ is a depolarizing channel and I_d is a d-dimensional identity map.

Result 1.5: Let us consider an arbitrary two-qubit state described by the density operator ρ_{AB}

$$\rho_{AB} = \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{12}^* & e_{22} & e_{23} & e_{24} \\ e_{13}^* & e_{23}^* & e_{33} & e_{34} \\ e_{14}^* & e_{24}^* & e_{34}^* & e_{44} \end{pmatrix}, \sum_{i=1}^{4} e_{ii} = 1 \tag{1.5.22}$$

where $(*)$ denotes the complex conjugate.

The SPA-PT of ρ_{AB} is then given by [33, 34]

$$\begin{aligned} \widetilde{\rho}_{AB} &= [\frac{1}{3}(I\otimes\widetilde{T}) + \frac{2}{3}(\widetilde{\Theta}\otimes D)](\rho_{AB}) \\ &= \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12}^* & E_{22} & E_{23} & E_{24} \\ E_{13}^* & E_{23}^* & E_{33} & E_{34} \\ E_{14}^* & E_{24}^* & E_{34}^* & E_{44} \end{pmatrix} \end{aligned} \tag{1.5.23}$$

where

$$
\begin{aligned}
&E_{11}=\frac{1}{9}(2+e_{11}), E_{12}=\frac{1}{9}e_{12}^{*}, E_{13}=\frac{1}{9}e_{13},\\
&E_{14}=\frac{1}{9}e_{23}, E_{22}=\frac{1}{9}(2+e_{22}), E_{23}=\frac{1}{9}e_{14},\\
&E_{24}=\frac{1}{9}e_{24}, E_{33}=\frac{1}{9}(2+e_{33}), E_{34}=\frac{1}{9}e_{34}^{*},\\
&E_{44}=\frac{1}{9}(2+e_{44})
\end{aligned}
\tag{1.5.24}
$$

$\widetilde{T}$ is the SPA of transpose map T and $\widetilde{\Theta}$ denotes the SPA of inversion map Θ. The inversion map Θ is defined as $\Theta(\rho_{AB}) = -\rho_{AB}$ and the depolarisation map is given by $D(\rho_{AB}) = \frac{I_2}{2}$. SPA-PT for a two-qubit photonic system has been demonstrated using single-photon polarization qubits and linear optical devices in [30].

Note: Like in two-qubit system, SPA-PT map for three-qubit system described by the density operator ρ_{ABC} can also be discussed with respect to the system A, B, C respectively.

Result 1.6: If a tripartite state described by the density operator ρ_{ABC} is a biseparable state in $A-BC$ (or $B-AC$ or $C-AB$) cut or a separable state then either of these holds [35]

$$
\lambda_{min}(\widetilde{\rho_{ABC}^{T_A}}) \;\; (or \;\; \lambda_{min}(\widetilde{\rho_{ABC}^{T_B}}) \;\; or \;\; \lambda_{min}(\widetilde{\rho_{ABC}^{T_C}})) \geq \frac{1}{10}
\tag{1.5.25}
$$

Result 1.7: Let us consider a tripartite state described by the density operator ρ_{ABC} [35]. For the discrimination of three-qubit system, the following inequality in terms of minimum eigenvalue of SPA-PT of ρ_{ABC} holds good.

a) If $max\{\lambda_{min}(\widetilde{\rho_{ABC}^{T_A}}), \lambda_{min}(\widetilde{\rho_{ABC}^{T_B}}), \lambda_{min}(\widetilde{\rho_{ABC}^{T_C}})\} < \frac{1}{10}$, then ρ_{ABC} is a genuine entangled state.

b) If $\lambda_{min}(\widetilde{\rho_{ABC}^{T_A}}) \geq \frac{1}{10}$, and either $\lambda_{min}(\widetilde{\rho_{ABC}^{T_B}}) < \frac{1}{10}$ or $\lambda_{min}(\widetilde{\rho_{ABC}^{T_C}}) < \frac{1}{10}$ or both are less than $\frac{1}{10}$ holds, then the tripartite state ρ_{ABC} is a $A-BC$ biseparable state.

Note: We can obtain the similar results for $B-AC \; or \; C-AB$ biseparable state.

c) If $\lambda_{min}(\widetilde{\rho_{ABC}^{T_i}}) \geq \frac{1}{10}$ holds for all $i = A, B, C$, then ρ_{ABC} is a fully separable state.

3. Witness operator: The primary technique for detecting entanglement experimentally is using the witness operator. Witness operators can detect the bipartite and multipartite entangled state. Entanglement witnesses are Hermitian operators with at least one negative eigenvalue [36].

An observable W is said to be an entanglement witness if it satisfies the following:

$$C1.\ \ Tr[W\sigma_{sep}] \geq 0, \forall \text{ separable state } \sigma_{sep} \tag{1.5.26}$$

$$C2.\ \ Tr[W\sigma_{ent}] < 0, \text{for at least one entangled state } \sigma_{ent} \tag{1.5.27}$$

Therefore, we can say that if the expectation value of the witness operator W with respect to any state ρ is less than zero, then the state ρ under investigation is an entangled state detected by W. Witness operator criteria is an experimentally realizable criterion, but it is not very easy to construct a witness operator to detect the entanglement of a quantum state, especially the state in a higher dimensional system. Also, different witness operators may be constructed to detect different classes of entangled states, otherwise, the separability problem would be solved just by constructing a single witness operator. Witness operators can be constructed by different methods which are available in the literature [37, 38].

1.5.4 Measures of entanglement

The amount of entanglement contained in the entangled state described by the density operator ρ can be quantified with entanglement measures $E(\rho)$. The entanglement measure $E(\rho)$ satisfies some properties, which are given below:
(i) $E(\rho)$ vanishes if ρ is separable.
(ii) $E(\rho)$ is invariant under local unitary transformation.
(iii) Local operations and classical communication cannot increase the expected entanglement.
(iv) $E(\rho)$ satisfies convexity property i.e.

$$E(\sum_k p_k\rho_k) \leq \sum_k p_k E(\rho_k) \tag{1.5.28}$$

It has been observed that some entanglement measures follow few properties but not all properties while on the other hand, some measures may not be applied in an experiment. Due to these difficulties, different entanglement measures have been introduced in the literature. Some of the entanglement measures are discussed below.

1.5.4.1 Measures of entanglement in bipartite system

1. Entanglement of formation: Consider the pure state decompositions of the bipartite system comprising of the subsystems A and B such that $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. The entanglement of formation for a state described by the density operator ρ_{AB} may be defined as [39]

$$E(\rho_{AB}) = min \sum_i p_i E(|\psi_i\rangle\langle\psi_i|) \tag{1.5.29}$$

The minimization in (1.5.29) is taken over all possible decompositions $\{p_i, |\psi_i\rangle\}$ of the density operator ρ_{AB}. Entanglement of formation for isotropic states in arbitrary dimension has been obtained [40]. The concept of entanglement of formation is important in the study of quantum communication protocols, and quantum cryptography [41]. The entanglement of formation is a valuable measure in quantum information theory, but it does have limitations. It is limited to bipartite systems only and calculating the exact entanglement of formation for mixed states is more complicated.

2. Negativity: Negativity of a bipartite state ρ_{AB} is defined as [42]

$$N(\rho_{AB}) = max[0, -2\lambda_{min}(\rho_{AB}^{T_A})] \tag{1.5.30}$$

where λ_{min} is the minimum eigenvalue of the partial transpose of ρ_{AB} denoted as $\rho_{AB}^{T_A}$. Negativity is a useful measure because it provides information about the amount of entanglement present in an arbitrary dimensional composite quantum system and can be computed for both pure and mixed states. It has applications in quantum information theory, quantum computing, and quantum communication protocols [42], but it has limitations too. Since negativity is computed with the help of partial transposition of the density operator and partial transposition is a positive map but not a completely positive map, so it may not be directly accessible in an experiment.

3. Concurrence: For a two-qubit pure state $|\psi\rangle_{AB}$, the concurrence of $|\psi\rangle_{AB}$ is defined as [43]

$$C(|\psi\rangle_{AB}) = \sqrt{2(1 - Tr(\rho_A^2))} \tag{1.5.31}$$

where $\rho_A = Tr_B(|\psi\rangle_{AB}\langle\psi|)$.

Let us define $\widetilde{\rho_{AB}} = (\sigma_y \otimes \sigma_y)\rho_{AB}^*(\sigma_y \otimes \sigma_y)$, where σ_y is the pauli matrices and ρ_{AB}^*

denotes the complex conjugate of ρ_{AB}. The concurence of two-qubit mixed state ρ_{AB} is defined as

$$C(\rho_{AB}) = max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}) \tag{1.5.32}$$

where λ_i's are the eigenvalues of $\rho_{AB}\widetilde{\rho_{AB}}$ which are arranged in descending order, i.e., $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$.
Concurrence lies between zero and one. When concurrence is zero, it corresponds to a separable state and when concurrence is one, it corresponds to a maximally entangled state. Entanglement of formation and concurrence are related to each other, as entanglement of formation is a monotonically increasing function of concurrence [44]. The formula for the calculation of the concurrence of higher dimensional mixed states is still not known, in spite of only a few analytic formulae of concurrence for the higher dimensional bipartite systems have been found [40, 45].

1.5.4.2 Measures of entanglement in tripartite system

Tangle and partial tangle: To define tangle and partial tangle, let us consider a general three-qubit pure state $|\psi\rangle_{ABC} \in \mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_C$ expressed in the computational basis as

$$|\psi\rangle_{ABC} = \frac{1}{N}(\lambda_0|000\rangle + \lambda_1|001\rangle + \lambda_2|010\rangle + \lambda_3|011\rangle + \lambda_4|100\rangle + \lambda_5|101\rangle + \lambda_6|110\rangle + \lambda_7|111\rangle) \tag{1.5.33}$$

where $N = \sqrt{\sum_{i=0}^{7} |\lambda_i|^2}$ denote the normalization factor.
Tangle: The tangle for the pure state $|\psi\rangle_{ABC}$ may be defined as [46]

$$\tau_{ABC} = 4|e_1 - 2e_2 + 4e_3| \tag{1.5.34}$$

$$\begin{aligned} \text{where,} \quad e_1 &= \lambda_0^2\lambda_7^2 + \lambda_1^2\lambda_6^2 + \lambda_2^2\lambda_5^2 + \lambda_3^2\lambda_4^2 \\ e_2 &= \lambda_0\lambda_7\lambda_3\lambda_4 + \lambda_0\lambda_7\lambda_2\lambda_5 + \lambda_0\lambda_7\lambda_1\lambda_6 + \lambda_5\lambda_2\lambda_3\lambda_4 + \lambda_6\lambda_1\lambda_3\lambda_4 + \lambda_5\lambda_2\lambda_6\lambda_1 \\ e_3 &= \lambda_0\lambda_6\lambda_3\lambda_5 + \lambda_7\lambda_1\lambda_2\lambda_4 \end{aligned} \tag{1.5.35}$$

Remark 1: A relation between bipartite entanglement in a three-qubit pure state has been presented in the form of an inequality which is known as Coffman-Kundu-

Wootters (CKW) inequality [44]

$$C_{AB}^2 + C_{AC}^2 \leq C_{A(BC)}^2 \tag{1.5.36}$$

where $C_{AB} = C(\rho_{AB})$ denotes the concurrence between the system A and B, $C_{AC} = C(\rho_{AC})$ denotes the concurrence between the system A and C. The term $C_{A(BC)}$ is given by

$$C_{A(BC)} = 2\sqrt{det[Tr_{BC}(\rho_{ABC})]} \tag{1.5.37}$$

where $Tr_{BC}(\rho_{ABC})$ denote the reduced system obtained after tracing out the system BC from ρ_{ABC}. $C_{A(BC)}$ may be called as the concurrence between system A with the joint system BC.

Residual entanglement: The term $C_{A(BC)}^2 - C_{AB}^2 - C_{AC}^2$ may be called as residual entanglement and it may be expressed in terms of e_1, e_2, e_3 as

$$C_{A(BC)}^2 - C_{AB}^2 - C_{AC}^2 = 4|e_1 - 2e_2 + 4e_3| \tag{1.5.38}$$

Therefore, the tangle can also be expressed in terms of residual entanglement as [44, 47]

$$\tau_{ABC} = C_{A(BC)}^2 - C_{AB}^2 - C_{AC}^2 \tag{1.5.39}$$

Remark 2: It may be noted that $\tau_{ABC} \neq 0$ for GHZ class of three-qubit states while it vanishes for a certain classes of three-qubit states such as W class, biseparable states, and separable states.

Partial tangle: The partial tangle of a tripartite pure state $|\psi\rangle_{ABC}$ can be expressed as [47]

$$\tau_{ij} = \sqrt{C_{i(jk)}^2 - C_{ik}^2} = \sqrt{\tau_{ABC} + C_{ij}^2} \tag{1.5.40}$$

for distinct i, j, k and $i, j, k \in \{A, B, C\}$.

1.6 Bell's non-locality

In 1935, Einstein, Podolsky, and Rosen (EPR) [16] presented a theory in which they

considered the wave function of two physical quantities and showed that the knowledge of one physical quantity is not sufficient to gain the knowledge about the other physical quantity, provided these two physical quantities are described by two non-commutative operators. From the above argument, they have concluded that the description of reality given by the wave function in quantum mechanics is not complete [16]. Thus, according to EPR, quantum mechanics lacks a very important property known as the element of reality, and hence quantum mechanics is an incomplete theory. It turns out later that the experiment performed by the researchers supports the theory of quantum mechanics and thus does not validate the EPR argument. The main argument for the experimental invalidation of the EPR argument was provided by John Bell in the form of an inequality, which is popularly known as Bell's inequality [17]. The inequality has been constructed exploiting the following two assumptions:
1) *Realism*:- It tells us about the real existence of the physical system. If a physical system exists then all the physical properties of it have a definite value independent of the measurement performed on the system.
2) *Locality*:- It means that the result of the measurement performed on one system does not influence the result of the measurement performed on another system.
In 1969, Clauser et. al. gave the first experimental form of Bell's theory and presented a generalized form of Bell's inequality known as CHSH inequality, which is given by [48]

$$B_{CHSH} = \langle A_1 B_1 \rangle \rho_{AB} + \langle A_1 B_2 \rangle \rho_{AB} + \langle A_2 B_1 \rangle \rho_{AB} - \langle A_2 B_2 \rangle \rho_{AB} \leq 2 \tag{1.6.1}$$

where $\langle A_i B_j \rangle \rho_{AB} = Tr[\rho(\hat{a}_i.\vec{\sigma}^A)(\hat{b}_j.\vec{\sigma}^B)]$ known as the correlation functions, ρ denote the two-qubit state shared between two distant parties, $\vec{\sigma}$ is the Pauli matrix vector, $\hat{a}_1$, $\hat{a}_2$, $\hat{b}_1$ and $\hat{b}_2$ are the unit vectors for the first and the second measurements performed on the subsystems A and B respectively. The Bell-CHSH inequality is tight, i.e. it defines one of the facets of the convex polytope of local-realistic (LR) models. Later, Freedman and Clauser [49] and after that Aspect, Grangier, and Roger [50] gave more convincing experimental predictions of Bell's inequality.

1.6.1 Experiment performed for the verification of Bell's non-locality

John S. Bell [17] succeeded in proving the fact that there exists a correlation between the outcome obtained by the measurement of one system with the outcome obtained by the measurement of another system. The discovered correlation may be termed as non-local correlation. He has provided the mathematical framework in the form of an inequality for the detection of non-locality in the bipartite system, which is known as Bell's inequality. In 1969, Clauser et. al. [48] generalized Bell's inequality in such a way that it may be implemented in the experiment. Also, they have proposed an extension of the Kocher and Commin's [51] experiment on the polarization correlation of a pair of optical photons. In 1972, Freedman et. al. [49] performed an experiment that was in agreement with the quantum mechanics with high accuracy and invalidated the local hidden variable theory. Later, Aspect et. al. [50] presented a new violation of Bell's inequality with the new experimental scheme using optical analogs of Stern-Gerlach filters and achieved the greatest violation of generalized Bell's inequality till the year 1982 [50]. In another experiment, Aspect et. al. [52] have shown that the correlation of linear polarizations of pair of photons can be measured with time-varying analyzers. Further, they found that their result violates Bell's inequality and is in good match with the predictions of quantum mechanics. The EPR paradox for the case of continuous variables has also been implemented in an experiment [53]. P. G. Kwiat et. al. [54] have shown experimentally, a violation of Bell non-locality by over 100 standard deviations in less than 5 min. In 1998, G. Weihs et. al. [55], performed an experiment for the first time when both the observers have no mutual influence on each other. This condition is imposed within the realm of locality and to achieve this condition, they separated both the observers by 400 m. They obtained a strong violation of Bell's inequality by considering the independent observers [55]. Tittel et. al. [56, 57] performed two experiments where they found a strong violation of Bell's inequality when the two distant partners are separated over 10 km apart. The experimental violation of Bell's inequality has been observed over more than 10 Kms using energy time entangled photons by W. Tittel et. al. [58].

1.7 Multipartite non-locality

John Bell [17] developed Bell's inequality to detect the signature of non-locality in the bipartite system but the concept of Bell's inequality can be generalized to the multipartite system also. Therefore, in multipartite systems, the violation of multipartite Bell inequalities is a signature of multipartite non-locality. It signifies that the observed correlations among multiple measurement outcomes cannot be accounted for by any local realistic theory. In particular, for a tripartite system, one can observe the non-locality property within reduced two-qubit and also one can look into the non-locality between all three-qubits. The non-locality in the reduced bipartite system may be detected by Mermin's inequality while the non-locality that existed between all three qubits may be detected by Svetlichny's inequality. The non-locality between all three qubits in the tripartite system is known as genuine non-locality.

1.7.1 Mermin's inequality

N. D. Mermin [59] presented a gedanken experiment of detecting the non-locality of three-qubits, which was based on GHZ Bell's theorem [60, 61]. He then demanded that his thought experiment demonstrated the non-locality of three-qubit better than the EPR experiment that analyzed the two-qubit non-locality [59]. Like Bell's inequality, Mermin also developed an inequality that may capture the non-locality of a three-qubit system. To proceed towards the development of an inequality, let us consider a three-qubit state described by the density operator ρ_{ABC}. N. D. Mermin [62] constructed an operator to detect the non-locality of a three-qubit state. The constructed operator is known as Mermin's Operator and can be written as

$$\begin{aligned} M = \vec{a}_1.\vec{\sigma} \otimes \vec{a}_2.\vec{\sigma} \otimes \vec{a}_3.\vec{\sigma} - \vec{a}_1.\vec{\sigma} \otimes \vec{b}_2.\vec{\sigma} \otimes \vec{b}_3.\vec{\sigma} \\ -\vec{b}_1.\vec{\sigma} \otimes \vec{a}_2.\vec{\sigma} \otimes \vec{b}_3.\vec{\sigma} - \vec{b}_1.\vec{\sigma} \otimes \vec{b}_2.\vec{\sigma} \otimes \vec{a}_3.\vec{\sigma} \end{aligned} \tag{1.7.1}$$

where $\vec{a}_i$ and $\vec{b}_j$ $(i, j = 1, 2, 3)$ are unit vectors in R^3 and the component of $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ denote the Pauli matrices. Using Mermin's operator, the non-locality of a three-qubit state ρ_{ABC} may be detected by the inequality given as

$$|\langle M \rangle_{\rho_{ABC}}| \leq 2 \tag{1.7.2}$$

The inequality (1.7.2) is known as Mermin's inequality and it can be considered as the generalized form of Bell-CHSH inequality [62]. If any three-qubit state violates (1.7.2) then we may say that state ρ_{ABC} has non-locality, but we cannot bifurcate the type of non-locality from the violation of Mermin's inequality. It can be biseparable non-locality or can be genuine non-locality. C. Pagonis et. al. [63] extended the proof of Mermin's inequality to n-particle case. The first experimental generalization for Bell's inequality using 3 photons was shown by Klyshko [64] and the experimental generalization of Bell's inequality using n photons was shown by Belinskii [65]. Later, the generalization of Bell's inequality for n-qubit state known as Bell-Klyshko inequality or Mermin-Klyshko inequality has been presented by Gisin et. al. [66]. The maximum violation of Bell-Klyshko inequality for n-qubit quantum state can be made up to a factor of $2^{\frac{(n-1)}{2}}$. If we assume m out of n-qubits to be independent, then the violation of Bell's inequality can be increased with increasing m exponentially [66] which is more than the maximum violation of Bell's inequality achieved by Mermin's work [62]. Bell's inequality for n-particle GHZ state has been studied by M. Ardehali [67]. The relation between the entanglement measures and the maximal violation of Mermin's inequality has been studied in [68, 69].

1.7.2 Svetlichny's inequality

A lot of research has already been done in studying the problem of two-qubit non-locality [70–79]. Therefore, the researcher turned on to the study of non-locality of multi-partite state [80–85]. As the number of qubits increases in the system, the complexity of the system also increases and thus, the study of the non-locality of the multipartite system may become a difficult problem but in spite of that, some progress has been achieved. In the literature, there exist inequality such as Mermin inequality [62] that may be used to detect the non-locality in the tripartite system. The problem is that the violation of it, which signifies the presence of non-locality, was not only observed for all the three-qubit in the tripartite system but also observed for the reduced two-qubit system. Thus, for a given three-qubit system, it is not always possible to discriminate between the non-locality of the reduced two-qubit system and the genuine non-locality of the three-qubit system. To sort out this problem, G. Svetlichny [86] introduced an inequality, which is effective for the detection of genuine non-locality in a tripartite system and is commonly known as Svetlichny's inequality. The inequality is given by [86]

$$|\langle S_v \rangle_{\rho_{ABC}}| \leq 4 \tag{1.7.3}$$

where S_v denote the Svetlichny operator, which may be defined as

$$\begin{aligned} S_v &= \vec{a}.\vec{\sigma}_1 \otimes [\vec{b}.\vec{\sigma}_2 \otimes (\vec{c}+\vec{c'}).\vec{\sigma}_3 + \vec{b'}.\vec{\sigma}_2 \otimes (\vec{c}-\vec{c'}).\vec{\sigma}_3] \\ &+ \vec{a'}.\vec{\sigma}_1 \otimes [\vec{b}.\vec{\sigma}_2 \otimes (\vec{c}-\vec{c'}).\vec{\sigma}_3 - \vec{b'}.\vec{\sigma}_2 \otimes (\vec{c}+\vec{c'}).\vec{\sigma}_3] \end{aligned} \tag{1.7.4}$$

Here $\vec{a},\vec{a}'$; $\vec{b},\vec{b}'$ and $\vec{c},\vec{c}'$ are the unit vectors and the $\vec{\sigma}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$ denote the spin projection operators. To obtain the maximal violation of the Svetlichny inequality, the expectation value of the Svetlichny operator must achieve the value $4\sqrt{2}$. In particular, the violation of Svetlichny inequality by three-qubit generalized GHZ state, maximal slice state, and W class state has been studied in [87, 88] and it has been found that the maximal violation $4\sqrt{2}$ may be obtained for GHZ state.

N. S. Jones et. al. [89] extended the 3-party non-locality described by Svetlichny's inequality to m-party Svetlichny's polynomial to detect the non-locality of m-qubit state. In 2009, Bancal et. al. also generalized the Svetlichny's inequality to n-qubit state and verified it by the violation on n-qubit GHZ state and for some parameters of n-qubit W state [84]. The experimental confirmation for the violation of Svetlichny's inequality has been reported by Lavoie et. al. [90]. The study of genuine non-locality of generalized GHZ state and maximally slice state in terms of three tangle has been studied and it has been found that maximally slice state always violates Svetlichny's inequality and generalized GHZ state violates Svetlichny's inequality when tangle is greater than $\frac{1}{4}$ [87]. D. Collins et. al. [91] re-derived the Svetlichny's inequality and derived a new Bell's inequality for three particle system with the help of Mermin-Klyshko (MK) inequalities [65, 66] and also further generalized their introduced Bell's inequality for n particle systems. The comparison between the re-derived Svetlichny's inequality and Mermin's inequality has been studied in [91]. Zhang et. al. [92] also investigated the genuine non-locality between three qubits possessed by Alice (A), Bob (B) and, Charlie (C), using Svetlichny's inequality and then showed that at most 2 Charlie can shared the genuine non-locality with an Alice and a Bob.

1.8 Non-locality and entanglement

Entanglement and non-locality can be considered as two faces of a single coin. Both of them represent the quantum correlation but one violates the Bell-type inequalities while the other does not. The quantum correlation that violates the inequality constructed on the assumptions of the local realism principle is called quantum non-local correlation but on the other hand, the correlation that existed in the entangled bipartite or multipartite system, may or may not violate the Bell type inequality. If a quantum state violates Bell's inequality then the state is said to have non-local features and hence entangled state. But the converse is not true. N. Gisin showed that all bipartite pure entangled states violate Bell's inequality [93]. This was generalized to multi-partite pure entangled states by S. Popescu and D. Rohrlich [94]. Gisin et. al. [95] proved the fact that there may exist some mixed states which violate Bell's inequality as well. They also have been shown that if the state represents a singlet state then the state maximally violates Bell's inequality. Braunstein et. al. [96] have shown that the maximum violation of CHSH inequality can even be achieved by mixed states. Later, Horodecki family [97] derived another form of Bell's inequality, which is a necessary and sufficient criterion for the violation of Bell-CHSH inequality by any arbitrary two-qubit quantum states. Let us now define a quantity $M(\rho_{AB})$ as

$$M(\rho_{AB}) = u_1 + u_2 \tag{1.8.1}$$

where u_1 and u_2 are the two maximum eigenvalues of $T^{\dagger}T$. T denotes the correlation matrix of order 3 and its entries t_{ij} can be calculated as

$$t_{ij} = Tr[\rho_{AB}(\sigma_i \otimes \sigma_j)], \quad i, j \in \{1,2,3\} \tag{1.8.2}$$

Now we are in a position to state the necessary and sufficient conditions for the violation of Bell's inequality. If a quantum state is described by the density operator ρ_{AB} then the necessary and sufficient condition that the state ρ_{AB} violate the Bell-CHSH inequality, is given by

$$M(\rho_{AB}) > 1 \tag{1.8.3}$$

Since the expectation value of the Bell-CHSH operator with respect to the state ρ_{AB} can go up to $2\sqrt{2}$ so the maximum value of $M(\rho_{AB})$ can go up to 2 only. Later, M. Zukowski and C. Brukner [98] generalized the necessary and sufficient condition (1.8.1) to n-qubit state, and derived a necessary and sufficient condition for the satisfaction of the generalized Bell inequality of an arbitrary n-qubit state.

For any two-qubit state ρ_{AB}, the upper bound of the non-locality can be expressed in terms of the concurrence as [99]

$$N_L(\rho_{AB}) \leq 2\sqrt{1 + C(\rho_{AB})^2} \tag{1.8.4}$$

where $N_L(\rho_{AB})$ denotes the non-locality of the state ρ_{AB} and $C(\rho_{AB})$ denotes the concurrence of the state ρ_{AB}.

The equality in (1.8.4) has been achieved for all the pure states. For the mixed two-qubit states, equality can be achieved when the state belongs to Q

$$Q \equiv \{(U_A \otimes U_B)(p|\psi_1\rangle\langle\psi_1| + (1-p)|\psi_2\rangle\langle\psi_2|)(U_A \otimes U_B)^\dagger\} \tag{1.8.5}$$

where U_A and U_B are the arbitrary unitary operators, $|\psi_1\rangle = Cos\theta|00\rangle + Sin\theta|11\rangle$ and $|\psi_2\rangle = Sin\theta|00\rangle + Cos\theta|11\rangle$, $\theta \in [0,\Pi)$ [99–101].

N. Gisin [102] showed that if we apply local filters on a quantum state that exhibits local properties, then the quantum state may become non-local. This is an implication of the hidden non-locality which may be revealed after the application of local filters. Hirsch et. al. proved that the entangled states from the class of Werner state [103] still admit local hidden variable theory even after applying local filters on the state. So, local filters cannot always reveal non-locality from entanglement [104]. There also exist some states whose non-locality is revealed after applying the most general local measurements [105]. Also, it has been found that there exist entangled states on which the application of a sequence of local measurements leads to the maximal violation of Bell's inequality [105]. Going one step forward, Masanes et. al. [106] have shown that the violation of the CHSH inequality can be seen in some kind of Bell experiment for all the entangled states. Thus, they have proved that all entangled states display some hidden non-locality.

From the above discussion, it is clear that entanglement and non-locality are very closely related concepts but they are not same. They can be considered as two different resources [107]. Before the work of Methot et. al. [71], it has been considered that maximally entangled states have more non-locality than non-maximally entangled states but this statement is not true in general. They discovered that in general, a non-maximally entangled state can give more non-locality than maximally entangled states with respect to all the measures such as Bell Inequalities, the Kullback Leibler distance, entanglement simulation with communication or with non-local boxes, the detection loophole, and efficiency of cryptography [71].

Non-locality and entanglement for two-qubits were studied by J. Batle and M. Cases [79] and obtained the class of two-qubit states that violate Bell's inequality maximally in terms of the degree of mixedness or maximum eigenvalue. They further established the relationship between the non-locality and distillability of three-qubit states. The Bell non-locality of higher dimensional quantum systems based on quantum entanglement has been studied by T. Zhang et. al. [108]. Su et. al. investigated the quantitative relationship between the entanglement and non-locality of a general two-qubit system and obtained the necessary and sufficient condition for the achievement of the upper bound on the non-locality of a general two-qubit system [99]. L. Tendick et. al. [109] have shown that it is not always necessary that if we increase the measurement resources then the requirement of the minimal state resources decreases for a fixed Bell violation.

1.9 Non-locality in communications

Many novel applications of non-locality have been developed for quantum computation and quantum communication [110], including communication complexity [111], quantum cryptography [112], randomness generation [113], and device-independent quantum computation [114] etc. In this thesis, we mainly focus on the role of quantum non-locality in communication protocols such as quantum teleportation, and controlled quantum teleportation.

1.10 Quantum teleportation

The process of transferring an unknown quantum state between two parties at two distant locations without transferring the physical information about the unknown quantum state itself is known as quantum teleportation [20]. In other words, it can also be understood as neither any physical information about the state is transferred nor a swap operation is performed between the sender and the receiver. Teleportation protocol makes use of the non-local correlations generated by using an entangled pair between the sender, the receiver, and the exchange of classical information between them. This concept plays a central role in quantum communication using quantum repeaters [22,115] and can also be used to implement logic gates for universal quantum computation [116].

1.10.1 Bennett et. al. quantum teleportation protocol for $2\otimes 2$ system

Originally, Bennett et. al. developed the quantum teleportation protocol in 1993, in which they have shown that an unknown quantum state can be teleported with the help of classical communication and a shared resource state, i.e., an entangled channel [20]. To understand the working principle of the protocol, we consider two parties, say, Alice (A) and Bob (B) residing far apart from each other. Suppose that Alice wants to teleport an unknown qubit $|\phi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A$, where $|\alpha|^2 + |\beta|^2 = 1$, from her location to Bob's location. To do this task, an entangled EPR-pair is generated at a source and then one qubit is sent to Alice and another to Bob respectively. Thus, in principle, Alice and Bob share a two-qubit maximally entangled state as a resource state that can be used in the teleportation protocol. Two-qubit maximally entangled states are also known as Bell states, which are of the form as follows

$$|\varphi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle_{AB} + |10\rangle_{AB}) \quad (1.10.1)$$

$$|\varphi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB}) \quad (1.10.2)$$

$$|\psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB}) \quad (1.10.3)$$

$$|\psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle_{AB} - |11\rangle_{AB}) \quad (1.10.4)$$

The set of states $\{|\varphi^+\rangle_{AB}, |\varphi^-\rangle_{AB}, |\psi^+\rangle_{AB}, |\psi^-\rangle_{AB}\}$ forms a basis and known as Bell basis. Let us now consider the Bell state $|\varphi^+\rangle_{AB}$ as the resource state shared between Alice and Bob. In the first step, Alice makes a joint measurement on her EPR particle and the unknown quantum state she wishes to teleport. The composite system can be written as

$$|\phi\rangle_A \otimes |\varphi^+\rangle_{AB} = (\alpha|0\rangle_A + \beta|1\rangle_A) \otimes \frac{1}{\sqrt{2}}(|01\rangle_{AB} + |10\rangle_{AB})$$

$$= \frac{1}{\sqrt{2}}(\alpha(|001\rangle_{AAB} + |010\rangle_{AAB}) + \beta(|101\rangle_{AAB} + |110\rangle_{AAB})) \quad (1.10.5)$$

The computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ in a four-dimensional Hilbert space can be transformed to the Bell basis using the transformation given as

$$|00\rangle_{AA} = \frac{1}{\sqrt{2}}(|\psi^+\rangle_{AA} + |\psi^-\rangle_{AA})$$

$$|11\rangle_{AA} = \frac{1}{\sqrt{2}}(|\psi^+\rangle_{AA} - |\psi^-\rangle_{AA})$$

$$|10\rangle_{AA} = \frac{1}{\sqrt{2}}(|\varphi^+\rangle_{AA} + |\varphi^-\rangle_{AA})$$

$$|01\rangle_{AA} = \frac{1}{\sqrt{2}}(|\varphi^+\rangle_{AA} - |\varphi^-\rangle_{AA}) \quad (1.10.6)$$

Using (1.10.6) in (1.10.5), and after re-arranging the terms, we get

$$
\begin{aligned}
|\phi\rangle_A \otimes |\varphi^+\rangle_{AB} &= \frac{1}{2}[|\varphi^+\rangle_{AA} \otimes (\alpha|0\rangle_B + \beta|1\rangle_B) + |\varphi^-\rangle_{AA} \otimes (-\alpha|0\rangle_B + \beta|1\rangle_B) \\
&+ |\psi^+\rangle_{AA} \otimes (\beta|0\rangle_B + \alpha|1\rangle_B) + |\psi^-\rangle_{AA} \otimes (-\beta|0\rangle_B + \alpha|1\rangle_B)] \quad (1.10.7)
\end{aligned}
$$

Alice then makes the measurement on Bell basis. After the measurement, a state will be projected at Bob's location and the state that Alice wanted to be teleported disappear from her site. This means that the no-cloning theorem [117] is not violated. Bob can reveal the state that appears on his site only after Alice communicates her result to Bob with the help of two bits of classical communication. After receiving two classical bits, Bob will apply the appropriate local unitary operation $\{I \otimes I, I \otimes \sigma_x, I \otimes \sigma_y, I \otimes \sigma_z\}$ to retrieve the state sent by Alice. For instance, if Alice's measurement outcome is $|\varphi^+\rangle_{AA}$ then the projected state is an exact replica of the state which was teleported, and therefore in this case, Bob has nothing to do to retrieve the state sent by Alice. The rest of the cases are discussed in Table 1.1.

Alice's measurement outcome	State appeared at Bob's location	Local operation performed by Bob	Final State at Bob's location
$\|\varphi^+\rangle_{AA}$	$\alpha\|0\rangle_B + \beta\|1\rangle_B$	I_2	$\alpha\|0\rangle_B + \beta\|1\rangle_B$
$\|\varphi^-\rangle_{AA}$	$-\alpha\|0\rangle_B + \beta\|1\rangle_B$	$-\sigma_z$	
$\|\psi^+\rangle_{AA}$	$\beta\|0\rangle_B + \alpha\|1\rangle_B$	σ_x	
$\|\psi^-\rangle_{AA}$	$-\beta\|0\rangle_B + \alpha\|1\rangle_B$	$-i\sigma_y$	

Table 1.1: Table of local operation to be applied on the received state to get the resulting state as the teleported state

Definition 1.10.1. The fidelity of quantum teleportation may be defined as the overlapping between the input state to be sent by the sender and the output state received by the receiver. For a shared channel ρ_{AB} in quantum teleportation protocol teleportation fidelity is denoted by (f(ρ_{AB})).

Remark: If the maximally entangled state is shared between two distant partners, then following Bennett's protocol, the fidelity of the teleportation is found to be equal to unity. Thus, it is called a perfect teleportation.

1.10.2 Revisiting quantum teleportation

In this section, let us revisit quantum teleportation and take a small tour of what has been already done on this topic. To proceed forward, let us define a few terms that may help us to go deeper into the detailed understanding of quantum teleportation.

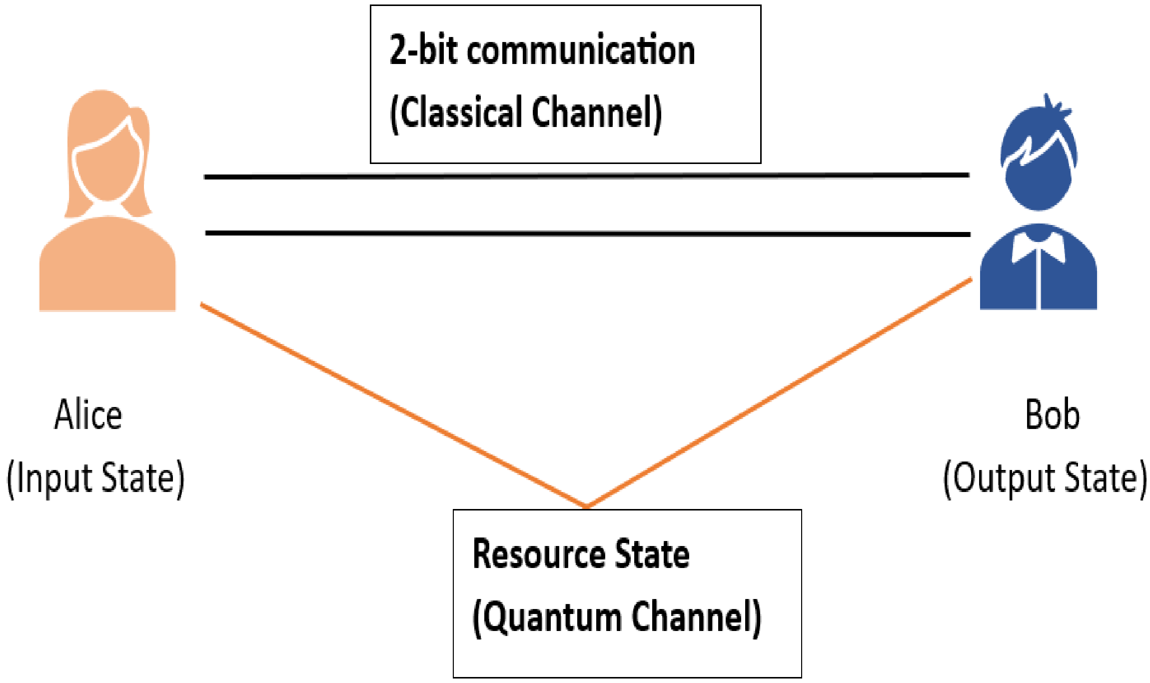

Figure 1.1: Pictorial representation of Bennett et. al. quantum teleportation protocol

Singlet fraction: It is defined as the maximum overlap between the quantum state described by the density operator ρ_{AB} and a maximally entangled state in a finite-dimensional Hilbert space. It is denoted by F(ρ_{AB}).
Mathematically, the singlet fraction F(ρ_{AB}) can be expressed as

$$F(\rho_{AB}) = Max_i(\langle\psi_i|\rho_{AB}|\psi_i\rangle, \quad i = 1, 2, \cdots, d^2 \tag{1.10.8}$$

where $|\psi_i\rangle$ denote the maximally entangled states lying in the Hilbert space of dimension d.
Alternatively, the singlet fraction may also be expressed as [118]

$$F(\rho_{AB}) \quad = \quad max_{U_A,U_B}\{F[\rho_{AB}, (U_A \otimes U_B)|\phi_d^+\rangle\langle\phi_d^+|(U_A^\dagger \otimes U_B^\dagger)]\} \tag{1.10.9}$$

where $|\phi_d^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$ and maximum is taken over all unitary operators U_A and U_B.
A quantity $V(\rho_{AB})$: It is used to detect whether the shared state ρ_{AB} is useful in quantum teleportation and it can be defined as

$$V(\rho_{AB}) = Tr\sqrt{T^\dagger T} = u_1 + u_2 + u_3 \tag{1.10.10}$$

where $u_i's$ are the eigenvalues of $\sqrt{T^{\dagger}T}$ and T denotes the correlation matrix of order 3. The entries t_{ij} of the correlation matrix T can be calculated as

$$t_{ij} = Tr[\rho_{AB}(\sigma_i \otimes \sigma_j)], \quad i,j \in \{1,2,3\} \tag{1.10.11}$$

Remark: If we follow the prepare and measure strategy to extract information from a given single copy of a quantum state then the maximum information we can gain about the given system by performing the optimal measurement on the system is $\frac{2}{3}$ [119].

1.10.2.1 Quantum teleportation: shared state lying in $2 \otimes 2$ dimensional Hilbert space

Braunstein et. al. [120] have shown that the teleportation of the polarization state of a photon can be achieved by measuring an optical version of the Bell operator. The conditional efficiency of the teleportation scheme has been shown to tend to 100%. K. Banaszek has derived the necessary and sufficient conditions to obtain the optimal teleportation in terms of maximal teleportation fidelity using arbitrary two-qubit pure states [121]. G. Rigolin also considered arbitrary two-qubit quantum state of the form $|\phi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ to demonstrate faithful quantum teleportation and further showed that if a multipartite state acts as a genuine teleportation channel then that state will have maximum entanglement [122]. Agrawal and Pati [123] reported that the Bennett et. al. quantum teleportation protocol can also be implemented with a non-maximally entangled pure state instead of the maximally entangled pure state as a resource state but the price has to be given in terms of the teleportation fidelity. The teleportation fidelity in this scenario reduces to a value, which is less than unity. Moreover, the teleportation protocol losses its deterministic property, and the protocol succeded with some non-zero probability. Another quantum teleportation scheme has been discussed in [124], where the shared entangled state between two distant parties is a non-maximally entangled state. Lee et. al. [125] studied quantum teleportation protocol using Werner state as a resource state for the teleportation of a two-qubit entangled state.

For $2 \otimes 2$ dimensional system, if the shared state is described by the density operator ρ_{AB} then the relation between singlet fraction $(F(\rho_{AB}))$ and teleportation fidelity $(f(\rho_{AB}))$ can be expressed as [126]

$$f(\rho_{AB}) = \frac{2F(\rho_{AB}) + 1}{3} \tag{1.10.12}$$

It is clear from the above relation (1.10.12) that if $F(\rho_{AB}) > \frac{1}{2}$ then the shared state ρ_{AB} is useful for teleportation [126]. Horodecki et. al. further expressed the teleportation fidelity in terms of the quantity $V(\rho_{AB})$ as [127]

$$f(\rho_{AB}) = \frac{1}{2}(1 + \frac{V(\rho_{AB})}{3}) \tag{1.10.13}$$

Using (1.10.13), Horodecki et. al. [127] also derived a general result for the usefulness of a two-qubit shared state as a resource state in quantum teleportation that can be stated as a result given below:

Result: Any general two-qubit state is useful for teleportation if and only if $V(\rho_{AB}) > 1$.

In an open quantum system, the shared two-qubit state between two distant partners is always a mixed state. Therefore, if we use the mixed two-qubit entangled state ρ_{AB} to implement the teleportation protocol then it may happen that either the teleportation fidelity $f(\rho_{AB}) \leq \frac{2}{3}$ or the singlet fraction $F(\rho_{AB}) \leq \frac{1}{2}$. Thus, the question arises that if the singlet fraction of the certain shared two-qubit entangled state is less than or equal to $\frac{1}{2}$, then can we increase the value of the singlet fraction of the shared state and go beyond the critical value $\frac{1}{2}$? Badziag et. al. [128] analyzed this question and applied local trace-preserving transformation on these two-qubit entangled states to increase the singlet fraction of the shared two-qubit states. They have derived this result for a particular class of states. F. Verstraete and H. Verschelde [129] analyzed the above question in a general setting and studied the teleportation of a single qubit using an arbitrary mixed two-qubit state. They have shown that any two-qubit entangled state described by the density operator ρ_{AB} can be useful as a resource state in quantum teleportation and derived an expression of the optimal singlet fraction denoted by $F_{LOCC}^{opt}(\rho_{AB})$. The optimal singlet fraction $F_{LOCC}^{opt}(\rho_{AB})$ is given by [129]

$$F_{LOCC}^{opt}(\rho_{AB}) = \frac{1}{2} - Tr(X^{opt}\rho_{AB}^{T_B}) \tag{1.10.14}$$

where X^{opt} is the optimal filtering operation of rank one and it can be written as $X^{opt} = (A \otimes I_2)|\phi\rangle\langle\phi|(A^{\dagger} \otimes I_2)$, $-I_2 \leq A \leq I_2$. The matrix A can be expressed in the form as

$$A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad -1 \leq a \leq 1 \tag{1.10.15}$$

It may be observed that the optimal singlet fraction given in (1.10.14) depends on the partial transposition operation, which is positive but not completely positive operation. Thus, it is very difficult to realize the optimal singlet fraction in the laboratory. To tackle this problem, S. Adhikari [33] re-expressed optimal singlet fraction $F_{LOCC}^{opt}(\rho_{AB})$ in terms of the minimum eigenvalue of the resulting state obtained after applying the operation known as structural physical approximation of the partial transpose (SPA-PT). The optimal singlet fraction $F_{LOCC}^{opt}(\rho_{AB})$ given in (1.10.14) then reduces to

$$F_{LOCC}^{opt}(\rho_{AB}) = \frac{1}{2} - \frac{9(a^2+1)}{2}[\lambda_{min} - \frac{2}{9}], \quad -1 \leq a \leq 1 \tag{1.10.16}$$

where λ_{min} denote the minimum eigenvalue of SPA-PT of state ρ_{AB}.
A lot of work has already been done by using different types of noise in the quantum teleportation protocol [130–132].

1.10.2.2 Quantum teleportation: shared state lying in $d \otimes d$ dimensional Hilbert space

Horodecki et. al. [126] derived the relation between singlet fraction $F(\rho_{AB})$ and teleportation fidelity $f(\rho_{AB})$ for the state ρ_{AB} lying in $d \otimes d$ dimensional Hilbert space. The obtained relation is then given by

$$f(\rho_{AB}) = \frac{dF(\rho_{AB})+1}{d+1} \tag{1.10.17}$$

The maximum acheivable singlet fraction of a shared separable state ρ_{AB} lying in $d \otimes d$ dimensional Hilbert space in quantum teleportation protocol cannot be greater than $\frac{1}{d}$. In other words, it can be restated as if a shared separable state ρ_{AB}^S lying in $d \otimes d$ dimensional Hilbert space is used as a resource state in a quantum teleportation protocol then $F(\rho_{AB}^S) \leq \frac{1}{d}$ [126]. Thus, for any bipartite separable state ρ_{AB} in $d \otimes d$ dimensional system, the teleportation fidelity $f(\rho_{AB})$ is always less than or equal to $\frac{2}{d+1}$, i.e., $f(\rho_{AB}) \leq \frac{2}{d+1}$. But there may also exist entangled states described by the density operator ρ_{AB}^e for which $f(\rho_{AB}^e) \leq \frac{2}{d+1}$. So, for a given bipartite state ρ_{AB}, it will not be possible to discriminate between separable and entangled state just by merely observing the inequality $f(\rho_{AB}) \leq \frac{2}{d+1}$. Thus, if we consider the contrapositive statement then it may be possible to say something about the entangled state. Therefore, the contrapositive statement may now be stated as: if $f(\rho_{AB}) > \frac{2}{d+1}$ or in terms of singlet fraction $F(\rho_{AB}) > \frac{1}{d}$, then the quantum state ρ_{AB} may be considered as an entangled state which may be useful in quantum teleportation.
From the above arguments, it is clear that the success of quantum teleportation de-

pends on the shared resource state, but it is not always feasible to check whether the shared resource state is useful for teleportation or not. N. Ganguly et. al. [133] partially solved this problem by constructing a witness operator for the possible detection of the shared entangled state useful for quantum teleportation. Later Adhikari et. al. [38] provided a systematic way to construct an optimal witness operator for qudit systems and in particular, they have shown that the constructed witness operator is optimal for both qubit-qubit and qutrit-qutrit system. Another higher dimensional teleportation scheme has been proposed using a partially entangled state as a resource state in which Alice uses a less entangled quantum channel but uses more classical bits to transfer a quantum state to Bob [134].

Till now, the teleportation of an arbitrary qubit has been studied in the quantum teleportation protocol but there is a scope of teleporting an arbitrary qudit using an entangled two-qudit state. This is addressed by Luo et. al. [135]. They have studied the teleportation of an arbitrarily high dimensional quantum state in a variant of the quantum teleportation scheme. They have illustrated their scheme for transferring an unknown qutrit via a maximally entangled two-qutrit state [135].

The effect of noise for d-dimensional bipartite state has been studied in [136] using four different types of noises such as dit-flip, d-phase-flip, dit-phase-flip, and depolarizing noise. The average fidelity of teleportation was derived using a different approach when the qudit undergoes the generalized amplitude damping channel.

1.10.3 Quantum teleportation via multipartite state as a shared state

It is known that there are two types of genuine three-qubit entangled states, namely, GHZ and W class of states, which are inequivalent under stochastic local operation and classical communication (SLOCC) [26]. Quantum teleportation using three particle GHZ state as a resource state has been introduced by Karlson et. al. [137]. It is an interesting fact that although the GHZ class of states can serve as a potential candidate for quantum teleportation but W class of states does not. This fact is evident from the work [138] where it has been shown that W class of states are not suitable for perfect quantum teleportation. The teleportation is perfect in the sense that a quantum state can be teleported with unit fidelity. In this direction of research, Gorbachev et. al. [139] have studied quantum teleportation by considering those W class of states as a shared state, which can be obtained after the application of non-local unitary operator on the inequivalent class of GHZ states. Further, Agrawal and

Pati introduced a new class of three-qubit states, which is given by [140]

$$|W_n\rangle_{ABC} = \frac{1}{\sqrt{2+2n}}(|100\rangle_{ABC} + \sqrt{n}e^{\iota\gamma}|010\rangle_{ABC} + \sqrt{n+1}e^{\iota\delta}|001\rangle_{ABC}) \qquad (1.10.18)$$

They have shown that the pure three-qubit state given by (1.10.18) falls under W class of state and this W class of state can be used for perfect teleportation.
Joo et. al. [141] developed two schemes S_1 and S_2 of teleportation protocol, when the W class state is shared as a resource state in quantum teleportation protocol. They have calculated the success probability and the average fidelity with respect to both the schemes and found that the average fidelity of the scheme S_2 is always less than the scheme S_1 [141]. Quantum teleportation of a two-qubit state using a constructed genuine four-qubit entangled state as a resource state has been studied in [142], where the constructed four-qubit state is not reducible to the pair of Bell state. Its properties are compared with the four-party GHZ and W state [142]. Jung et. al. [143] discussed the behavior of GHZ and W state when shared as a resource state in quantum teleportation protocol under the effect of different kinds of noise and derived a relation between fidelity and entanglement under different noise situtations [143]. Another scheme of teleportation using a GHZ-like state as a resource state which is quite similar to W state has been proposed [144]. In the scheme they considered one sender and two receivers and the unknown qubit can be teleported to any of the receivers depending upon the outcome of the sender and the third party other than the receiver [144]. Quantum teleportation using three-party non-symmetric or asymmetric states as a resource state has been studied in [145]. J. Bae et. al. further have shown that teleportation with an asymmetric three-qubit state would carry more information than teleportation using a three-qubit symmetric state [145].

1.10.4 Realization of quantum teleportation in an experiment

First experimental approach of quantum teleportation has been reported by D. Bouwmeester et. al. [146]. The implementation of quantum teleportation of any arbitrary quantum state has been achieved in an experiment. They have used parametric down-conversion and two-photon interferometry for generating entanglement and analyzing the Bell state in an experiment [147]. Experimental implementation of quantum teleportation over interatomic distances using liquid-state nuclear magnetic resonance has also been reported in [148]. In 2001, an experimental approach of deterministic teleportation of an arbitrary polarization state has been reported in [149]. In this experiment, the Bell state measurements were distinguished completely, which assured a perfect quantum teleportation scheme [149]. An experimental demonstration of teleportation with an average fidelity of 0.84 has been given in [150]. This experiment has been performed to achieve teleportation using a C-not gate, which opens the way toward quantum computing [150]. The teleportation of a two-qubit composite system can be realized in an experiment [151]. The high-fidelity teleportation of photons over a distance of 600 meters using linear optics is presented in [152]. A free-space implementation of quantum teleportation over 16 km and later for 100 km and 143 km respectively has been proclaimed in [153–155]. Deterministic quantum teleportation of photonic quantum bits has also been shown to be realized in an experiment using a hybrid technique [156]. Recently, an experimental realization of quantum teleportation of a high dimensional state has been recorded in [157].

1.11 Quantum teleportation and non-locality

S. Popescu [158] remarked that the non-locality given out by teleportation and the non-locality of correlation may be considered as the two faces of the same physical property but they are inequivalent. Further, he conjectured that if a bipartite entangled state ρ_{AB} violates some Bell's inequality, then the state ρ_{AB} can be used for teleportation, and vice versa [158]. But this conjecture may not be true as there exist some two-qubit mixed entangled states that are useful in teleportation but do not violate any Bell's inequality. Popescu [158, 159] also raised a few questions on the topics of teleportation and non-locality such as (i) whether there exists any two-qubit mixed state which violate Bell's inequality but not useful in quantum teleportation? (ii) what is the relation between the fidelity of teleportation and non-locality?

N. Gisin [160] obtained the lower bound of the teleportation fidelity $f(\rho_{AB})$, where the state ρ_{AB} shared between two distant partners is non-local. Thus, the state ρ_{AB} is non-local if the teleportation fidelity $f(\rho_{AB})$ satisfies

$$f(\rho_{AB}) > 0.87 \tag{1.11.1}$$

Gisin [160] also cited an example of a two-qubit mixed state such as Werner state [103] as a resource state in a quantum teleportation protocol and found that the fidelity of the Werner state is more than the classical limit i.e. $\frac{2}{3}$ but is not greater than 0.87.

He then concluded that it does not mean that there exists non-locality in the Werner state which is hidden. In 1996, Horodecki et. al. [127] took an important step in this direction and derived a general relation between teleportation fidelity and Bell's inequality for any arbitrary two-qubit state. They stated that any two-qubit quantum state which violates Bell's inequality is useful for teleportation and provided the lower bound of maximum teleportation fidelity denoted by $f_{max}(\rho_{AB})$ in terms of $M(\rho_{AB})$, is given by

$$\frac{1}{2}[1+\frac{M(\rho_{AB})}{3}] \leq f_{max}(\rho_{AB}) \tag{1.11.2}$$

where M(ρ_{AB}) may be considered as a quantifier for the estimation of non-locality and is defined in (1.8.1). Also, it can be easily found out that the maximal value of $M(\rho_{AB})$ is equal to $\frac{B_{max}^2}{4}$, where B_{max} denotes the maximal value of Bell's inequality of the state ρ_{AB}. In 2013, Cavalcanti et. al. [161] showed that all the entangled states which are useful for teleportation do violate Bell-CHSH inequality deterministically. Thus they act as non-local resources in quantum teleportation. In this way, they established a linkage between teleportation and non-locality. Chakrabarty et. al. [162] used the output of the Pati-Braunstein deletion machine as a resource state for quantum teleportation and found that it is useful for teleportation but it follows the LHV model.
Wang et. al. [163] studied the effect of two-qubit noisy channels on quantum teleportation, entanglement, and non-locality. In particular, they have considered the effect of noise on the Bell state and the Werner state and derived the relation between non-locality, entanglement and teleportation. They also found that there exists a critical value of the correlation for which non-classical teleportation fidelity, non-vanishing entanglement, and Bell non-local states have been achieved [163]. T. Jennewein et. al. [164] proved that quantum teleportation indeed exhibits non-local nature by performing an experiment. Non-locality and teleportation for three-qubit states were first studied by S. Lee et. al. and provided a general result that if any three-qubit state violates Mermin's inequality then they are useful in three-qubit quantum teleportation [165].

1.12 Controlled quantum teleportation (CQT)

The original quantum teleportation scheme proposed by Bennett et. al. [20] is applicable for two parties. In 1998, Karlsson et. al. [137] introduced another teleportation protocol that works for three parties, which is famously known as controlled quantum teleportation (CQT). In this scheme, there are three parties Alice (the sender), Bob (the receiver), and Charlie (the controller), shares a three-qubit entangled state known as the resource state between them. Alice wants to teleport an unknown qubit to Bob via the shared state with the participation of Charlie. Karlsson et. al. [137] considered a genuine three-qubit GHZ state as a resource state. Therefore, the teleportation protocol via GHZ state may be described as follows: firstly, Charlie will apply Von Neumann measurement on his particle. After his measurement, an entangled state will be projected between Alice and Bob in terms of Charlie's measurement. After his measurement, he uses classical bits to communicate his measurement outcome to Alice and Bob. Then, Alice will perform a Bell-state measurement on her shared qubit and the unknown quantum state which she wants to be teleported. Consequently, a single qubit state will be projected at Bob's location, which contains Charlie's measurement parameter. Finally, Alice will send one classical bit to Bob to inform him about her measurement outcome so that he can apply the appropriate unitary operator to obtain the teleported state at his location. This protocol is known as controlled quantum teleportation. This teleportation scheme is controlled by Charlie in the sense that, he can adjust his parameter in such a way that it can increase or decrease the fidelity of the teleportation.

Now the question arises that under what circumstances, we will be able to successfully teleport a qubit following the procedure of controlled quantum teleportation protocol with three-qubit state? Like in quantum teleportation, the success rate of controlled quantum teleportation also depends on fidelity. Unlike teleportation, there are two types of fidelities in controlled quantum teleportation. These fidelities may be termed as (i) Conditioned fidelity and (ii) Non-conditioned fidelity.

Now, we are in a position to describe the CQT scheme. In CQT scheme, we may consider that Alice, Bob and Charlie shared a three-qubit pure/mixed state described by the density operator ρ_{ABC}. Throughout the thesis, we assume that Charlie act as a controller who perform Von Neumann measurement on his qubit. A single qubit Von Neumann measurement in the computational basis may be described as $\{\pi_k = |k\rangle\langle k|, k = 0,1\}$. Generally, a single qubit measurement in any arbitrary basis is denoted by B_k, which is given in (1.4.22) and (1.4.23). When Charlie perform Von Neumann measurement B_k on his qubit, the three-qubit state ρ_{ABC} projected onto the two-qubit state

$$\rho_{AB}^{(k)} = \frac{1}{p_k}(I \otimes I \otimes B_k)\rho_{ABC}(I \otimes I \otimes B_k^{\dagger}), \quad k = 0,1 \tag{1.12.1}$$

where $p_k = tr((I \otimes I \otimes B_k)\rho_{ABC}(I \otimes I \otimes B_k^{\dagger}))$ denote the probability of collapsing the three-qubit state to two-qubit state after the measurement performed on the third subsystem. The two-qubit state $\rho_{AB}^{(k)}$ shared between Alice and Bob may be used as a resource state when teleporting an arbitrary single qubit state possessed by Alice. We further assume that in the process of single qubit teleportation using a shared two-qubit state, Alice act as a sender and Bob, a receiver. Alternatively, we may also describe the above equivalent situation with the reduced two-qubit state described by the density operator $\rho_{AB} = Tr_C(\rho_{ABC})$. The resulting two-qubit state described by the density operator ρ_{AB} may also be used in transmitting an arbitrary single qubit state through conventional teleportation scheme.

In CQT scheme, the faithfulness of the teleportation may be quantified by the conditioned fidelity denoted by $f_C(\rho_{AB}^{(k)})$ and the non-conditioned fidelity $f_{NC}(\rho_{AB})$.

(i) **Conditioned fidelity:** It is the fidelity of the state $\rho_{AB}^{(k)}$ acheived with the participation of the controller. It is denoted by $f_C(\rho_{AB}^{(k)})$.

(ii) **Non-conditioned fidelity:** It is the fidelity estimated without the controller's participation. It is denoted by $f_{NC}(\rho_{AB})$.

Assumptions: For a successful controlled quantum teleportation, we assume the following:

(i) For a qubit system, conditioned fidelity should be greater than the classical limit, i.e. $f_C(\rho_{AB}^{(k)}) > \frac{2}{3}$.

(ii) Non-conditioned fidelity should be less than or equal to classical fidelity, i.e. $f_{NC}(\rho_{AB}) \leq \frac{2}{3}$.

There are lot of work that can be found in the literature for the development of the CQT scheme. In 2008, Gao et. al. [166] used partially entangled states called maximal slice (MS) states as a resource channel for CQT protocol. A deterministic CQT scheme with the MS states and GHZ state as the resource state has been studied in [137, 167]. The efficiency of CQT protocol may be measured by the quantity known as controller's power ($P_{CT}^{(k)}$), which can be expressed as the difference between the conditioned fidelity and non-conditioned fidelity. Mathematically, it can be written in the form as

$$P_{CT}^{(k)} = f_C(\rho_{AB}^{(k)}) - f_{NC}(\rho_{AB}) \tag{1.12.2}$$

Initially, Li et. al. [167] have considered $f_C(\rho_{AB}^{(k)}) = 1$ in their work and defined controller's power ($P_{CT}^{(k)}$) as $1 - f_{NC}(\rho_{AB})$. One important point to note here that, we need to minimize non-conditioned fidelity of teleportation to achieve the maximum value of controller's power. In this respect, they found that the non-conditioned fidelity cannot attained the better value than the fidelity using a classical channel, when CQT is considered with the GHZ state. Thus the maximally entangled GHZ state is considered as a suitable channel for the CQT scheme. On the other hand, when MS states are used for controlled quantum teleportation, the non-conditioned fidelity can be greater than the classical limit and hence these states are not suitable channels for the CQT of arbitrary single-qubit states [167]. Later, they have generalized their CQT protocol for multiqubit pure system [168]. A more general form of controller's power for perfect controlled quantum teleportation of an entangled three-qubit pure state has been studied by Jeong et. al. [169]. They defined a term known as the minimal control power, and calculated the minimal control power for a class of GHZ states and W states.

Further, Artur et. al. [170] characterize the three-qubit states with extreme properties and use them to derive tight lower and upper bounds for both the teleportation fidelity and control power for a given amount of entanglement. In another piece of work, Paulson et. al. [171] analyzed the CQT protocol using X maximally and non-maximally entangled mixed state for a given spectrum and mixedness of the state. Wang et. al. [172] studied d-dimensional control power, by teleporting a qudit using 2^N-dimensional standard three-qudit GHZ state or GHZ-type state channels in the perfect CQT scheme. Recently, controlled quantum teleportation was experimentally realized using cluster states [173]. The potential application of controlled quantum teleportation may be found in quantum computing algorithms, quantum communication protocols, and quantum error correction schemes [174]. The concept of CQT may also be used in quantum networks [175], entanglement swapping [176], quantum reapters [177], quantum key distribution [178], and quantum cryptography [179–181].

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Chapter 2

Eigenvalue Criteria for Quantum Teleportation Protocol

"Teleportation is the closest we can get to a magic trick. It's like saying, let's make this object disappear here and reappear there. And it's not easy. "

- Anton Zeilinger

In this chapter [1]*, we study the problem of estimation of singlet fraction in higher dimensional bipartite system. We derive criteria for the detection of $d \otimes d$ dimensional negative partial transpose (NPT) entangled state useful for teleportation. The criteria derived here are based on the maximum eigenvalue of the NPT entangled state, which is in principle easier to determine experimentally than to completely reconstruct the state via tomography. We then illustrate our criteria by considering a class of qubit-qubit system and qutrit-qutrit system.*

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[1]This chapter is based on a published research paper " Teleportation criteria based on maximum eigenvalue of the shared $d \otimes d$ dimensional mixed state: Beyond Singlet Fraction, *International Journal of Theoretical Physics* **60**, 1038 (2021)".

2.1 Introduction

Quantum teleportation is an important topic to study in quantum information science. It plays a vital role in the development of quantum information theory and quantum technologies [1, 3]. Bennett et. al. [20] have developed the first protocol of quantum teleportation for a two-qubit system. The developed protocol talks about the transfer of information contained in a qubit from a sender (say, Alice) to a receiver (say, Bob). To execute this protocol, firstly Alice and Bob shared an entangled state between them. Then she performs a two-qubit Bell-state measurement on particles in her possession. After that, she communicates the measurement result to Bob by sending two classical bits. The receiver Bob then reconstruct the quantum state at his place by applying suitable unitary operation such as $I, \sigma_x, \sigma_y, \sigma_z$ on his qubit according to the measurement outcome sent by the sender Alice.

The quantum teleportation protocol described above can be considered as a basic protocol for other quantum schemes such as quantum repeater [115], quantum gate teleportation [116], port-based teleportation [182]. The most important ingredient in quantum teleportation protocol is the resource state, which is shared between two distant parties because it would not be possible to realize quantum teleportation without a shared entangled state. In a realistic situation, the shared entangled state in general becomes a mixed entangled state. The usefulness of the shared entangled state between two distant partners in a teleportation protocol depends on the value of the singlet fraction [126]. For $d \otimes d$ dimensional system, the shared state is useful in quantum teleportation if the singlet fraction of the shared state is greater than $\frac{1}{d}$.

The singlet fraction also characterizes the nature of the quantum state in the sense that if the given state is separable then the singlet fraction of the given two-qubit mixed state is less than or equal to $\frac{1}{2}$ [126]. In another way, it can be stated tha if the singlet fraction of an arbitrary given state is greater than $\frac{1}{2}$ then the state is entangled. But the converse of the statement is not true. This means that there exists a two-qubit mixed entangled state whose singlet fraction is less than or equal to $\frac{1}{2}$ and hence not useful in quantum teleportation. In this perspective, Badziag et. al. [128] have shown that a dissipative interaction with the environment is sufficient to improve the value of the singlet fraction. They have presented a class of entangled quantum states whose singlet fraction is exactly equal to $\frac{1}{2}$ before interaction with the environment but after the interaction with the environment, the value of the singlet fraction improves. Even getting this result also, the question remains whether interaction with the environment increases the singlet fraction of any two-qubit mixed state? The answer is in affirmative. Verstraete et. al. [129] have studied this problem and obtained trace-preserving LOCC that enhances the singlet fraction and makes its value greater than $\frac{1}{2}$ for any two-qubit mixed entangled state. They have derived a connection between the optimal singlet fraction and the partial transpose of a given state. The established relation tells us that the two-qubit state is useful as a resource state for teleportation if and only if the optimal singlet fraction is greater than $\frac{1}{2}$.

Till now, we have discussed about the resource state useful in teleportation for $2\otimes 2$ dimensional system. We now continue our discussion with a higher dimensional system. Generally, it has been proved that to teleport an arbitrary d-dimensional pure state, only a maximally entangled pure state in $d\otimes d$ is required [183]. Zhao et. al. [184] have derived the necessary and sufficient conditions of faithful teleportation of an arbitrary d-dimensional pure state with $m\otimes d$ and $d\otimes n$ dimensional entangled resource, where m and n denoting the dimension of the first subsystem in $m\otimes d$ and second subsystem in $d\otimes n$ dimensional entangled resource states respectively. A general expression for the output state of the quantum channel associated with the original teleportation protocol with an arbitrary $d\otimes d$ dimensional mixed resource state has been obtained in [185].

The motivation of this chapter is as follows: (i) Since partial transposition is a non-physical operation and cannot be implemented in a laboratory so we apply SPA on partial transposition of the given state. Being SPA a completely positive map, the expression of singlet fraction gets free from partial transposition operation and thus it can be implemented in an experiment. (ii) The second motivation comes from the problem of estimation of singlet fraction in a higher dimensional bipartite system. Since singlet fraction depends on bipartite maximally entangled states but in higher dimensional system, it is very difficult to construct bipartite maximally entangled state so it would not be an easy task to get the experimentally estimated value of singlet fraction for higher dimensional bipartite system.

The above mentioned motivation enabled us to establish another criterion for the detection of entangled state useful in teleportation and that must be easy to implement in an experiment. Therefore, instead of singlet fraction if the criterion is expressed in terms of the eigenvalue then it requires a lesser number of measurements than to completely reconstruct the state via tomography [186]. Thus, in terms of a number of required measurements, the criterion based on eigenvalue is more efficient than quantum tomography.

2.2 Revisiting maximal singlet fraction of mixed two-qubit state

Verstraete et. al. [129] have derived the optimal trace-preserving local operation together with classical communication and have shown that it optimally increases the singlet fraction of a mixed quantum state ρ_{AB} and hence maximizes its teleportation fidelity. They have studied the case of a two-qubit system and proved that if the state is entangled then it is always possible to increase the singlet fraction above $\frac{1}{2}$ and thus make teleportation fidelity greater than $\frac{2}{3}$. Since their result is based on the partial transposition operation so it is not possible to realize it in the experimental setup. In this section, we revisit their result and apply structural physical approximation (SPA) of partial transposition. By doing this, the final expression of the singlet fraction gets modified, and since SPA of partial transposition is a completely positive map so the singlet fraction can be estimated experimentally. However, we find that in this case, the value of the singlet fraction is not always greater than $\frac{1}{2}$. Therefore, the result of this section motivates us further to investigate new teleportation criteria that will be studied in the following section.
The expression of optimal singlet fraction after LOCC for a two-qubit state ρ_{AB} is already discussed in (1.10.14). We may note here an important point is that the expression of the singlet fraction contains the partial transposition operation, which is not a completely positive map so the partial transposition of state ρ_{AB} denoted by ρ_{AB}^{Γ} is not a physically realizable operation. Thus the value of singlet fraction found in the Verstraete et. al. [129] work may not be directly accessible in an experiment. To overcome this problem, we use SPA-PT of ρ_{AB} and re-express $Tr(X^{opt}\rho_{AB}^{\Gamma})$ given in (1.10.14) as [33, 34]

$$Tr(X^{opt}\rho_{AB}^{\Gamma}) = 9Tr(X^{opt}\widetilde{\rho}_{AB}) - 2 \tag{2.2.1}$$

where $\widetilde{\rho}_{AB}$ is the SPA-PT of ρ_{AB}.
Using (2.2.1), equation (1.10.14) can be re-expressed as

$$F_{LOCC}^{opt}(\rho_{AB}) = \frac{5}{2} - 9Tr(X^{opt}\widetilde{\rho}_{AB}) \tag{2.2.2}$$

Now it is possible to estimate the value of $F_{LOCC}^{opt}(\rho_{AB})$ experimentally but we have to pay a cost in a way that the obtained value of $F_{LOCC}^{opt}(\rho_{AB})$ may not always be greater than $\frac{1}{2}$. If it may happen that $Tr(X^{opt}\widetilde{\rho}_{AB}) < \frac{2}{9}$ then only $F_{LOCC}^{opt}(\rho_{AB})$ is greater than $\frac{1}{2}$. This implies that if there exist state for which $Tr(X^{opt}\widetilde{\rho}_{AB}) \geq \frac{2}{9}$ then $F_{LOCC}^{opt}(\rho_{AB}) \leq \frac{1}{2}$. To illustrate, let us consider a two-qubit state described by the density operator

$$\sigma_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b & f & 0 \\ 0 & f^* & d & 0 \\ 0 & 0 & 0 & e \end{pmatrix}, \quad b+d+e=1 \tag{2.2.3}$$

where $*$ denotes the complex conjugation.

The density matrix σ_{12} has been studied by many authors in different contexts [39, 187–191]. The state σ_{12} is an entangled state and its concurrence is given by [39,187]

$$C(\sigma_{12}) = 2|f| \tag{2.2.4}$$

Using (1.5.23), we can obtain the SPA-PT of σ_{12} as

$$\widetilde{\sigma}_{12} = \begin{pmatrix} \frac{2}{9} & 0 & 0 & \frac{f}{9} \\ 0 & \frac{2+b}{9} & 0 & 0 \\ 0 & 0 & \frac{2+d}{9} & 0 \\ \frac{f^*}{9} & 0 & 0 & \frac{2+e}{9} \end{pmatrix} \tag{2.2.5}$$

Now if we consider the filter A of the form as $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $0 \leq a \leq 1$, then X^{opt} is given by

$$X^{opt} = \begin{pmatrix} \frac{a^2}{2} & 0 & 0 & \frac{a}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{2.2.6}$$

The optimal singlet fraction of σ_{12} is given by

$$F_{LOCC}^{opt}(\sigma_{12}) = \frac{5}{2} - 9Tr(X^{opt}\widetilde{\sigma}_{12}) \tag{2.2.7}$$

where $Tr(X^{opt}\widetilde{\sigma}_{12})$ is given by

$$Tr(X^{opt}\widetilde{\sigma}_{12}) = \frac{2a^2+2aRe(f)+2+e}{18} \tag{2.2.8}$$

The inequality $\frac{2a^2+2aRe(f)+2+e}{18} \geq \frac{2}{9}$ holds if the filtering parameter a satisfies

$$\frac{-Re(f)+\sqrt{Re(f)^2-2e+4}}{2} \leq a \tag{2.2.9}$$

Therefore, $F_{LOCC}^{opt}(\sigma_{12})$ is less than equal to $\frac{1}{2}$ iff (2.2.9) holds.
Let us now consider a particular case where we can set the values of the state parameter as: $b=0.2$, $d=0.4$, $e=0.4$, and $f=0.25+0.1i$. Using these values, the density matrix given in (2.2.3) reduces to

$$\sigma_{12}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.25+0.1i & 0 \\ 0 & 0.25-0.1i & 0.4 & 0 \\ 0 & 0 & 0 & 0.4 \end{pmatrix} \tag{2.2.10}$$

The SPA-PT of $\sigma_{12}^{(1)}$ is given by

$$\tilde{\sigma}_{12}^{(1)} = \begin{pmatrix} \frac{2}{9} & 0 & 0 & \frac{0.25+0.1i}{9} \\ 0 & \frac{2.2}{9} & 0 & 0 \\ 0 & 0 & \frac{2.4}{9} & 0 \\ \frac{0.25-0.1i}{9} & 0 & 0 & \frac{2.4}{9} \end{pmatrix} \tag{2.2.11}$$

The value of $Tr(X^{opt}\widetilde{\sigma}_{12}^{(1)})$ is given by

$$Tr(X^{opt}\widetilde{\sigma}_{12}^{(1)}) = \frac{2a^2+0.5a+2.4}{18} \tag{2.2.12}$$

Thus, the optimal singlet fraction of $\sigma_{12}^{(1)}$ is given by

$$F_{LOCC}^{opt}(\sigma_{12}^{(1)}) = \frac{2.6-2a^2-0.5a}{2}, \quad 0.78 \leq a \leq 1 \tag{2.2.13}$$

Figure 2.1 illustrate the fact that $F_{LOCC}^{opt}(\sigma_{12}^{(1)})$ is always less than or equal to $\frac{1}{2}$.

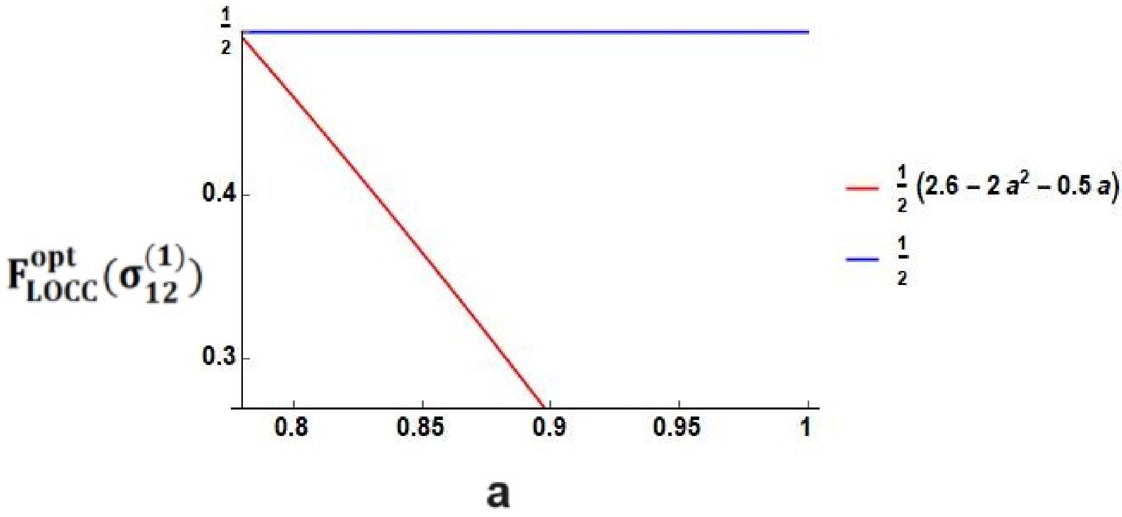

Figure 2.1: Plot of optimal singlet fraction obtained after LOCC operation versus the filtering parameter a.

This motivates us to investigate new teleportation criteria that may go beyond singlet fraction and identify not only a two-qubit entangled state but also a higher dimensional NPT entangled state, which may be useful in quantum teleportation.

2.3 Teleportation criteria in terms of maximum eigenvalue

In this section, we derive a criterion for the usefulness of the shared $d \otimes d$ dimensional NPT entangled states in quantum teleportation. In particular, for $2 \otimes 2$ dimensional entangled states, the derived criterion may be useful in a situation when the singlet fraction calculated before sending a qubit through the local environment is less than or equal to $\frac{1}{2}$. This means that if we don't use the local environment to increase the value of the singlet fraction, our criterion can still detect whether the shared resource state is useful for teleportation or not. To achieve our main result, we need to go through a few lemmas that we have discussed below.

Lemma 2.3.1. If $\lambda_{max}(\rho_{AB})$ denotes the maximum eigenvalue of $d \otimes d$ dimensional quantum state ρ_{AB}, then

$$\frac{1}{d^2} \leq \lambda_{max}(\rho_{AB}) \leq 1 \tag{2.3.1}$$

Proof: Let us consider a $d \otimes d$ dimensional quantum state ρ_{AB}. Therefore, the density matrix ρ_{AB} has d^2 eigenvalues and let they are denoted by λ_1, λ_2,.........., λ_{d^2}. Using the properties of a density matrix, we have

$$1 = Tr(\rho_{AB}) = \sum_{i=1}^{d^2} \lambda_i \leq d^2 \lambda_{max}(\rho_{AB}) \tag{2.3.2}$$

Also, since it is known that $\lambda_{max}(\rho_{AB}) \leq 1$, for a density matrix ρ_{AB} and using (2.3.2), we have the inequality (2.3.1). ■

Lemma 2.3.2. The maximum eigenvalue of an arbitrary $d \otimes d$ dimensional quantum state ρ_{AB} is always greater than or equal to the singlet fraction of ρ_{AB}. Mathematically, it can be expressed as

$$\lambda_{max}(\rho_{AB}) \geq F(\rho_{AB}) \tag{2.3.3}$$

where $\lambda_{max}(\rho_{AB})$ denote the maximum eigenvalue of ρ_{AB}.

Proof: Let us start with the definition of the singlet fraction given in (1.10.9), which can be re-expressed as

$$\begin{aligned} F(\rho_{AB}) &= max_{U_A,U_B} Tr[\rho_{AB}(U_A \otimes U_B)|\phi_d^+\rangle\langle\phi_d^+|(U_A^\dagger \otimes U_B^\dagger)] \\ &= max_{U_A,U_B} Tr[(U_A^\dagger \otimes U_B^\dagger)\rho_{AB}(U_A \otimes U_B)|\phi_d^+\rangle\langle\phi_d^+|] \\ &\leq \{max_{U_A,U_B} \lambda_{max}[(U_A \otimes U_B)\rho_{AB}(U_A^\dagger \otimes U_B^\dagger)]\}\{Tr[|\phi_d^+\rangle\langle\phi_d^+|]\} \\ &= max_{U_A,U_B} \lambda_{max}[(U_A \otimes U_B)\rho_{AB}(U_A^\dagger \otimes U_B^\dagger)] \\ &= \lambda_{max}(\rho_{AB}) \end{aligned} \tag{2.3.4}$$

The inequality in the third step is a consequence of the Result 1.1 and the last equality follows from a well known fact that the two quantum states $(U_A \otimes U_B)\rho_{AB}(U_A^\dagger \otimes U_B^\dagger)$ and ρ_{AB} have the same set of eigenvalues [192]. Therefore, we have provided the alternative proof of this result that has already been obtained in [193]. ■

Now, we are in a position to relate the usefulness of the entangled state ρ_{AB} as a resource state in quantum teleportation and the maximum eigenvalue of ρ_{AB}. The relation may be stated in the following way: In order to be useful in quantum teleportation protocol, an arbitrary $d \otimes d$ dimensional entangled quantum state ρ_{AB} must have $\lambda_{max}(\rho_{AB}) > \frac{1}{d}$. The proof of the statement can be done using Lemma 2.3.2.

Corollary 2.3.1. The upper bound of the maximum achievable teleportation fidelity from a given bipartite state ρ_{AB} in $d \otimes d$ dimensional Hilbert space is given by

$$f(\rho_{AB}) \leq \frac{\lambda_{max}(\rho_{AB})d+1}{d+1} \tag{2.3.5}$$

Proof: Applying Lemma 2.3.2 in (1.10.17), the above inequality can be achieved. ■

Corollary 2.3.2. If an arbitrary $d \otimes d$ dimensional NPT entangled mixed state described by the density operator ρ_{AB} and the maximum eigenvalue denoted by $\lambda_{max}(\rho_{AB})$ satisfies

$$\lambda_{max}(\rho_{AB}) > \frac{1}{d}$$

then the maximum achievable teleportation fidelity can be written as

$$f(\rho_{AB}) < \frac{2\lambda_{max}(\rho_{AB})}{1+\lambda_{max}(\rho_{AB})}$$

Proof: Applying $\lambda_{max}(\rho_{AB}) > \frac{1}{d}$ in Corollary 2.3.1, we get the upper bound of the maximum achievable teleportation fidelity in terms of maximum eigenvalue of ρ_{AB}. ■

Next, we will show that the separability condition is necessary and sufficient for only the isotropic state.

Theorem 2.3.1. An arbitrary $d \otimes d$ dimensional isotropic quantum state ρ_p shared between two distant partners is separable if and only if

$$\lambda_{max}(\rho_p) \leq \frac{1}{d} \tag{2.3.6}$$

Proof: Let us consider the noisy singlet state of the form

$$\rho_p = p|\phi^+\rangle\langle\phi^+| + (1-p)\frac{I \otimes I}{d^2}, 0 \leq p \leq 1 \tag{2.3.7}$$

where $|\phi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$.
The maximum eigenvalue of the density matrix ρ_p is given by

$$\begin{aligned}
\lambda_{max}(\rho_p) &= \lambda_{max}[p|\phi^+\rangle\langle\phi^+| + (1-p)\frac{I \otimes I}{d^2}], \;\; 0 \leq p \leq 1 \\
&= p + \frac{1-p}{d^2}
\end{aligned} \tag{2.3.8}$$

Now, since it is known that the state ρ_p is separable if and only if $0 \leq p \leq \frac{1}{d+1}$ [194] so we can say that the state ρ_p is separable if and only if $\lambda_{max}(\rho_p) \leq \frac{1}{d}$. ■

2.3.1 Examples

In this subsection, we will study a few quantum states ρ_{AB} for which singlet fraction $F(\rho_{AB})$ and maximum eigenvalue $\lambda_{max}(\rho_{AB})$ satisfies the inequality

$$F(\rho_{AB}) \leq \frac{1}{d} < \lambda_{max}(\rho_{AB}) \tag{2.3.9}$$

When the left part of the inequality (2.3.9) holds, then we are uncertain about the usefulness of the entangled state ρ_{AB} as a resource state in quantum teleportation. If the right part of the inequality (2.3.9) holds true, then we can say that the state ρ_{AB} can be useful in teleportation. This means that when the singlet fraction is unable to detect the useful of states for quantum teleportation, then the maximum eigenvalue can serve the purpose. Also, we should note that for the above quantum state ρ_{AB}, interaction with the environment is not taken into account.

Example 2.1: Let us now recall the quantum state described by the density operator $\sigma_{12}^{(1)}$ given in (2.2.10). Firstly, we need to check whether the state $\sigma_{12}^{(1)}$ is entangled. For the detection of the entangled state $\sigma_{12}^{(1)}$, let us consider a witness operator of the form [195]

$$W_1 = \frac{1}{4}(I \otimes I + I \otimes \sigma_z + \sigma_z \otimes I - \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y) \tag{2.3.10}$$

The expectation value of the operator W_1 with respect to the state $\sigma_{12}^{(1)}$ is given by

$$Tr(W_1 \sigma_{12}^{(1)}) = -0.55 \tag{2.3.11}$$

Since the expectation value of the witness operator W_1 is negative for the state $\sigma_{12}^{(1)}$ so the state $\sigma_{12}^{(1)}$ is an entangled state. Now we are in a position to say whether the entangled state is useful for teleportation by calculating its maximum eigenvalue. The maximum eigenvalue of the state is $\lambda_{max}(\sigma_{12}^{(1)}) = 0.587 > \frac{1}{2}$. Thus the state can be useful in quantum teleportation. This example is important in the sense that the value of the singlet fraction (after applying SPA-PT operation) is unable to detect the state as a resource state in quantum teleportation but on the other hand, maximum eigenvalue can help us to reach the correct conclusion.

Example 2.2: Let us consider a quantum state described by the density matrix ρ_1, which is given by [128]

$$\rho_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{3-2\sqrt{2}}{2} & \frac{1-\sqrt{2}}{2} & 0 \\ 0 & \frac{1-\sqrt{2}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \sqrt{2}-1 \end{pmatrix} \tag{2.3.12}$$

To detect whether the state ρ_1 is entangled, let us construct a witness operator as [195]

$$W_2 = \frac{1}{4}(I \otimes I + \sigma_z \otimes I + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y) \tag{2.3.13}$$

$Tr(W_2\rho_1)$ can be calculated as

$$Tr(W_2\rho_1) = -1.406 \tag{2.3.14}$$

The negative value of $Tr(W_2\rho_1)$ indicates that the state ρ_1 is entangled. Also the singlet fraction of ρ_1 is found to be $\frac{1}{2}$. Since $F(\rho_1) = \frac{1}{2}$ so it can be concluded that the state ρ_1 is not useful as a resource state for teleportation. But it is known that all entangled two-qubit mixed states are useful for teleportation [129]. Hence, the inference from the singlet fraction that the state ρ_1 is not useful as a resource state for teleportation is not correct. Let us now calculate the eigenvalues of ρ_1 and they are given by {0.5858,0.4142,0,0}. The maximum eigenvalue is found to be $\lambda_{max}(\rho_1) = 0.5858$. Since $\lambda_{max}(\rho_1) > 1/2$, we can conclude that the state ρ_1 can be useful for teleportation.

Example 2.3: Let us take another quantum state from $3 \otimes 3$ dimensional Hilbert space described by the density matrix ρ_2

$$\rho_2 = \begin{pmatrix} \frac{1-a}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.22 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}-a & -0.22 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.22 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.22 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a}{2} \end{pmatrix}, \quad 0.35 \leq a \leq 0.369 \tag{2.3.15}$$

The witness operator that detects the state described by the density operator ρ_2 is given by [196]

$$\begin{aligned} W_3 &= -\frac{3}{2}S_6\otimes S_6+\frac{3}{2}S_3\otimes S_3-\frac{2}{\sqrt{3}}S_8\otimes S_7+\frac{2}{3}I\otimes S_7-\frac{1}{2}S_7\otimes S_7+\frac{5}{6}S_8\otimes S_8+\frac{1}{2}iS_4\otimes S_1 \\ &+ \frac{1}{2}iS_1\otimes S_4-\frac{1}{2}S_2\otimes S_2+\frac{1}{2}S_5\otimes S_5+\frac{2}{3}S_7\otimes I-\frac{2}{3}I\otimes I+\frac{2\sqrt{3}}{9}S_8\otimes I-\frac{2}{\sqrt{3}}S_7\otimes S_8 \\ &+ \frac{2\sqrt{3}}{9}I\otimes S_8 \end{aligned} \tag{2.3.16}$$

where S_1, S_2 and S_3 are three symmetric Gell-Mann matrices given by (1.4.6), S_4, S_5 and S_6 are three anti-symmetric Gell-Mann matrices given by (1.4.7) and S_7 and S_8 are two diagonal Gell-Mann matrices given by (1.4.8).
We note that $Tr(W_3\rho_2) = 0.44 - 3a$, where $0.35 \leq a \leq 0.369$. Thus, $Tr(W_3\rho_2) < 0$ for $0.35 \leq a \leq 0.369$. Hence the state ρ_2 is an entangled state.
Let us calculate the singlet fraction of ρ_2. To do this, we need maximally entangled basis states in $3\otimes 3$ dimensional Hilbert space. The maximally entangled basis for the two-qutrit system is given by [197]

$$\begin{aligned} &|B_0\rangle = \frac{1}{\sqrt{3}}[|00\rangle+|22\rangle-e^{i\frac{\pi}{3}}|11\rangle], \;\; |B_1\rangle = \frac{1}{\sqrt{3}}[|01\rangle+|20\rangle-e^{i\frac{\pi}{3}}|12\rangle], \\ &|B_2\rangle = \frac{1}{\sqrt{3}}[|02\rangle+|21\rangle-e^{i\frac{\pi}{3}}|10\rangle], \;\; |B_3\rangle = \frac{1}{\sqrt{3}}[|11\rangle+|00\rangle-e^{i\frac{\pi}{3}}|22\rangle], \\ &|B_4\rangle = \frac{1}{\sqrt{3}}[|12\rangle+|01\rangle-e^{i\frac{\pi}{3}}|20\rangle], \;\; |B_5\rangle = \frac{1}{\sqrt{3}}[|10\rangle+|02\rangle-e^{i\frac{\pi}{3}}|21\rangle], \\ &|B_6\rangle = \frac{1}{\sqrt{3}}[|11\rangle+|22\rangle-e^{i\frac{\pi}{3}}|00\rangle], \;\; |B_7\rangle = \frac{1}{\sqrt{3}}[|20\rangle+|12\rangle-e^{i\frac{\pi}{3}}|01\rangle], \\ &|B_8\rangle = \frac{1}{\sqrt{3}}[|21\rangle+|10\rangle-e^{i\frac{\pi}{3}}|02\rangle] \end{aligned} \tag{2.3.17}$$

Then the singlet fraction of ρ_2 can be calculated using the maximally entangled basis (2.3.17) as

$$\begin{aligned} F(\rho_2) &= max_{B_i}\langle B_i|\rho_2|B_i\rangle, \;\; i=0,1,....,8 \\ &= \frac{1.22-a}{3} \end{aligned} \tag{2.3.18}$$

Figure 2.2 shows that $F(\rho_2)$ decreases as the state parameter a increases. The singlet fraction $F(\rho_2)$ is always less than $\frac{1}{3}$ when the state parameter a lying in the interval [0.35,0.369]. Therefore, according to the singlet fraction criterion, the state described by the density operator ρ_2 may or may not be useful in quantum teleportation.

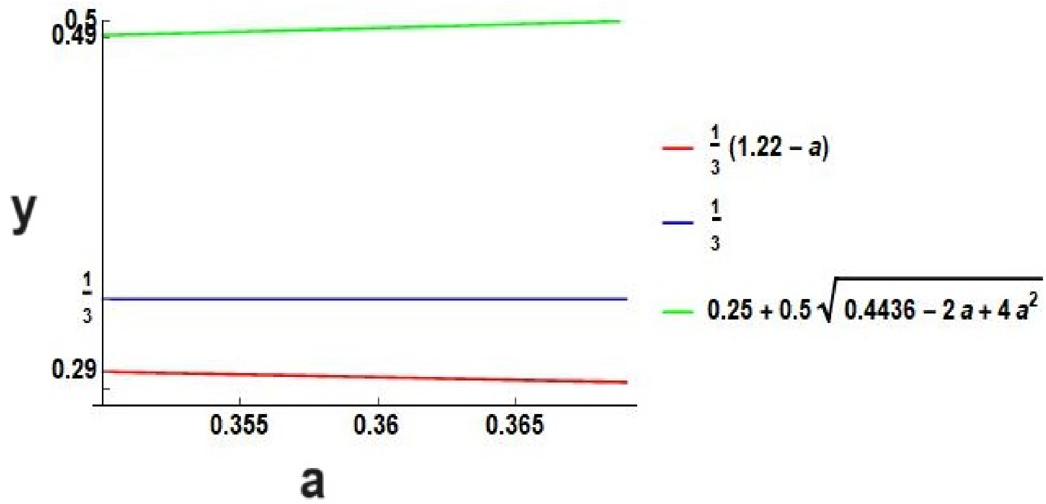

Figure 2.2: Plot of singlet fraction and maximum eigenvalue, i.e., $y = (F(\rho_2))/(\lambda_{max}(\rho_2))$ versus the state parameter a. Green line denotes the maximum eigenvalue of ρ_2, blue line denotes the classical limit of teleportation in terms of singlet fraction for $3 \otimes 3$ dimensional system, i.e $\frac{1}{3}$ and red line denotes the singlet fraction of state ρ_2

Let us now calculate the eigenvalues of ρ_2. The maximum eigenvalue of ρ_2 is given by

$$\lambda_{max}(\rho_2) = \frac{1}{4} + \frac{1}{2}\sqrt{0.4436 - 2a + 4a^2}, \quad 0.35 \leq a \leq 0.369 \tag{2.3.19}$$

We have also shown in Figure 2.2 that $\lambda_{max}(\rho_2)$ is always greater than $\frac{1}{3}$ when $a \in [0.35, 0.369]$. Thus, maximum eigenvalue ρ_2 can help us to infer that the state ρ_2 can be useful in quantum teleportation.

2.4 Teleportation criteria in terms of upper bound of the maximum eigenvalue in Dembo's bound

In this section, we will study those cases where the maximum eigenvalue of ρ_{AB} is unable to infer anything about the usefulness of ρ_{AB} in quantum teleportation, i.e., the case where the maximum eigenvalue of a given quantum state satisfies the inequality

$$\lambda_{max}(\rho_{AB}) \leq \frac{1}{d} \tag{2.4.1}$$

To overcome this problem, we have provided a criterion which is based on Dembo's bound to detect whether the state is useful for teleportation or not.

Let us consider a qutrit-qutrit system described by the density operator

$$\rho_3 = \begin{pmatrix} \frac{a}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.015 \\ 0 & \frac{a}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-a}{2} & 0 \\ 0.015 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-a}{2} \end{pmatrix}, \quad 0.5 \leq a \leq 0.65 \tag{2.4.2}$$

We find that $Tr(W_3\rho_3) = -1.53 + 2a < 0$, for $0.5 \leq a \leq 0.65$. Thus, the witness operator W_3 detect the state ρ_3 as an entangled state. But the question is whether the entangled state ρ_3 is useful in quantum teleportation.
Eigenvalues of ρ_3 are given by: $[0,0,0,0,0,\frac{1-a}{2},\frac{a}{2},0.125(2-\sqrt{16a^2-16a+4.0144}),0.125(2+\sqrt{16a^2-16a+4.0144})]$. The maximum eigenvalue is given by $\lambda_{max}(\rho_3) = 0.125(2+\sqrt{16a^2-16a+4.0144})$. We can observe that $\lambda_{max}(\rho_3) \leq \frac{1}{3}$ when $0.5 \leq a \leq 0.65$. Therefore, our criterion based on maximum eigenvalue fails to detect whether the state ρ_3 is useful in teleportation. It motivates us to search for the maximal bound of maximum eigenvalue that can be greater than $\frac{1}{d}$ for $d \otimes d$ dimensional system.
To start our search, let us consider the upper bound of the maximal eigenvalue of the $d \otimes d$ dimensional quantum state ρ_{AB} under investigation. The upper bound may be denoted as $\lambda_{max}^{D}(\rho_{AB})$ and it is given by R.H.S of the inequality (1.1.4)

$$\lambda_{max}^{D}(\rho_{AB}) = \frac{c+\eta_{d^2-1}}{2} + \sqrt{\frac{(c-\eta_{d^2-1})^2}{2} + (b^*)^T b} \tag{2.4.3}$$

where $R_{d^2} = \begin{pmatrix} R_{d^2-1} & b \\ (b^*)^T & c \end{pmatrix}$, η_1 is the lower bound on minimal eigenvalue of R_{d^2-1}, η_{d^2-1} is the upper bound on maximal eigenvalue of R_{d^2-1} and b is a vector of dimension d^2-1. We are now in a position to provide a criterion in terms of the upper bound of the maximal eigenvalue of the $d \otimes d$ dimensional quantum state ρ_{AB}.

Corollary 2.4.1. In order to be useful in quantum teleportation protocol, an arbitrary $d \otimes d$ dimensional entangled quantum state ρ_{AB} must have $\lambda_{max}^{D}(\rho_{AB}) > \frac{1}{d}$.
Proof: We know that $\lambda_{max}^{D}(\rho_{AB})$ is the upper bound of maximum eigenvalue. Hence, $\lambda_{max}(\rho_{AB}) \leq \lambda_{max}^{D}(\rho_{AB})$. After applying this fact to the relation between the usefulness of ρ_{AB} in quantum teleportation and the maximum eigenvalue of ρ_{AB}, the result will be proved. ■

Example 2.4: Recalling, the qutrit-qutrit system described by the density operator ρ_3

given in (2.4.2). The maximal bound of maximum eigenvalue of $3\otimes 3$ dimensional density matrix ρ_3 is given by $\lambda^D_{max}(\rho_3)$. For the 9×9 order density matrix ρ_3, we have $c = 0.1750$, $(b^*)^T = \begin{pmatrix} 0.015 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $\eta_8 = 0.325$. Using the above values, we find that $\lambda^D_{max}(\rho_3) = 0.357$. Thus using Corollary 2.4.1, we are able to show that the state ρ_3 can be useful in teleportation.

Example 2.5: Let us consider another qutrit-qutrit NPT entangled state, which is given by the density matrix [198]

$$\rho_\alpha = \frac{2}{7}|\phi_3^+\rangle\langle\phi_3^+| + \frac{\alpha}{7}\sigma_+ + \frac{5-\alpha}{7}\sigma_-, 4 < \alpha \leq 5 \tag{2.4.4}$$

where $|\phi_3^+\rangle = \frac{1}{\sqrt{3}}\sum_{i=0}^{2}|ii\rangle$, $\sigma_+ = \frac{1}{3}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|)$, $\sigma_- = \frac{1}{3}(|10\rangle\langle 10| + |21\rangle\langle 21| + |02\rangle\langle 02|)$. For the density matrix ρ_α, we have $c = \frac{2}{21}$, $(b^*)^T = \begin{pmatrix} \frac{2}{21} & 0 & 0 & 0 & \frac{2}{21} & 0 & 0 & 0 \end{pmatrix}$ and $\eta_8 = \frac{5}{21}$. In this case, $\lambda^D_{max}(\rho_\alpha) = 0.3346$, which is greater than $\frac{1}{3}$. Therefore, in this case, also we can use Corollary 2.4.1 to conclude that the state described by the density operator ρ_α can be useful in quantum teleportation.

2.5 Conclusion

In this chapter, we have modified the relationship between the optimal singlet fraction and the partial transpose of a given state by approximating the partial transposition operation through the procedure of structural physical approximation. The modification is required because partial transposition is a non-physical operation and thus it cannot be implemented in the laboratory. By using the SPA-PT method, we are able to show that the modified value of the optimal singlet fraction can be estimated in an experiment. Further, we have proposed a criterion for the detection of an entangled state useful in quantum teleportation, which are based on the maximum eigenvalue of the given state. Moreover, we have studied our criteria for the detection of $d\otimes d$ dimensional NPT entangled states useful in quantum teleportation in the given two cases: (i) $F(\rho_{AB}) \leq \frac{1}{d} < \lambda_{max}(\rho_{AB})$ or (ii) $\lambda_{max}(\rho_{AB}) \leq \frac{1}{d} < \lambda^D_{max}(\rho_{AB})$. Our criteria can in principle be determined in an experiment because maximum eigenvalue can be estimated experimentally [186, 199, 200].

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Chapter 3

Quantification of Non-locality of two-qubit Entangled state and its application in Controlled Quantum Teleportation

"The phenomenon of non-locality in quantum mechanics shows that the world is much more mysterious and interconnected than we previously thought."

- David Bohm

In this chapter [1], we quantify the non-locality of such entangled state ρ_{AB} which are neither detected by W_{CHSH} witness operator or nor by the quantity $M(\rho_{AB})$ given by (1.8.1). Non-locality is a feature of quantum mechanics that cannot be explained by local realistic theory. It can be detected by the violation of Bell's inequality. In this chapter, we have considered the evaluation of Bell's inequality with the help of the XOR game. In the XOR game, a two-qubit entangled state is shared between the two distant players. It may generate a non-local correlation between the players which contributes to the maximum probability of winning the game. We have aimed to determine the strength of the non-locality through the XOR game. Thus, we have defined a quantity $S_{NL}(\rho_{AB})$ called the strength of non-locality, purely on the basis

[1] This chapter is based on a research paper "Strength of the nonlocality of two-qubit entangled state and its applications, *Physica Scripta* **98**, 055101 (2023)".

of the maximum probability of winning the XOR game. We have also derived the relation between the introduced quantity $S_{NL}(\rho_{AB})$ and the quantity $M(\rho_{AB})$, to study in depth, the problem of non-locality of a two-qubit entangled state. Interestingly, we have found that the newly defined quantity $S_{NL}(\rho_{AB})$ fails to detect the non-locality of the entangled state, when the witness operator constructed from $CHSH$ operator cannot detect the entangled state. To overcome this problem, we have modified the definition of the strength of non-locality and have shown that the modified definition may detect the non-locality of such entangled states, which were earlier undetected by $S_{NL}(\rho_{AB})$. Furthermore, we have provided two applications of the strength of the non-locality $S_{NL}(\rho_{AB})$: (i) establishment of a link between the two-qubit non-locality determined by $S_{NL}(\rho_{AB})$ and the three-qubit non-locality determined by the Svetlichny operator and (ii) determination of the upper bound of the power of the controller in terms of $S_{NL}(\rho_{AB})$ in controlled quantum teleportation.

3.1 Introduction

In 1964, J. S. Bell [17] derived a criterion to detect the non-local correlation that may exist in Einstein, Podolski, and Rosen (EPR) pair of particles. His work proved that the predictions of quantum mechanics are incompatible with the local realistic theory. Bell's criterion for detecting non-locality can be expressed in terms of a mathematical inequality, which is popularly known as Bell's inequality [1, 3, 48], derived using the local-realism principle. Thus, any classical system making local choices will produce a classical correlation satisfying this inequality. In the late 1960s, many experiments were performed to show the violation of Bell's inequality for the EPR pair, but none were successful. An experiment performed by Alain Aspect et. al. successfully shows the violation of Bell's Inequality [50, 52]. After Bell's seminal work, many studies were devoted to non-locality.

The study of non-locality is relevant for many reasons. One is that it can be used as a resource for the development of device-independent quantum information processing [201]. A few other reasons that may attract the study of non-locality is that it may have much application in a variety of quantum information processing tasks such as self-testing [202, 203], secure communication [21], randomness certification [204], and distributed computing [205]. In recent work, a marginal problem has been studied in the context of the computation of Bell inequalities [206].

Detection of an observed non-local correlation is one of the prime problems in the study of non-locality. The foremost tool to detect non-locality is Bell's inequality [17], and it may be considered as the standard approach for detecting non-locality. The violation of Bell's inequality may indicate the presence of a non-local feature in a two-qubit state described by the density operator ρ_{AB}. Therefore, if any two-qubit state ρ_{AB} violates Bell's inequality, then the state may exhibit a non-local correlation, and thus, the state can be identified as an entangled state. But the converse of the statement is not true. This means that there exists a two-qubit entangled state that may satisfy Bell's inequality. This shows that although, there is a connection between quantum entanglement and non-locality [207, 208], conceptually they are very much distinct [103]. These two counterintuitive features of quantum mechanics can be used as a physical resource to enhance our computational power [209]. Thus, detecting these quantum mechanical features before using them as a resource is necessary. Along this line of research, I. S. Eliens et. al. [210] have studied the non-locality detection problem and represent it as a tensor network problem. In [211], the generalized R-matrix has been used to study the non-locality and entanglement of the three-qubit state. In [207], the uncertainty-induced non-locality measure has been used to detect the non-locality of a two-qubit state. Further, the classification and quantification of a pure three-qubit state have been studied using the concurrence of a generic two-qubit pure state [212]. The witness operator method may also be used to detect non-locality, and it is very useful because it can be implemented in the experiment.

In the last few years, testing of Bell's inequality has been viewed as a Bell game [213]. In this game, Alice and Bob may be considered players, and Charlie acts as a referee or verifier. There are many rounds of the game, and in each round, Charlie, who acts as a verifier, sends a query (input) to other members, Alice and Bob. They will have to send an answer (output) to Charlie. Before starting the game, the following assumptions are made: (i) Players know the set of possible queries, (ii) Players know the rules of the game (iii) Players know the common strategy in deciding the process in each round of the game. Here we can consider an entangled state as a resource that may be used in these processes. Therefore, in the perspective of a game, Bell locality may be defined as the process by which the output generated by each player is independent of the input of other players. Thus, if there is any correlation found between the players, then it is due to the presence of correlation in the shared entangled state. When this definition of Bell's locality does not hold, then we can talk of Bell's non-locality. Initially, Bell constructed the inequality in which two parties are there in the composite system and each party measures dichotomic observables in two different settings. Later, researchers have started generalizing the Bell's non-locality with N parties, k measurement settings, and d outcomes of the measurement [65, 214–218]. It has been observed that two or more different non-local quantum behaviors may be responsible for the maximal violation of Bell's inequality. However, the extremal quantum behavior can be realized by a unique (up to unitary equivalence) quantum representation [219]. The non-local correlation that violate Bell's inequalities maximally by unique quantum behaviors has been studied in [78]. These Bell's inequalities are maximally violated by non-maximally entangled states, thus showing that these states are necessary to characterize the boundary of the quantum region. The non-local correlation characterized by Bell's inequalities could be used as a resource for quantum optics, quantum computation, and quantum information. In this direction, Obada et. al. [220] have studied the link between non-locality and entanglement and have shown that the entangled state may possess the phenomenon of the sudden death of entanglement and non-locality under the effect of thermal noise. The influence of the dissipation rate of the dissipative system on the quantum correlation has been studied in [221], using the Hilbert–Schmidt distance and Bell's inequality correlations. They found that the quantum correlation can be enhanced for some specific values of the dipole–dipole interaction. In another work [222], it has been shown that the Bell's non-locality can be enhanced when the two-mode parametric amplifier cavity is initially prepared in the coherent states.

It is known that in any theory, the degree of steering is an equally important part of the uncertainty principle to measure the degree of non-locality [223]. But J. Oppenheim and S. Wehner [224] have used the uncertainty principle alone to establish the relation between the maximum probability of winning the XOR game and the expectation value of the Bell-CHSH operator with respect to the shared state between the players. Thus they have shown that the degree of non-locality can be determined by the uncertainty principle alone. Therefore, one may ask whether only one factor, i.e., uncertainty principle is enough to measure the degree of non-locality for all non-local games. The answer is negative because R. Ramanathan et. al. [223] have shown that non-local games exist where the uncertainty principle and the degree of steering are needed to measure the degree of non-locality. In particular, the degree of non-locality for the XOR game can be measured using the uncertainty principle alone. In the literature, there is a related work [225] where it has been shown that some points that cannot maximize any XOR game lie on the quantum boundary.

The main motivation of this chapter is to investigate the following question: If the Bell's inequality, the quantity $M(\rho_{AB})$, and the maximum probability of winning of XOR game fail to determine the non-locality of an entangled state, and if we further restrict the usage of the filtering operation, then can we measure the strength of the non-locality by any other means?

To address the above stated question, we first consider the evaluation of Bell's inequality as an XOR game. The relation established in [224] suggests that if Bell's inequality is violated, then the maximum probability of winning the game is greater than $\frac{3}{4}$. Thus, there is a relation between the non-locality of the shared state and the maximum probability of winning the game. We found that there exists an entangled shared state with which if players played the game, then the probability of winning the game may be less than or equal to $\frac{3}{4}$. This indicates that the XOR game may be won by adopting any local realistic theory, but this is not the case. We have investigated this loophole and tried to fix it by defining the strength of the non-locality through the maximum probability of winning the game.

3.2 Revisiting the non-locality of two-qubit system

In a two-player Bell test game [110], the players may be referred to as Alice and Bob who are far apart from each other. Each player will receive a query (input) and will have to provide an answer (output). The game may be repeated in many rounds. The players are allowed to prepare a common strategy before the game but after the game starts, the players are not allowed to communicate with each other. The rules of the game and the list of possible queries are known in advance. If the rules are set in a way that the players must produce different answers if both receive a query "1" and otherwise, the answer is the same, then the game cannot be trivially won with a list of pre-determined answers. With respect to the defined game, Bell locality means that the process by which both the players generate the output without considering the other player's input. Thus, if any correlations are generated between the players then this is due to a shared resource. The Bell non-locality came into the picture when Bell locality doesn't hold. Bell non-locality can be demonstrated by the violation of Bell-CHSH inequality. Generally, it has been shown by R. Horodecki et. al. [127] that any two-qubit state described by the density matrix ρ_{AB} violates CHSH inequality if and only if $M(\rho_{AB}) > 1$ and the quantity $M(\rho_{AB})$ is defined in (1.8.1). In this section, we revisit the non-locality of a two-qubit entangled state ρ_{AB}^{ent} by introducing a measure of the strength of the non-locality of ρ_{AB}^{ent}. The motivation of this section is to develop a measure that may detect the non-local nature of the given entangled state ρ_{AB}^{ent}, which is neither detected by Bell-CHSH inequality (for a particular setting) nor detected by any general setting described by the criterion $M(\rho_{AB}^{ent}) > 1$.

3.2.1 A definition of the strength of the non-locality of two-qubit entangled state

In this subsection, we will define the strength of the non-locality of two-qubit entangled state ρ_{AB}^{ent} in terms of the maximum probability of winning the game played between two distant players which are sharing an entangled state ρ_{AB}^{ent}.

Let us consider an XOR game played between two distant players Alice (A) and Bob (B) [224, 226]. In this game, the winner is decided by the XOR of the answers $a \oplus b = a + b \ (mod 2)$, where $a, b \in \{0, 1\}$ and it denote the answers given by the players A and

B, when the referee asks them randomly selected questions $(s,t) \in S \times T$, where S and T denote finite non-empty sets. The winning condition of the game may be expressed in terms of the predicate given by

$$V(a \oplus b/s,t) = 1, \text{ if and only if } a \oplus b = s.t \tag{3.2.1}$$

The players A and B obtain outcomes (answers) a and b after performing measurement operators A_s^a and B_t^b on their respective qubits. Here, we may consider s and t as the corresponding measurement settings. The measurement operators A_s^a and B_t^b may be expressed in terms of the observables as

$$A_s^a = \frac{1}{2}(I + (-1)^a A_s), \;\; B_t^b = \frac{1}{2}(I + (-1)^b B_t) \tag{3.2.2}$$

The operators A_s and B_t are given by

$$A_s = \sum_j a_s^{(j)} \Gamma_j, \;\; B_t = \sum_j b_t^{(j)} \Gamma_j \tag{3.2.3}$$

where $\vec{a}_s = (a_s^{(1)}, a_s^{(2)}, ..., a_s^{(N)}) \in R^N$ and $\vec{b}_t = (b_t^{(1)}, b_t^{(2)}, ..., b_t^{(N)}) \in R^N$ denote real unit vectors of dimension $N = min\{|S|,|T|\}$. $\Gamma_1, \Gamma_2, ..., \Gamma_N$ are the anti-commuting generators of a Clifford algebra.

If we assume that the two distant players, A and B, play the game using the shared state ρ_{AB} given in (1.5.5), then the maximum probability P^{max} of winning the game overall strategy is given by [224]

$$P^{max} = \frac{1}{2}[1 + \frac{\langle B_{CHSH} \rangle_{\rho_{AB}}}{4}] \tag{3.2.4}$$

where $\langle B_{CHSH} \rangle_{\rho_{AB}} = Tr[(A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1)\rho_{AB}]$ denotes the expectation value of the Bell operator B_{CHSH} with respect to the state ρ_{AB}. Since, the maximum probability of winning the game depends on the expectation value of the Bell operator B_{CHSH}, P^{max} is somehow related to the non-locality of the state ρ_{AB}. Thus, to determine the non-locality of any arbitrary two-qubit state ρ_{AB}, we define here the strength of the non-locality. The strength of the non-locality of ρ_{AB} denoted by $S_{NL}(\rho_{AB})$ in terms of P^{max} may be defined as

$$S_{NL}(\rho_{AB}) = max\{P^{max} - \frac{3}{4}, 0\} \tag{3.2.5}$$

Therefore, $S_{NL}(\rho_{AB})$ can be considered as the quantifier of the strength of the non-locality for any theory, and it can be calculated by calculating P^{max} for different theories such as (i) classical theory, (ii) theory based on quantum mechanics, and (iii) for any non-signaling theory. For any classical theory, $P^{max} \leq \frac{3}{4}$ and hence $S_{NL}(\rho_{AB}) = 0$. For quantum mechanical theory and for non-signaling correlation, we have $P^{max} > \frac{3}{4}$ and thus $S_{NL}(\rho_{AB}) \neq 0$.

Furthermore, we can consider the situation where the players performed their measurements in different measurement settings, such as measurements performed along $xy-$, $xz-$, and $yz-$ planes. In this scenario, the maximum probability of winning the game depends upon the expectation value of the Bell operators in different planes. To further illuminate this point, consider the Bell operators B_{xy}, B_{xz}, and B_{yz} in $xy-$, $xz-$, and $yz-$ planes. In these planes, the maximum probability of winning the game is denoted by P_{xy}, P_{xz}, and P_{yz}, respectively. Therefore, the relation between the expectation value of the Bell operators defined in different planes with respect to the two-qubit quantum state described by the density operator ρ_{AB} and the corresponding maximum probability of winning may be expressed as

$$P_{ij}^{max} = \frac{1}{2}[1 + \frac{\langle B_{ij} \rangle_{\rho_{AB}}}{4}], i, j = x, y, z \;\; \& \;\; i \neq j \tag{3.2.6}$$

The Bell operators B_{xy}, B_{xz}, and B_{yz} can be written in terms of the observables σ_x, σ_y, and σ_z as [227]

$$B_{ij} = \sigma_i \otimes \frac{\sigma_i + \sigma_j}{\sqrt{2}} + \sigma_i \otimes \frac{\sigma_i - \sigma_j}{\sqrt{2}} + \sigma_j \otimes \frac{\sigma_i + \sigma_j}{\sqrt{2}} - \sigma_j \otimes \frac{\sigma_i - \sigma_j}{\sqrt{2}}, i, j = x, y, z \; \& \; i \neq j \tag{3.2.7}$$

For the case discussed above, the strength of the non-locality $S_{NL}^{(ij)}(\rho_{AB})$ may be defined as

$$S_{NL}^{(ij)}(\rho_{AB}) = max\{P, 0\} \tag{3.2.8}$$

where $P = \{P_{xy}^{max} - \frac{3}{4}, P_{xz}^{max} - \frac{3}{4}, P_{yz}^{max} - \frac{3}{4}\}$.

From the definition $S_{NL}^{(ij)}(\rho_{AB})$ given in (3.2.8), it is clear that if ρ_{AB} is an entangled state, and further if it satisfies the Bell-CHSH inequality in every xy, yz, xz setting, then all quantities $P_{ij}^{max} - \frac{3}{4}, (i \neq j;\; i, j = x, y, z)$ will be negative. Hence, the value of $S_{NL}^{(ij)}(\rho_{AB})$ for $i \neq j;\; i, j = x, y, z$ will be equal to zero. This gives an absurd result because ρ_{AB} represents an entangled state. Thus, we can apply the definition (3.2.8) only when at least one $i \neq j$; $(i, j = x, y, z)$ for which the quantity $P_{ij}^{max} - \frac{3}{4}$ is/are positive.

3.2.2 Dependence of the strength of non-locality on witness operator

Let us consider the game discussed above, which is played between two distant partners Alice and Bob, using a shared state between them. If the shared state is any entangled state described by ρ_{AB}^{ent} and the maximum probability P^{max} of winning the game using the shared state ρ_{AB}^{ent} satisfies the inequality $P^{max} > \frac{3}{4}$, then as per the definition of the strength of the non-locality of ρ_{AB}^{ent} given in (3.2.5) could be non-zero. Otherwise, if the players are playing the game with the classical state shared between them, then $P^{max} \in [0, \frac{3}{4}]$ and then the strength of the non-locality will be equal to zero. This event may occur even if the players choose their measurement settings in different planes. In this perspective, we can ask the following question: Is it possible to determine the strength of the non-locality when the maximum probability of winning the game played with an entangled state, lies between 0 and $\frac{3}{4}$?

To investigate the above question, we first express the maximum probability P^{max} of winning the game in terms of the expectation value of the witness operator with respect to the general two-qubit state described by the density operator ρ_{AB}. Also, we find that when ρ_{AB} represents an entangled state which is not detected by the witness operator, then the maximum probability of winning the game lies between 0 and $\frac{3}{4}$. On the contrary, if there exists any witness operator that detects the entangled state, then $P^{max} \geq \frac{3}{4}$.

Now, our task is to first establish the relationship between the maximum probability of winning the game played using a two-qubit state ρ_{AB}, and the expectation value of the witness operator with respect to the state ρ_{AB}. The relationship may be stated as:

Result 3.2.1. If ρ_{AB} denotes any arbitrary two-qubit bipartite state shared between the two distant players Alice and Bob and P^{max} denotes the maximum probability of winning the game overall strategy taken by the players, then P^{max} is given by

$$P^{max} = \frac{3}{4} - \frac{Tr[W_{CHSH}\rho_{AB}]}{8} \tag{3.2.9}$$

where, $W_{CHSH} (= 2I - B_{CHSH})$ denotes the witness operator.

Proof:- If any bipartite two-qubit state ρ_{AB} is shared between the players Alice and Bob, then the maximum probability of winning the game is given by [224]

$$P^{max} = \frac{1}{2}[1 + \frac{Tr[B_{CHSH}\rho_{AB}]}{4}] = \frac{3}{4} - \frac{Tr[W_{CHSH}\rho_{AB}]}{8} \tag{3.2.10}$$

In the second line of the proof, we have used $B_{CHSH} = 2I - W_{CHSH}$. Hence proved. ■

In the same spirit, we can relate the maximum probability of winning the game with the expectation value of the witness operator in different xy, yz, and xz settings as

$$P^{max}_{xy} = \frac{3}{4} - \frac{Tr[W^{xy}_{CHSH}\rho_{AB}]}{8} \tag{3.2.11}$$

$$P^{max}_{yz} = \frac{3}{4} - \frac{Tr[W^{yz}_{CHSH}\rho_{AB}]}{8} \tag{3.2.12}$$

$$P^{max}_{xz} = \frac{3}{4} - \frac{Tr[W^{xz}_{CHSH}\rho_{AB}]}{8} \tag{3.2.13}$$

3.2.2.1 Strength of the non-locality when two-qubit entangled state detected by the witness operator W_{CHSH}

In this subsection, we will discuss the case when the witness operator detects the entangled state ρ^{ent}_{AB} and then we show that the strength of the non-locality denoted by $S_{NL}(\rho^{ent}_{AB})$ can be determined in this case.

Result 3.2.1 provides the relationship between the expectation value of the witness operator W_{CHSH} with respect to any arbitrary two-qubit state ρ_{AB}, and the maximum winning probability P^{max}. Therefore, the strength of the non-locality $S_{NL}(\rho_{AB})$ defined in (3.2.5) may be re-expressed in terms of witness operator W_{CHSH} as

$$S_{NL}(\rho_{AB}) = max\{-\frac{Tr[W_{CHSH}\rho_{AB}]}{8}, 0\} \tag{3.2.14}$$

Let us discuss three cases when ρ_{AB} represents (i) a separable state, (ii) an entangled state not detected by W_{CHSH}, and (iii) an entangled state detected by W_{CHSH}.

Case I: If any separable state is described by the density operator ρ^{sep}_{AB}, then $Tr[W_{CHSH}\rho^{sep}_{AB}] \geq 0$ and hence $P^{max} \leq \frac{3}{4}$. In this case, $S_{NL}(\rho^{sep}_{AB}) = 0$.

Case II: If the state ρ^{entnd}_{AB} denotes an entangled state not detected by witness operator W_{CHSH}, then also we obtain $Tr[W_{CHSH}\rho^{entnd}_{AB}] \geq 0$ and hence $P^{max} \leq \frac{3}{4}$. In this case, the amount of non-locality of the state ρ^{entnd}_{AB} can be estimated to be zero. Although the state ρ^{entnd}_{AB} is an entangled state and thus may possess non-local properties but its non-locality may not be revealed by the non-local quantifier S_{NL}. Further, we may note that the state ρ_{AB}^{entnd} may not be detected by W_{CHSH}, but there may exist other witness operators that may detect it, and in that case, it may be possible to quantify its non-locality through S_{NL}.

Case III: If the state ρ_{AB}^{entd} represent an entangled state detected by the witness operator W_{CHSH}, then $Tr[W_{CHSH}\rho_{AB}^{entd}] < 0$ and hence $P^{max} > \frac{3}{4}$. In this case, the amount of non-locality of ρ_{AB}^{entd} can be calculated by the formula $S_{NL} = -\frac{Tr[W_{CHSH}\rho_{AB}^{entd}]}{8}$.
Now, our aim is to show through the example that the two-qubit state under investigation is a quantum correlated state, and thus its strength of non-locality can be determined. To proceed with our discussion, let us consider the two-qubit quantum state described by the density operator $\rho_{AB}^{(1)}$

$$\rho_{AB}^{(1)} = \frac{1}{4}[I \otimes I + 0.001\sigma_x \otimes I + 0.8\sigma_1 \otimes \sigma_1 + 0.89\sigma_2 \otimes \sigma_2 - 0.9\sigma_3 \otimes \sigma_3] \qquad (3.2.15)$$

The state $\rho_{AB}^{(1)}$ is an entangled state. In this case, we can construct the witness operator $W_{CHSH}^{(1)}$ as

$$W_{CHSH}^{(1)} = 2I \otimes I - A_0^{(1)} \otimes B_0^{(1)} + A_0^{(1)} \otimes B_1^{(1)} - A_1^{(1)} \otimes B_0^{(1)} - A_1^{(1)} \otimes B_1^{(1)} \qquad (3.2.16)$$

where, $A_0^{(1)} = \sigma_x,\ A_1^{(1)} = \sigma_y,$

$$B_0^{(1)} = 0.8\sigma_x + 0.4\sigma_y + 0.447\sigma_z,\ B_1^{(1)} = -0.4\sigma_x + 0.8\sigma_y + 0.447\sigma_z \qquad (3.2.17)$$

Therefore, the expectation value of $W_{CHSH}^{(1)}$ with respect to the state $\rho_{AB}^{(1)}$ is given by

$$Tr[W_{CHSH}^{(1)}\rho_{AB}^{(1)}] = -0.028 < 0 \qquad (3.2.18)$$

Hence, in this example, we can see the state $\rho_{AB}^{(1)}$ is detected as an entangled state by the witness operator $W_{CHSH}^{(1)}$. Thus, the strength of the non-locality of the state $\rho_{AB}^{(1)}$ can be calculated using (3.2.14) as

$$S_{NL}(\rho_{AB}^{(1)}) = 0.0035 \qquad (3.2.19)$$

3.2.2.2 Strength of the non-locality when the witness operator W_{CHSH} does not detect the two-qubit entangled state

Till now, we don't have sufficient information to make a definite conclusion about the non-locality of an entangled state described by the density operator ρ_{AB}^{entnd}, which is not detected by the witness operator W_{CHSH}. Let us take an example to understand

what we mean to say:
Consider the entangled state $\rho_{AB}^{(2)}$ and witness operator $W_{CHSH}^{(2)}$, which are given by

$$\rho_{AB}^{(2)} = \frac{1}{4}[I\otimes I+0.7\sigma_1\otimes\sigma_1+0.2\sigma_2\otimes\sigma_2-0.5\sigma_3\otimes\sigma_3] \quad (3.2.20)$$

$$W_{CHSH}^{(2)} = 2I\otimes I-A_0^{(2)}\otimes B_0^{(2)}+A_0^{(2)}\otimes B_1^{(2)}-A_1^{(2)}\otimes B_0^{(2)}-A_1^{(2)}\otimes B_1^{(2)} \quad (3.2.21)$$

where $A_0^{(2)}$, $A_1^{(2)}$, $B_0^{(2)}$, $B_1^{(2)}$ are given by

$$A_0^{(2)} = A_1^{(2)} = 0.7\sigma_x+0.5\sigma_y+0.5099\sigma_z$$

$$B_0^{(2)} = 0.4\sigma_x+0.4\sigma_y+0.8246\sigma_z,\ B_1^{(2)} = 0.5\sigma_x+0.3\sigma_y+0.812404\sigma_z \quad (3.2.22)$$

The expectation value of $W_{CHSH}^{(2)}$ with respect to the state $\rho_{AB}^{(2)}$ can be calculated as

$$Tr[W_{CHSH}^{(2)}\rho_{AB}^{(2)}] = 1.9845 \geq 0 \quad (3.2.23)$$

Thus, this example shows that there may exist entangled states which are not detected by $W_{CHSH}^{(2)}$ operator given in (3.2.21), and from Result 3.2.1, we can remark $P^{max} \leq \frac{3}{4}$. Hence, we conclude that there exist entangled states for which $P^{max} \leq \frac{3}{4}$. Therefore, for those entangled states which are not detected by W_{CHSH}, we find $S_{NL} = 0$, and thus S_{NL} is unable to measure the true strength of non-locality of such entangled states. This problem may be sorted out if we construct another witness operator that may detect such entangled states which are not detected by W_{CHSH}. Since there does not exist any general relationship between the maximum probability P^{max} and the expectation value of any arbitrary witness operator, it is not possible to define the strength of the non-locality in terms of any arbitrary witness operator. Therefore, we need to redefine the strength of the non-locality using a different approach.
It is known from (3.2.9) that if W_{CHSH} fails to detect the entangled state ρ_{AB}^{ent}, then the value of the expression $P^{max}-\frac{3}{4}$ will be negative. Thus, our idea is to calculate the upper bound of the expression $P^{max}-\frac{3}{4}$ and if we find that the calculated upper bound is positive, then we may infer that there may be a possibility to get the non-zero value of $S_{NL}(\rho_{AB}^{ent})$. To do this, recall (3.2.9) and re-express it as

$$Tr[W_{CHSH}\rho_{AB}^{ent}] = 6-8P^{max} \quad (3.2.24)$$

We should note that in this scenario, it is assumed that W_{CHSH} does not detect the state ρ_{AB}^{ent} and thus $Tr[W_{CHSH}\rho_{AB}^{ent}] \geq 0$, hence $P^{max} \leq \frac{3}{4}$.

Let us now re-start with the quantity $Tr[W_{CHSH}\rho_{AB}^{ent}(\rho_{AB}^{ent})^{T_B}]$, where T_B denotes the partial transposition with respect to the subsystem B and using the Result 1.1 given in (1.1.11), we may get the following inequality

$$Tr[W_{CHSH}\rho_{AB}^{ent}(\rho_{AB}^{ent})^{T_B}] \geq \lambda_{min}((\rho_{AB}^{ent})^{T_B})Tr[W_{CHSH}\rho_{AB}^{ent}] \tag{3.2.25}$$

Using (3.2.24) and (3.2.25), we get

$$Tr[W_{CHSH}\rho_{AB}^{ent}(\rho_{AB}^{ent})^{T_B}] \geq \lambda_{min}((\rho_{AB}^{ent})^{T_B})(6-8P^{max}) \tag{3.2.26}$$

If ρ_{AB}^{ent} is a bipartite two-qubit entangled state, then $\lambda_{min}((\rho_{AB}^{ent})^{T_B}) < 0$, and its entanglement may be quantified by negativity, which may be defined as

$$N(\rho_{AB}^{ent}) = -2\lambda_{min}((\rho_{AB}^{ent})^{T_B}) \tag{3.2.27}$$

Therefore, for the entangled state ρ_{AB}^{ent}, the inequality (3.2.26) reduces to

$$\begin{aligned} &Tr[W_{CHSH}\rho_{AB}^{ent}(\rho_{AB}^{ent})^{T_B}] \geq -\frac{1}{2}N(\rho_{AB}^{ent})(6-8P^{max}) \\ \implies &P^{max} - \frac{3}{4} \leq \frac{Tr[W_{CHSH}\rho_{AB}^{ent}(\rho_{AB}^{ent})^{T_B}]}{4N(\rho_{AB}^{ent})} \end{aligned} \tag{3.2.28}$$

The inequality (3.2.28) motivates us to re-define the strength of the non-locality $S_{NL}(\rho_{AB}^{ent})$ of the entangled state ρ_{AB}^{ent} undetected by W_{CHSH}. Therefore, if the state ρ_{AB}^{ent} is not detected by W_{CHSH} and then $S_{NL}^{New}(\rho_{AB}^{ent})$ may be defined as

$$S_{NL}^{New}(\rho_{AB}^{ent}) \quad = \quad q(P^{max} - \frac{3}{4}) + (1-q)K \tag{3.2.29}$$

where $K = \frac{Tr[W_{CHSH}\rho_{AB}^{ent}(\rho_{AB}^{ent})^{T_B}]}{4N(\rho_{AB}^{ent})}$ and q $(0 \leq q < 1)$ is chosen in such a way that $S_{NL}^{New}(\rho_{AB}^{ent}) > 0$.

The upper bound of q can be obtained by employing the condition $S_{NL}^{New}(\rho_{AB}^{ent}) > 0$. Therefore, the upper bound of q is given by

$$q < \frac{K}{\frac{3}{4} - P^{max} + K} \tag{3.2.30}$$

To illustrate our result, let us consider the two-qubit state described by the density operator ρ_{AB}, which is given by

$$\rho_{AB} = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \frac{1}{3} & x & 0 \\ 0 & x & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3}-x \end{pmatrix}, \quad 0 \leq x \leq \frac{1}{3} \tag{3.2.31}$$

It can be easily verified that ρ_{AB} is an entangled state for $x \in (0.167, 0.333)$. Also, we found that for the same range of x, we have $Tr[W^{xy}_{CHSH}\rho_{AB}] = 2 - 4\sqrt{2}x \geq 0$. Therefore, the state ρ_{AB} is undetected by the witness operator W^{xy}_{CHSH}.

To calculate the strength of the non-locality of ρ_{AB}, we follow the definition (3.2.29) and accordingly determine the following quantities,

$$P^{max} - \frac{3}{4} = 4\sqrt{2}x - 2, K = \frac{1 - 2(1+\sqrt{2})x + 6x^2}{2(\sqrt{(72x^2 - 12x + 1)} - 1)} \tag{3.2.32}$$

Therefore, using (3.2.30), we find that

$$q < [0.55, 1], \text{ when } x \in (0.1667, 0.333) \tag{3.2.33}$$

Therefore, the strength of the non-locality of the state ρ_{AB} is given by

$$S^{New}_{NL}(\rho_{AB}) = q(P^{max} - \frac{3}{4}) + (1-q)K, \; 0 < q < 0.55 \tag{3.2.34}$$

where the expressions $P^{max} - \frac{3}{4}$ and K are given in (3.2.32). The value of $S^{New}_{NL}(\rho_{AB})$ for x and q satisfying (3.2.33) are shown in Figure 3.1.

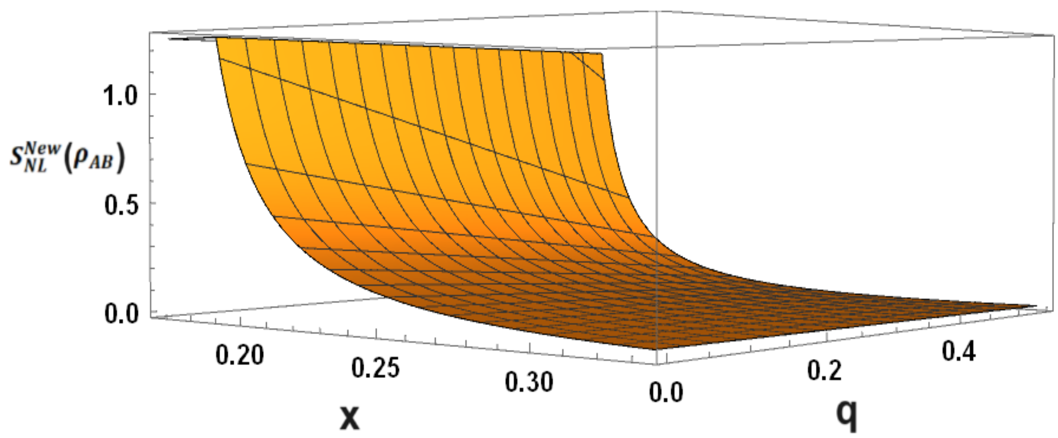

Figure 3.1: The curve represents the non-zero value of $S^{New}_{NL}(\rho_{AB})$ for the state ρ_{AB}. Here, x denotes the state parameter, and q lies in the range (0,0.55).

So, by exploiting the above procedure, we are able to calculate the strength of non-locality for the entangled states probabilistically which are not detected by W_{CHSH}.

3.2.3 Relation between $S_{NL}(\rho_{AB}^{ent})$ and the quantity $M(\rho_{AB}^{ent})$

In this subsection, we consider a two-qubit entangled state described by the density operator ρ_{AB}^{ent} and obtain the relationship between the strength of the non-locality $S_{NL}(\rho_{AB}^{ent})$ and the quantity $M(\rho_{AB}^{ent})$. To derive the required relationship, we need a few lemmas which are given below:

Lemma 3.2.1. If $P^{max}(\rho_{AB}^{ent})$ denotes the maximum probability of winning the game via the shared state ρ_{AB}^{ent} between the two players, then the upper bound of $P^{max}(\rho_{AB}^{ent})$ in terms of $M(\rho_{AB}^{ent})$ is given by

$$P^{max}(\rho_{AB}^{ent}) \leq \frac{1}{2}(\frac{\sqrt{M(\rho_{AB}^{ent})}}{2}+1) \tag{3.2.35}$$

Proof: Recalling (3.2.4), $P^{max}(\rho_{AB}^{ent})$ can be re-written as

$$P^{max}(\rho_{AB}^{ent}) = \frac{1}{2}(1+\frac{\langle B_{CHSH}\rangle_{\rho_{AB}^{ent}}}{4}) \tag{3.2.36}$$

Let us denote $\langle B_{max}\rangle_{\rho_{AB}^{ent}} = max_{B_{CHSH}}\langle B_{CHSH}\rangle_{\rho_{AB}^{ent}}$. Therefore, $P^{max}(\rho_{AB}^{ent})$ given in (3.2.36) reduces to the inequality as

$$\begin{aligned} P^{max}(\rho_{AB}^{ent}) &\leq \frac{1}{2}(1+\frac{\langle B_{max}\rangle_{\rho_{AB}^{ent}}}{4}) \\ &= \frac{1}{2}(1+\frac{\sqrt{M(\rho_{AB}^{ent})}}{2}) \end{aligned} \tag{3.2.37}$$

In the last line, we have used $\langle B_{max}\rangle_{\rho_{AB}^{ent}} = 2\sqrt{M(\rho_{AB}^{ent})}$ [127]. ■

Using Lemma 3.2.1 and taking the upper bound of the inequality $M(\rho_{AB}^{ent}) \leq 2$, it can be easily observed that $P^{max}(\rho_{AB}^{ent}) \leq \frac{1}{2}(1+\frac{1}{\sqrt{2}})$.

Lemma 3.2.2. If W_{CHSH} denotes the witness operator detecting the two-qubit entangled state ρ_{AB}^{ent}, then the lower bound of $M(\rho_{AB}^{ent})$ is given by

$$M(\rho_{AB}^{ent}) \geq [1-\frac{1}{2}Tr[W_{CHSH}\rho_{AB}^{ent}]]^2 \tag{3.2.38}$$

Proof: From (3.2.4), $\langle B_{CHSH}\rangle_{\rho_{AB}^{ent}}$ can be expressed as

$$\langle B_{CHSH}\rangle_{\rho_{AB}^{ent}} = 8P^{max}(\rho_{AB}^{ent}) - 4 \tag{3.2.39}$$

Using $\langle B_{CHSH}\rangle_{\rho_{AB}^{ent}} \leq \langle B_{max}\rangle_{\rho_{AB}^{ent}}$, the equation (3.2.39) can be re-expressed as

$$8P^{max}(\rho_{AB}^{ent}) - 4 \leq \langle B_{max}\rangle_{\rho_{AB}^{ent}} = 2\sqrt{M(\rho_{AB}^{ent})} \tag{3.2.40}$$

Using Result 3.2.1 and simplifying (3.2.40), we get the result given in (3.2.38). ■

Now, we are in a position to connect $S_{NL}(\rho_{AB}^{ent})$ and $M(\rho_{AB}^{ent})$.

Result 3.2.2. If ρ_{AB}^{ent} denotes any two-qubit entangled state, which violates the CHSH inequality and is detected by W_{CHSH}, then

$$S_{NL}(\rho_{AB}^{ent}) < \frac{\sqrt{M(\rho_{AB}^{ent})} - 1}{4} \tag{3.2.41}$$

Proof: Since the CHSH witness operator W_{CHSH} detects the entangled state ρ_{AB}^{ent}, so $S_{NL}(\rho_{AB}^{ent})$ is given by

$$S_{NL}(\rho_{AB}^{ent}) = -\frac{Tr[W_{CHSH}\rho_{AB}^{ent}]}{8} \tag{3.2.42}$$

Using Lemma 3.2.2, $S_{NL}(\rho_{AB}^{ent})$ can be re-expressed in terms of $M(\rho_{AB}^{ent})$ as

$$S_{NL}(\rho_{AB}^{ent}) < \frac{\sqrt{M(\rho_{AB}^{ent})} - 1}{4} \tag{3.2.43}$$

Hence Proved. ■

Using Result 3.2.2, and the fact $M(\rho_{AB}^{ent}) \leq 2$, we get the upper bound of $S_{NL}(\rho_{AB}^{ent})$, which is given by

$$S_{NL}(\rho_{AB}^{ent}) < \frac{\sqrt{2} - 1}{4} \tag{3.2.44}$$

So far, we have discussed the relationship between $S_{NL}(\rho_{AB}^{ent})$ and $M(\rho_{AB}^{ent})$ when $M(\rho_{AB}^{ent}) > 1$. But what, if $M(\rho_{AB}^{ent}) \leq 1$? Let us now discuss this case in the form of another result that can be stated as:

Result 3.2.3. If we suppose that the two-qubit entangled state ρ_{AB}^{ent} satisfies the CHSH inequality i.e., $M(\rho_{AB}^{ent}) \leq 1$, and further, if it is not detected by the witness operator W_{CHSH}, then the relation between $S_{NL}(\rho_{AB}^{ent})$ and $M(\rho_{AB}^{ent})$ is given by

$$0 < S_{NL}(\rho_{AB}^{ent}) \leq q(\frac{\sqrt{M(\rho_{AB}^{ent})}-1}{4}) + (1-q)K \quad (3.2.45)$$

where $K = \frac{Tr[W_{CHSH}\rho_{AB}^{ent}(\rho_{AB}^{ent})^{T_B}]}{4N(\rho_{AB}^{ent})}$ and q satisfy the inequality

$$0 \leq q < \frac{4K}{1-\sqrt{M(\rho_{AB}^{ent})}+4K} \quad (3.2.46)$$

Proof: If the two-qubit entangled state ρ_{AB}^{ent} is not detected by the witness operator W_{CHSH} then $P^{max} \leq \frac{3}{4}$. Thus, the strength of the non-locality $S_{NL}^{New}(\rho_{AB}^{ent})$ of ρ_{AB}^{ent} may be defined by (3.2.29). Therefore recalling (3.2.29), we get

$$\begin{aligned} S_{NL}(\rho_{AB}^{ent}) &= q(P^{max} - \frac{3}{4}) + (1-q)K \\ &\leq q(\frac{\sqrt{M(\rho_{AB}^{ent})}-1}{4}) + (1-q)K \end{aligned} \quad (3.2.47)$$

In the second line, we have used inequality (3.2.35). Since the inequality (3.2.47) gives the upper bound of $S_{NL}(\rho_{AB}^{ent})$ in terms of $M(\rho_{AB}^{ent})$, so it may happen that the value of $S_{NL}(\rho_{AB}^{ent})$ may be negative also, which is not acceptable. Thus, to make it positive, we have to put some restrictions on q. Therefore, We can choose q in such a way that the inequality (3.2.30) holds. The inequality (3.2.30) may be re-expressed in terms of $M(\rho_{AB}^{ent})$ as

$$0 \leq q < \frac{4K}{1-\sqrt{M(\rho_{AB}^{ent})}+4K} \quad (3.2.48)$$

Hence Proved. ■

Further, employing the condition $M(\rho_{AB}^{ent}) \leq 1$ again, it can be easily shown that the inequality (3.2.47) reduces to

$$S_{NL}^{New}(\rho_{AB}^{ent}) \leq (1-q)K \quad (3.2.49)$$

Hence, we have shown here that we are capable of detecting the non-locality of ρ_{AB}^{ent} even if $M(\rho_{AB}^{ent}) \leq 1$, for some entangled state ρ_{AB}^{ent}.

3.3 Strength of the non-locality of two-qubit entangled system determined by optimal witness operator

In Section 3.2, we found that there may exist a shared entangled state ρ_{AB}^{ent} which is not detected by witness operator W_{CHSH}, and as a consequence, the maximum probability P^{max} of winning the game played between two distant players with ρ_{AB}^{ent} must be less than or equal to $\frac{3}{4}$. But just by merely observing this fact, we cannot say that the strength of the non-locality of the state ρ_{AB}^{ent} is zero as there exist other witness operators that may detect it. But the problem is that there does not exist any general relationship between P^{max} and any witness operator W^a different from W_{CHSH}. Thus, in this perspective, we can ask the following question: for any two-qubit entangled state ρ_{AB}^{ent} shared between two distant players playing the XOR game and if, $Tr(W_{CHSH}\rho_{AB}^{ent}) \geq 0$ and $Tr(W^a\rho_{AB}^{ent}) < 0$, then can we measure the strength of the non-locality of two-qubit entangled state ρ_{AB}^{ent}? We investigate this question for a particular case, $W^a = W^{opt}$, and W^{opt} denotes the optimal witness operator. The reason behind this choice is that the optimal witness operator detects the maximum number of entangled states.

3.3.1 Derivation of witness operator inequality

In this subsection, we start with the derivation of witness operator inequality using Bell-CHSH inequality. To achieve this inequality, we may consider the optimal witness operator as $W^{opt} = (|\psi\rangle_{AB}\langle\psi|)^{T_B}$, where $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and T_B denote the partial transposition with respect to subsystem B. In the second step, we establish a relationship between the optimal witness operator W^{opt} and the CHSH witness operator W_{CHSH}, and then we derive the lower and upper bound of $W_{CHSH}^{xy} + W_{CHSH}^{xz} + W_{CHSH}^{yz}$, when the optimal witness operator W^{opt} detects the entangled state ρ_{AB}^{ent}.
To start with, let us consider W^{opt} that may be expressed in terms of the Bell operators B_{xy}, B_{xz}, and B_{yz} as [227]

$$W^{opt} = \frac{1}{4}[I_4 + \frac{1}{2\sqrt{2}}(B_{xy} + B_{xz} + B_{yz})] \tag{3.3.1}$$

The expectation value of W^{opt} with respect to the two-qubit density operator ρ_{AB}^{ent} is given by

$$Tr[W^{opt}\rho_{AB}^{ent}] = \frac{1}{4}[1+\frac{1}{2\sqrt{2}}(\langle B_{xy}\rangle_{\rho_{AB}^{ent}} + \langle B_{xz}\rangle_{\rho_{AB}^{ent}} + \langle B_{yz}\rangle_{\rho_{AB}^{ent}})]$$

Recalling (3.2.6) and adding the expression of P_{ij}^{max} for different i & j, we get

$$\sum_{\substack{i,j=x,y,z\\ i\neq j}} P_{i,j}^{max} = \frac{3}{2} + \frac{\sum_{\substack{i,j=x,y,z\\ i\neq j}} \langle B_{ij}\rangle_{\rho_{AB}^{ent}}}{8} \tag{3.3.2}$$

Using (3.3.1), we can re-express (3.3.2) in terms of the expectation value of W^{opt} with respect to the state ρ_{AB}^{ent} as

$$P_{xy}^{max} + P_{yz}^{max} + P_{zx}^{max} = \frac{3}{2} - \frac{1}{2\sqrt{2}} + \sqrt{2}Tr[W^{opt}\rho_{AB}^{ent}] \tag{3.3.3}$$

We should note an important fact that the expectation value of CHSH witness operator W_{CHSH} is positive i.e., $\langle W_{CHSH}\rangle \geq 0$ when $\langle B_{CHSH}\rangle$ lying in the subinterval $[-2\sqrt{2},0]$, while it is positive or negative according to $\langle B_{CHSH}\rangle \in (0,2]$ or $\langle B_{CHSH}\rangle \in (2,2\sqrt{2}]$. Since we assume that the state ρ_{AB}^{ent} satisfies the Bell's inequality in every setting, so we consider $-2 \leq \langle B_{ij}\rangle_{\rho_{AB}^{ent}} \leq 2,\ \ i,j=x,y,z;\ i\neq j$. Thus, using (3.2.6) in the interval $[-2,2]$, we get

$$\frac{1}{4} \leq P_{ij}^{max} \leq \frac{3}{4},\ \ \forall i,j=x,y,z\ \ \&\ \ i\neq j \tag{3.3.4}$$

Therefore, using (3.3.4) in (3.3.3) and after simplifying it, we get

$$-0.28033 \leq Tr[W^{opt}\rho_{AB}^{ent}] \leq 0.78033 \tag{3.3.5}$$

Since the inequality (3.3.5) is derived using the Bell-CHSH inequality, and it involves the expectation value of the witness operator, so it may be termed as witness operator inequality. This inequality clearly shows that there exists a witness operator such as W^{opt} that may detect the entangled state ρ_{AB}^{ent}, which may not be identified by the Bell operator $B_{ij}\ \ (i,j=x,y,z;\ i\neq j)$. The existence of the subinterval $[-0.28033,0]$ of the witness operator inequality indicates the fact that we may have entangled states ρ_{AB}^{ent} that can be detected by W^{opt}, although it satisfies the Bell-CHSH inequality.
Now we are in a position to derive the lower and upper bound of $W_{CHSH}^{xy} + W_{CHSH}^{xz} + W_{CHSH}^{yz}$. To derive the required lower and upper bound, we are exploiting the subinterval $[-0.28033,0]$, where W^{opt} detects the entangled state ρ_{AB}^{ent}. We should note here a

crucial point that the state ρ_{AB}^{ent} is not detected by any of the operators W_{CHSH}^{xy}, W_{CHSH}^{xz}, and W_{CHSH}^{yz}.

Result 3.3.1. If ρ_{AB}^{ent} denotes an entangled state which is not detected by W_{CHSH}^{xy}, W_{CHSH}^{yz}, and W_{CHSH}^{zx}, and W^{opt} is an optimal witness operator such that $Tr[W^{opt}\rho_{AB}^{ent}] \in [-0.28033, 0]$, then

$$8.82843 \leq \langle W_{CHSH}^{xy}\rangle_{\rho_{AB}^{ent}} + \langle W_{CHSH}^{yz}\rangle_{\rho_{AB}^{ent}} + \langle W_{CHSH}^{xz}\rangle_{\rho_{AB}^{ent}} \leq 11.9997 \tag{3.3.6}$$

Proof: To start the derivation of the bounds, let us first express the expectation value of W^{opt} in terms of the expectation value of W_{CHSH}^{xy}, W_{CHSH}^{xz} and W_{CHSH}^{yz}. It is given by

$$\begin{aligned} Tr[W^{opt}\rho_{AB}^{ent}] &= \frac{1}{4}[1+\frac{1}{2\sqrt{2}}(\langle B_{xy}\rangle_{\rho_{AB}^{ent}} + \langle B_{yz}\rangle_{\rho_{AB}^{ent}} + \langle B_{xz}\rangle_{\rho_{AB}^{ent}})] \\ &= \frac{1}{4}[1+\frac{1}{2\sqrt{2}}(6 - \langle W_{CHSH}^{xy}\rangle_{\rho_{AB}^{ent}} - \langle W_{CHSH}^{yz}\rangle_{\rho_{AB}^{ent}} - \langle W_{CHSH}^{xz}\rangle_{\rho_{AB}^{ent}})] \end{aligned} \tag{3.3.7}$$

Considering the witness operator inequality in the negative subinterval, i.e. when $Tr[W^{opt}\rho_{AB}^{ent}] \in [-0.2803, 0]$, (3.3.7) reduces to the inequality

$$8.82843 \leq \langle W_{CHSH}^{xy}\rangle_{\rho_{AB}^{ent}} + \langle W_{CHSH}^{yz}\rangle_{\rho_{AB}^{ent}} + \langle W_{CHSH}^{xz}\rangle_{\rho_{AB}^{ent}} \leq 11.9997 \qquad \blacksquare$$

Thus, the witness operator inequality (3.3.6) in the negative region gives the lower and upper bound of $\langle W_{CHSH}^{xy}\rangle_{\rho_{AB}^{ent}} + \langle W_{CHSH}^{yz}\rangle_{\rho_{AB}^{ent}} + \langle W_{CHSH}^{xz}\rangle_{\rho_{AB}^{ent}}$, provided $\langle W_{CHSH}^{xy}\rangle_{\rho_{AB}^{ent}} \geq 0, \langle W_{CHSH}^{yz}\rangle_{\rho_{AB}^{ent}} \geq 0, \langle W_{CHSH}^{zx}\rangle_{\rho_{AB}^{ent}} \geq 0$.

To illustrate our Result 3.3.1, let us consider the state described by the density operator $\rho_{AB}^{(3)}$

$$\rho_{AB}^{(3)} = \frac{1}{4}[I\otimes I - 0.01\sigma_1\otimes I + 0.002 I\otimes\sigma_3 - 0.7\sigma_1\otimes\sigma_1 - 0.7\sigma_2\otimes\sigma_2 - 0.67\sigma_3\otimes\sigma_3] \tag{3.3.8}$$

We find that the state $\rho_{AB}^{(3)}$ is an entangled state, but it satisfies the Bell-CHSH inequality in different settings as $\langle B_{xy}\rangle_{\rho_{AB}^{(3)}} = -1.9799$, $\langle B_{yz}\rangle_{\rho_{AB}^{(3)}} = -1.93747$, $\langle B_{xz}\rangle_{\rho_{AB}^{(3)}} = -1.93747$. Further, we find that the state $\rho_{AB}^{(3)}$ is not detected by the CHSH witness operator as $\langle W_{CHSH}^{xy}\rangle_{\rho_{AB}^{(3)}} = 3.9799 \geq 0, \langle W_{CHSH}^{xz}\rangle_{\rho_{AB}^{(3)}} = 3.93747 \geq 0$ and $\langle W_{CHSH}^{yz}\rangle_{\rho_{AB}^{(3)}} = 3.93747 \geq 0$.

Let us now probe whether the state $\rho_{AB}^{(3)}$ is detected by W^{opt} or not. To investigate this, let us calculate the expectation value of W^{opt} with respect to the state $\rho_{AB}^{(3)}$ as

$$\begin{aligned} Tr[W^{opt}\rho_{AB}^{(3)}] &= \frac{1}{4}[1+\frac{1}{2\sqrt{2}}(\langle B_{xy}\rangle_{\rho_{AB}^{(3)}} + \langle B_{xz}\rangle_{\rho_{AB}^{(3)}} + \langle B_{yz}\rangle_{\rho_{AB}^{(3)}})] \\ &= -0.2675 \end{aligned} \tag{3.3.9}$$

Therefore, $Tr[W^{opt}\rho_{AB}^{(3)}]$ satisfies the witness operator inequality, and thus, one can easily verify that $\langle W_{CHSH}^{xy}\rangle_{\rho_{AB}^{(3)}} + \langle W_{CHSH}^{yz}\rangle_{\rho_{AB}^{(3)}} + \langle W_{CHSH}^{xz}\rangle_{\rho_{AB}^{(3)}} = 11.85484$ satisfy the inequality (3.3.6).

3.3.2 Upper bound for the strength of non-locality of two-qubit entangled system detected by optimal witness operator

In this subsection, we first derive the inequality that provides the upper bound of the maximum probability of winning in terms of the expectation value of W^{opt}. By doing this, we can establish the connection between the maximum probability of winning and the expectation value of W^{opt}. This connection enables us to estimate the strength of the non-locality of an entangled state which is undetected by W_{CHSH} but detected by W^{opt}. The following result educates us about the question that we have in the starting paragraph of this section.

Result 3.3.2. If the quantum state ρ_{AB}^{ent} satisfies the Bell-CHSH inequality in $xy-$, $yz-$ and $zx-$ setting i.e. if $-2 \leq \langle B_{ij}\rangle_{\rho_{AB}^{ent}} \leq 2,\ \forall\ i,j = x,y,z;\ i \neq j$, and if the state ρ_{AB}^{ent} may be identified as an entangled state by the witness operator W^{opt} given in (3.3.1), then the strength of the non-locality of ρ_{AB}^{ent} may be estimated by the inequality

$$S_{NL}(\rho_{AB}^{ent}) \leq \frac{3}{4} - \frac{1}{2\sqrt{2}} + \sqrt{2}Tr[W^{opt}\rho_{AB}^{ent}] \qquad (3.3.10)$$

Proof: Without any loss of generality, we can assume $max\{P_{xy}^{max}, P_{xz}^{max}, P_{yz}^{max}\} = P_{xy}^{max}$. Then we can have the following inequality

$$P_{xy}^{max} \leq P_{xy}^{max} + P_{yz}^{max} + P_{zx}^{max} \qquad (3.3.11)$$

Recalling the expression given in (3.3.3) and using (3.3.11), we get

$$\begin{aligned} P_{xy}^{max} &\leq \frac{3}{2} - \frac{1}{2\sqrt{2}} + \sqrt{2}Tr[W^{opt}\rho_{AB}^{ent}] \\ \Rightarrow P_{xy}^{max} - \frac{3}{4} &\leq U = \frac{3}{4} - \frac{1}{2\sqrt{2}} + \sqrt{2}Tr[W^{opt}\rho_{AB}^{ent}] \end{aligned} \qquad (3.3.12)$$

■

Our task is now to check whether the upper bound of $P_{xy}^{max} - \frac{3}{4}$ is positive when W^{opt} detects the entangled state ρ_{AB}^{ent}. We have to check this because it may indicate the fact that there is a possibility of detecting non-locality via W^{opt}. The truthfulness of the above statement is given in Table 3.1:

We are now in a position to estimate the non-locality of the entangled state described

S.No.	$Tr[W^{opt}\rho_{AB}^{ent}]$	$U=\frac{3}{4}-\frac{1}{2\sqrt{2}}+\sqrt{2}Tr[W^{opt}\rho_{AB}^{ent}]$
1	0	0.39645
2	-0.05	0.325736
3	-0.10	0.255025
4	-0.15	0.184315
5	-0.20	0.113604
6	-0.25	0.0428932
7	-0.28033	0.0001

Table 3.1: The table provides the different values of $U=\frac{3}{4}-\frac{1}{2\sqrt{2}}+\sqrt{2}Tr[W^{opt}\rho_{AB}^{ent}]$, which is the upper bound of $S_{NL}(\rho_{AB}^{ent})$ when $Tr[W^{opt}\rho_{AB}^{ent}]\in[-0.28033,0]$

by the density operator ρ_{AB}^{ent}. Therefore, using the definition of the strength of the non-locality $S_{NL}^{(xy)}(\rho_{AB}^{ent})$ given in (3.2.8), the inequality (3.3.12) reduces to

$$S_{NL}^{(xy)}(\rho_{AB}^{ent})\leq\frac{3}{4}-\frac{1}{2\sqrt{2}}+\sqrt{2}Tr[W^{opt}\rho_{AB}^{ent}] \tag{3.3.13}$$

Similarly, if we assume either $max\{P_{xy}^{max},P_{xz}^{max},P_{yz}^{max}\}=P_{xz}^{max}$ or $max\{P_{xy}^{max},P_{xz}^{max},P_{yz}^{max}\}=P_{yz}^{max}$ then, we obtain the same result. Since the upper bound of the strength of the non-locality does not depend on any particular setting, so the inequality (3.3.13) may be re-expressed as

$$S_{NL}(\rho_{AB}^{ent})\leq\frac{3}{4}-\frac{1}{2\sqrt{2}}+\sqrt{2}Tr[W^{opt}\rho_{AB}^{ent}] \tag{3.3.14}$$

Hence the theorem is proved.

To illustrate our result, let us consider the state described by the density operator ρ_n, which is given by

$$\rho_n=\begin{pmatrix}\frac{1-a}{6} & 0 & 0 & 0.0005\\ 0 & \frac{5}{6}-a & -0.251 & 0\\ 0 & -0.251 & a & 0\\ 0.0005 & 0 & 0 & \frac{a}{6}\end{pmatrix},\quad \frac{1}{10}<a<\frac{13}{20} \tag{3.3.15}$$

Applying the partial transposition criterion, we can say that the state ρ_n is an entangled state. The state satisfies the Bell-CHSH inequality, as we find that $\langle B_{xy}\rangle_{\rho_n}=-1.41987$, $\langle B_{yz}\rangle_{\rho_n}=-1.65416$, and $\langle B_{xz}\rangle_{\rho_n}=-1.65133$. But, the state ρ_n is detected by W^{opt} as $Tr[W^{opt}\rho_n]=-0.167667<0$. Although the state ρ_n satisfies the Bell-CHSH inequality in different settings, but it is detected by W^{opt}. Thus, we can use our Result 3.3.2, for the estimation of the non-locality of ρ_n and the strength of the non-locality is given by

$$S_{NL}(\rho_n) \leq 0.15933 \tag{3.3.16}$$

3.4 Expression for the strength of the non-locality of two-qubit entangled state in terms of measurement parameter and state parameter

Theorem 3.4.1. If Alice (A) and Bob (B) share any arbitrary two-qubit entangled state described by the density operator ρ_{AB}^{ent} given in (1.5.5), and if the maximized winning probability P^{max} satisfies $P^{max} \leq \frac{3}{4}$ then

$$\begin{aligned} & c_1[\lambda_1^{(0)}(\mu_1^{(0)} - \mu_1^{(1)}) + \lambda_1^{(1)}(\mu_1^{(0)} + \mu_1^{(1)})] \\ + \quad & c_2[\lambda_2^{(0)}(\mu_2^{(0)} - \mu_2^{(1)}) + \lambda_2^{(1)}(\mu_2^{(0)} + \mu_2^{(1)})] \\ + \quad & c_3[\lambda_3^{(0)}(\mu_3^{(0)} - \mu_3^{(1)}) + \lambda_3^{(1)}(\mu_3^{(0)} + \mu_3^{(1)})] \leq 2. \end{aligned} \tag{3.4.1}$$

where $\lambda_j^i \in \mathbb{R}^3$ and $\mu_j^i \in \mathbb{R}^3$ $(i = 0,1; j = 1,2)$ denote the real parameter of the Bell operator, which satisfies

$$(\lambda_1^{(i)})^2 + (\lambda_2^{(i)})^2 + (\lambda_3^{(i)})^2 = 1, \ \ (\mu_1^{(i)})^2 + (\mu_2^{(i)})^2 + (\mu_3^{(i)})^2 = 1, \ \ i = 0,1 \tag{3.4.2}$$

Proof:- Let us start with the Bell-CHSH operator B_{CHSH}, which is given by

$$B_{CHSH} = A_0 \otimes B_0 - A_0 \otimes B_1 + A_1 \otimes B_0 + A_1 \otimes B_1$$

The witness operator W_{CHSH} can be constructed from the Bell-CHSH operator as

$$W_{CHSH} = 2I \otimes I - A_0 \otimes B_0 + A_0 \otimes B_1 - A_1 \otimes B_0 - A_1 \otimes B_1 \tag{3.4.3}$$

where the Hermitian operators A_0, A_1, B_0, B_1 can be expressed in terms of the Pauli matrices $\sigma_i, \ i = x,y,z$ as

$$A_0 \ = \ \lambda_1^0 \sigma_x + \lambda_2^0 \sigma_y + \lambda_3^0 \sigma_z, \ A_1 = \lambda_1^1 \sigma_x + \lambda_2^1 \sigma_y + \lambda_3^1 \sigma_z$$

$$B_0 = \mu_1^0\sigma_x + \mu_2^0\sigma_y + \mu_3^0\sigma_z,\; B_1 = \mu_1^1\sigma_x + \mu_2^1\sigma_y + \mu_3^1\sigma_z \tag{3.4.4}$$

Recalling the two-qubit state ρ_{AB}^{ent} given in (1.5.5), and then let us calculate the expectation value of W_{CHSH} with respect to the state ρ_{AB}^{ent}. The expectation value is given by

$$Tr[W_{CHSH}\rho_{AB}^{ent}] = 2 - \frac{1}{4}[\sum_{j=x,y,z} c_j\{Tr(A_0\sigma_j)Tr[(B_0 - B_1)\sigma_j] + Tr(A_1\sigma_j)Tr[(B_0+B_1)\sigma_j]\}] \tag{3.4.5}$$

Using (3.4.4) in (3.4.5), we get

$$\begin{aligned} Tr[W_{CHSH}\rho_{AB}^{ent}] &= 2 - \{c_1[\lambda_1^{(0)}(\mu_1^{(0)} - \mu_1^{(1)}) + \lambda_1^{(1)}(\mu_1^{(0)} + \mu_1^{(1)})] + c_2[\lambda_2^{(0)}(\mu_2^{(0)} - \mu_2^{(1)}) \\ &+ \lambda_2^{(1)}(\mu_2^{(0)} + \mu_2^{(1)})] + c_3[\lambda_3^{(0)}(\mu_3^{(0)} - \mu_3^{(1)}) + \lambda_3^{(1)}(\mu_3^{(0)} + \mu_3^{(1)})]\} \end{aligned} \tag{3.4.6}$$

From Result 3.2.1, it is clear that $P^{max} \leq \frac{3}{4}$, only when $Tr[W_{CHSH}\rho_{AB}^{ent}] \geq 0$. Therefore,

$$\begin{aligned} Tr[W_{CHSH}\rho_{AB}^{ent}] \geq 0 \implies & c_1[\lambda_1^{(0)}(\mu_1^{(0)} - \mu_1^{(1)}) + \lambda_1^{(1)}(\mu_1^{(0)} + \mu_1^{(1)})] \\ + & c_2[\lambda_2^{(0)}(\mu_2^{(0)} - \mu_2^{(1)}) + \lambda_2^{(1)}(\mu_2^{(0)} + \mu_2^{(1)})] \\ + & c_3[\lambda_3^{(0)}(\mu_3^{(0)} - \mu_3^{(1)}) + \lambda_3^{(1)}(\mu_3^{(0)} + \mu_3^{(1)})] \leq 2. \end{aligned} \tag{3.4.7}$$

■

Corollary 3.4.1. If the following inequality is satisfied by any two-qubit arbitrary entangled state ρ_{AB}^{ent},

$$\begin{aligned} & c_1[\lambda_1^{(0)}(\mu_1^{(0)} - \mu_1^{(1)}) + \lambda_1^{(1)}(\mu_1^{(0)} + \mu_1^{(1)})] + c_2[\lambda_2^{(0)}(\mu_2^{(0)} - \mu_2^{(1)}) \\ + & \lambda_2^{(1)}(\mu_2^{(0)} + \mu_2^{(1)})] + c_3[\lambda_3^{(0)}(\mu_3^{(0)} - \mu_3^{(1)}) + \lambda_3^{(1)}(\mu_3^{(0)} + \mu_3^{(1)})] > 2 \end{aligned} \tag{3.4.8}$$

and the state is detected by W_{CHSH}, then $P^{max} > \frac{3}{4}$.

Proof: This corollary follows from Result 3.2.1. ■

Now we are in a position to measure the strength of the non-locality of any general two-qubit entangled state. The expression of the strength of the non-locality can be expressed in terms of the measurement parameters and state parameters, and it is given in the result below:

Result 3.4.1. If any arbitrary two-qubit state described by the density operator ρ_{AB}^{ent} given in (1.5.5) represents an entangled state, which is detected by the witness operator W_{CHSH} then its non-locality can be determined using the following formula

$$\begin{aligned} S_{NL}(\rho_{AB}^{ent}) &= \frac{1}{8}[c_1(\lambda_1^{(0)}(\mu_1^{(0)}-\mu_1^{(1)})+\lambda_1^{(1)}(\mu_1^{(0)}+\mu_1^{(1)})) \\ &+ c_2(\lambda_2^{(0)}(\mu_2^{(0)}-\mu_2^{(1)})+\lambda_2^{(1)}(\mu_2^{(0)}+\mu_2^{(1)})) \\ &+ c_3(\lambda_3^{(0)}(\mu_3^{(0)}-\mu_3^{(1)})+\lambda_3^{(1)}(\mu_3^{(0)}+\mu_3^{(1)}))-2] \end{aligned} \tag{3.4.9}$$

Proof: Substituting (3.4.6) in (3.2.14), we get the desired result. ■

3.5 Applications

In this section, we will discuss two applications of the introduced quantity $S_{NL}(\rho_{AB}^{ent})$ such as (i) application of $S_{NL}(\rho_{AB}^{ent})$ in the determination of the genuine non-locality of two particular classes of three-qubit GHZ state and W state, and (ii) application of $S_{NL}(\rho_{AB}^{ent})$ in finding the upper limit of the power of the controller in controlled quantum teleportation.

3.5.1 Linkage between the strength of the non-locality of two-qubit entangled state and the expectation value of the Svetlichny operator with respect to a pure three-qubit state

In this section, we give a brief discussion about the non-locality of the three-qubit state, and then we establish a relationship between the two-qubit non-locality with the non-locality of the pure three-qubit state. We measure the strength of the two-qubit non-locality by S_{NL}, and the pure three-qubit non-locality is measured by the expectation value of the Svetlichny operator.

Let us consider a tripartite system describing a pure three-qubit state. In a three-qubit state, there may exist different types of correlation. The correlation may exist either between two subsystems only or between all three subsystem. The correlations are genuinely tripartite non-local if the correlations cannot be simulated by a hybrid (non-local)-local ensemble of a three-qubit system. Here, a hybrid (non-local)-local ensemble of a three-qubit system means that any two subsystems are non-locally correlated but it is locally correlated, with the third subsystem. The genuine tripartite non-local correlation exists in the three-qubit state ρ_{ABC} that may be detected by

Svetlichny inequality, which can be read as [86]

$$|\langle S_v\rangle_{\rho_{ABC}}| \leq 4 \quad (3.5.1)$$

where S_v denotes the Svetlichny operator, which may be defined as

$$\begin{aligned} S_v &= \vec{a}.\vec{\sigma}_1 \otimes [\vec{b}.\vec{\sigma}_2 \otimes (\vec{c}+\vec{c'}).\vec{\sigma}_3 + \vec{b'}.\vec{\sigma}_2 \otimes (\vec{c}-\vec{c'}).\vec{\sigma}_3] \\ &+ \vec{a'}.\vec{\sigma}_1 \otimes [\vec{b}.\vec{\sigma}_2 \otimes (\vec{c}-\vec{c'}).\vec{\sigma}_3 - \vec{b'}.\vec{\sigma}_2 \otimes (\vec{c}+\vec{c'}).\vec{\sigma}_3] \end{aligned} \quad (3.5.2)$$

Here $\vec{a},\vec{a'}$, $\vec{b},\vec{b'}$, and $\vec{c},\vec{c'}$ are the unit vectors and $\vec{\sigma}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$ denote the spin projection operators.

The expectation value of the Svetlichny operator with respect to the three-qubit state ρ_{ABC} is given by [228, 229]

$$\begin{aligned} \langle S_v\rangle_{\rho_{ABC}} &= Max_{\vec{a},\vec{b},\vec{c},\vec{a'},\vec{b'},\vec{c'}}([\vec{a}.\vec{\sigma}_1 \otimes \vec{b}.\vec{\sigma}_2 - \vec{a'}.\vec{\sigma}_1 \otimes \vec{b'}.\vec{\sigma}_2]^T M(\vec{c}+\vec{c'}).\vec{\sigma}_3 \\ &+ [\vec{a}.\vec{\sigma}_1 \otimes \vec{b'}.\vec{\sigma}_2 + \vec{a'}.\vec{\sigma}_1 \otimes \vec{b}.\vec{\sigma}_2]^T M(\vec{c}-\vec{c'}).\vec{\sigma}_3) \end{aligned} \quad (3.5.3)$$

where $M = (M_{j,ik})$ represents a matrix with the entries $M_{ijk} = Tr(\sigma_i \otimes \sigma_i \otimes \sigma_k)$, $i, j, k = 1, 2, 3$.

If any three-qubit state ρ_{ABC} violates the inequality (3.5.1) then ρ_{ABC} can be considered as a genuine tripartite non-local state. M. Li et. al. [228] derived the upper bound of the expectation value of the Svetlichny operator S_v with respect to any three-qubit state and it is given by

$$|\langle S_v\rangle_{\rho_{ABC}}| \leq 4\mu_1 \quad (3.5.4)$$

where μ_1 denotes the maximum singular value of the matrix M.

We are now in a position to establish a relationship between $S_{NL}(\rho_{AB})$ and $\langle S_v\rangle_{\rho_{ABC}}$. To do this, let us first consider a canonical form of a pure three-qubit state, which is given by [230]

$$|\psi\rangle_{ABC} = \lambda_0|000\rangle_{ABC} + \lambda_1 e^{i\theta}|100\rangle_{ABC} + \lambda_2|101\rangle_{ABC} + \lambda_3|110\rangle_{ABC} + \lambda_4|111\rangle_{ABC} \quad (3.5.5)$$

where $\sum_{i=0}^{4} \lambda_i^2 = 1$, $0 \leq \lambda_i \leq 1$ and $0 \leq \theta \leq \pi$.

To achieve the required relation, we take into account the two-qubit state described by the density operator ρ_{AB}, whose purification is the three-qubit state $|\psi\rangle_{ABC}$ [231].

The state ρ_{AB} is given by

$$\rho_{AB} = \begin{pmatrix} \lambda_0^2 & 0 & \lambda_0\lambda_1 e^{i\theta} & \lambda_0\lambda_3 \\ 0 & 0 & 0 & 0 \\ \lambda_0\lambda_1 e^{-i\theta} & 0 & \lambda_1^2+\lambda_2^2 & \lambda_1\lambda_3 e^{-i\theta}+\lambda_2\lambda_4 \\ \lambda_0\lambda_3 & 0 & \lambda_1\lambda_3 e^{i\theta}+\lambda_2\lambda_4 & \lambda_3^2+\lambda_4^2 \end{pmatrix} \tag{3.5.6}$$

We can make an observation that it is not very easy to obtain the analytical relationship between the expectation value of the Svetlichny operator with respect to the pure three-qubit state $|\psi\rangle_{ABC}$, and the strength of the non-locality of two-qubit mixed state ρ_{AB} by keeping all the parameters. Thus to obtain the required relationship, we consider a few particular types of three-qubit states.

3.5.1.1 A family of pure three-qubit states: GHZ class

Let us consider a pure three-qubit state, which can be expressed as

$$|\psi^{MS}\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle_{ABC} + cos\theta|110\rangle_{ABC} + sin\theta|111\rangle_{ABC}),\ 0<\theta<\frac{\pi}{2} \tag{3.5.7}$$

It is known as the maximal slice (MS) state [87]. The inherent symmetries of the MS state make it very useful for quantum communication purposes [167]. The expectation value of Svetlichny operator S_v with respect to the state $|\psi^{MS}\rangle_{ABC}$ is given by [87]

$$\langle S_v\rangle_{|\psi^{MS}\rangle_{ABC}} = 4\sqrt{2-Cos^2\theta} \tag{3.5.8}$$

Using (3.5.6), we can obtain the two-qubit state described by the density operator ρ_{AB}^{MS}, whose purification is the state $|\psi^{MS}\rangle_{ABC}$. The state ρ_{AB}^{MS} is given by

$$\rho_{AB}^{MS} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{cos\theta}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{cos\theta}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{3.5.9}$$

The negativity of the state ρ_{AB}^{MS} is given by

$$N(\rho_{AB}^{MS}) = \sqrt{\frac{1+cos2\theta}{2}} \tag{3.5.10}$$

Consider the Bell-CHSH witness operator W_{CHSH}^{ij} $(i,j=x,y,z; i\neq j)$ in $xy-$, $yz-$ and

$zx-$ plane to detect the state ρ_{AB}^{MS}. The witness operator W_{CHSH}^{ij} is given by

$$W_{CHSH}^{ij} = 2I - B_{ij}, \ i,j = x,y,z; i \neq j \tag{3.5.11}$$

where $B_{ij} = \sqrt{2}[\sigma_i \otimes \sigma_i + \sigma_j \otimes \sigma_j], \ i,j = x,y,z$ and $i \neq j$.
Let us now discuss different cases by considering the witness operator in different two-dimensional planes.

Case I: $xy-$ **plane**.
The witness operator defined in this plane is given by

$$W_{CHSH}^{xy} = 2I - \sqrt{2}[\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y] \tag{3.5.12}$$

The expectation value of W_{CHSH}^{xy} with respect to the state ρ_{AB}^{MS} is given by

$$Tr[W_{CHSH}^{xy} \rho_{AB}^{MS}] = 2 > 0, \forall \ \ \theta \in (0, \frac{\pi}{2}) \tag{3.5.13}$$

The witness operator W_{CHSH}^{xy} does not detect the state ρ_{AB}^{MS} for any value of $\theta \in (0, \frac{\pi}{2})$. Therefore, the strength of the non-locality of ρ_{AB}^{MS} can be obtained as

$$S_{NL}(\rho_{AB}^{MS}) = q.(P_{xy}^{max} - \frac{3}{4}) + (1-q).K \tag{3.5.14}$$

where $K = \frac{Tr[W_{CHSH}^{xy} . \rho_{AB}^{MS} . (\rho_{AB}^{MS})^{T_B}]}{4.N(\rho_{AB}^{MS})} = \frac{1}{4cos\theta}$ and $q < (0.5, 1]$. Further, we have $P_{max}^{xy} - \frac{3}{4} = -\frac{1}{4}$. Using these values, we can get the expression for the strength of the non-locality of ρ_{AB}^{MS} as

$$S_{NL}(\rho_{AB}^{MS}) = \frac{1 - q(1 + cos\theta)}{4cos\theta}, \ q < (0.5, 1] \tag{3.5.15}$$

In particular, considering $q = 0.3$, the expression for $S_{NL}(\rho_{AB}^{MS})$ given in (3.5.15) reduces to

$$S_{NL}(\rho_{AB}^{MS}) = \frac{0.7 - 0.3cos\theta}{4cos\theta}, \ 0 < \theta < \frac{\pi}{2} \tag{3.5.16}$$

As θ varies from 0 to $\frac{\pi}{2}$, $S_{NL}(\rho_{AB}^{MS}) \in (0.1, 1.2]$.
Using (3.5.8) and (3.5.16), we obtain a relation between $S_{NL}(\rho_{AB}^{MS})$ and $\langle S_v \rangle_{|\psi^{MS}\rangle_{ABC}}$ as

$$\langle S_v \rangle_{|\psi^{MS}\rangle_{ABC}} = 4\sqrt{2 - \frac{49}{1600} . \frac{1}{(S_{NL}((\rho_{AB}^{MS}) + \frac{3}{40})^2}}, \ 0.1 < S_{NL}(\rho_{AB}^{MS}) \leq 1.2 \tag{3.5.17}$$

and the values of $\langle S_v \rangle_{|\psi^{MS}\rangle_{ABC}}$ with respect to $S_{NL}(\rho_{AB}^{MS})$ in xy-plane are shown in Figure 3.2.

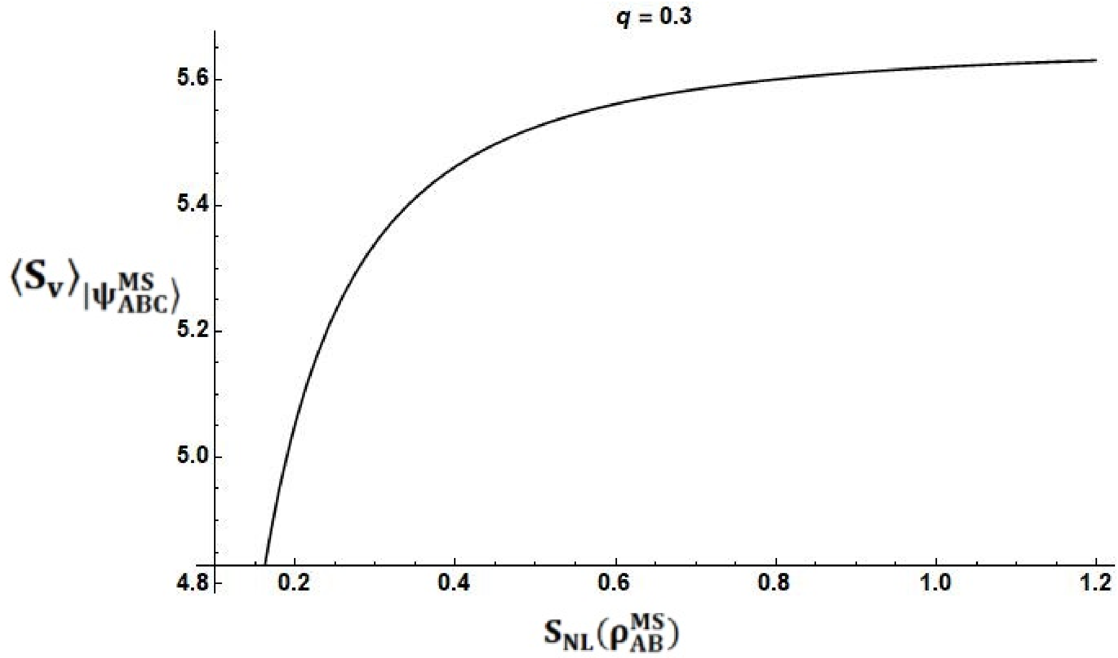

Figure 3.2: The graph depicts the relationship between $\langle S_v \rangle_{|\psi^{MS}\rangle_{ABC}}$ and $S_{NL}(\rho_{AB}^{MS})$. It is clear from the graph that for $S_{NL}(\rho_{AB}^{MS})$ belongs to (0.1,1.2] when q is taken as 0.3, $\langle S_v \rangle_{|\psi^{MS}\rangle_{ABC}}$ is always greater than 4, i.e., S_v inequality is violated.

Case II: $yz-$ **plane**.

The witness operator defined in yz plane is given by

$$W_{CHSH}^{yz} = 2I - \sqrt{2}[\sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z] \tag{3.5.18}$$

The expectation value of W_{CHSH}^{yz} with respect to the state ρ_{AB}^{MS} is given by

$$Tr[W_{CHSH}^{yz} \rho_{AB}^{MS}] = 2 - \sqrt{2} + \sqrt{2}cos\theta > 0, \forall\ \theta \in (0, \frac{\pi}{2}) \tag{3.5.19}$$

In this case also, the witness operator W_{CHSH}^{yz} does not detect the state ρ_{AB}^{MS} for any value of $\theta \in (0, \frac{\pi}{2})$.

Therefore, the strength of the non-locality of ρ_{AB}^{MS} can be obtained as

$$S_{NL}(\rho_{AB}^{MS}) = q.(P_{max}^{yz} - \frac{3}{4}) + (1-q).K \tag{3.5.20}$$

where $K = \frac{Tr[W_{CHSH}^{yz} . \rho_{AB}^{MS} . (\rho_{AB}^{MS})^{T_B}]}{4.N(\rho_{AB}^{MS})} = \frac{2-\sqrt{2}+\sqrt{2}cos\theta}{8cos\theta}$ and $q < (0.5, 1)$ for $\theta \in (0, \frac{\pi}{2})$. Further, we have $P_{max}^{yz} - \frac{3}{4} = -\frac{2-\sqrt{2}+\sqrt{2}cos\theta}{8}$. Using these values, we can get the expression for the strength of the non-locality of ρ_{AB}^{MS} as

$$S_{NL}(\rho_{AB}^{MS}) = \frac{2-2\sqrt{2}sin^2\frac{\theta}{2} - q(4cos^2\frac{\theta}{2} - \sqrt{2}Sin^2\theta))}{8cos\theta} \tag{3.5.21}$$

In particular, considering $q = 0.001$, the expression for $S_{NL}(\rho_{AB}^{MS})$ given in (3.5.21) reduces to

$$S_{NL}(\rho_{AB}^{MS}) = -0.001(\frac{2-\sqrt{2}+\sqrt{2}cos\theta}{8}) + 0.999(\frac{2-\sqrt{2}+\sqrt{2}cos\theta}{8cos\theta}) \tag{3.5.22}$$

As $\theta \in (0, \frac{\pi}{2}]$, $S_{NL}(\rho_{AB}^{MS}) \in (0.25, 0.7]$.
Using (3.5.8) and (3.5.22), we obtain a relation between $S_{NL}(\rho_{AB}^{MS})$ and $\langle S_v \rangle_{|\psi^{MS}\rangle_{ABC}}$ as

$$\langle S_v \rangle_{|\psi^{MS}\rangle_{ABC}} = \sqrt{32-u^2},\ 0.25 < S_{NL}(\rho_{AB}^{MS}) \leq 0.7$$

where $u = \frac{-(8S_{NL}-1.41221)+\sqrt{(8S_{NL}-1.41221)^2+0.00330521}}{0.007}$. The values of $\langle S_v \rangle_{|\psi^{MS}\rangle_{ABC}}$ with respect to $S_{NL}(\rho_{AB}^{MS})$ in yz-plane are shown in Figure 3.3.

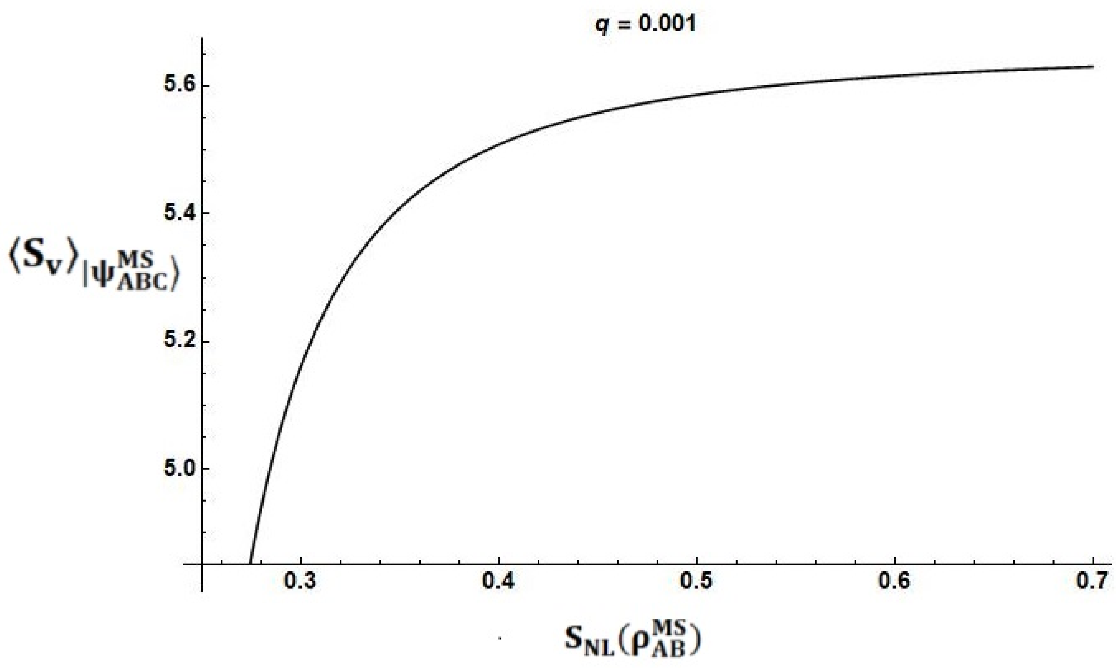

Figure 3.3: The graph depicts the relationship between $\langle S_v \rangle_{|\psi^{MS}\rangle_{ABC}}$ and $S_{NL}(\rho_{AB}^{MS})$. It is clear from the graph that for $S_{NL}(\rho_{AB}^{MS})$ belongs to (0.25,0.7] when q is taken as 0.001, $\langle S_v \rangle_{|\psi^{MS}\rangle_{ABC}}$ is always greater than 4, i.e., S_v inequality is violated.

Case III: $zx-$ **plane:** In a similar fashion, we can obtain the relationship between $\langle S_v \rangle_{|\psi^{MS}\rangle_{ABC}}$ and $S_{NL}(\rho_{AB}^{MS})$ when the witness operator W_{CHSH} defined in $zx-$ plane.

3.5.1.2 A family of pure three-qubit states: W-class of Type-I

Let us consider a family of pure three-qubit states, which can be expressed in the form as

$$|\psi_1\rangle_{ABC} = \lambda_0|000\rangle_{ABC} + 0.3|101\rangle_{ABC} + \sqrt{0.91-\lambda_0^2}|110\rangle_{ABC},\ \lambda_0 \in [0, 0.953939]$$

The state $|\psi_1\rangle_{ABC}$ belongs to the $W-$ class of states. Let us consider a two-qubit state described by the density operator $\rho_{AB}^{(t1)}$ which when purified, gives rise to the three-qubit pure state $|\psi_1\rangle_{ABC}$. The two-qubit state $\rho_{AB}^{(t1)}$ is given by [231]

$$\rho_{AB}^{(t1)} = \begin{pmatrix} \lambda_0^2 & 0 & 0 & \lambda_0\sqrt{0.91-\lambda_0^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.09 & 0 \\ \lambda_0\sqrt{0.91-\lambda_0^2} & 0 & 0 & 0.91-\lambda_0^2 \end{pmatrix},\ \lambda_0 \in [0, 0.953939] \quad (3.5.23)$$

In this interval of λ_0, the state $\rho_{AB}^{(t1)}$ is an entangled state, but it is not detected by the CHSH witness operators W_{CHSH}^{xy} and W_{CHSH}^{yz}. The entangled state $\rho_{AB}^{(t1)}$ is only detected by the CHSH witness operator W_{CHSH}^{xz}.

In the $xz-$ plane, the expectation value of CHSH witness operator W_{CHSH}^{xz} with respect to the state $\rho_{AB}^{(t1)}$ is given by

$$Tr[W_{CHSH}^{xz}\rho_{AB}^{(t1)}] = 0.840345 - 2.82843\lambda_0\sqrt{0.91-\lambda_0^2} < 0,\quad \lambda_0 \in [0.335, 0.85] \quad (3.5.24)$$

where $W_{CHSH}^{xz} = 2I - B_{xz}$ and $B_{xz} = \sqrt{2}[\sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z]$.

Therefore, in this case, the non-locality of the two-qubit state $\rho_{AB}^{(t1)}$ can be calculated via the formula

$$\begin{aligned} S_{NL}(\rho_{AB}^{(t1)}) &= -\frac{Tr[W_{CHSH}^{xz}\rho_{AB}^{(t1)}]}{8} \\ &= \frac{-(0.840345 - 2.82843\lambda_0\sqrt{0.91-\lambda_0^2})}{8}, \lambda_0 \in [0.335, 0.85] \quad (3.5.25) \end{aligned}$$

It can be easily found that the value of $S_{NL}(\rho_{AB}^{(t1)})$ lies in the interval $[0, 0.06]$ when $\lambda_0 \in [0.335, 0.85]$.

The expression (3.5.25) can be re-expressed as

$$\lambda_0^2 = \frac{0.91 \pm \sqrt{(0.91)^2 - k}}{2} \tag{3.5.26}$$

where $k = [\frac{8S_{NL}(\rho_{AB}^{(1)})+0.840305}{2.82843}]^2$.

Now, our task is to calculate the expectation value of the Svetlichny operator with respect to the state $|\psi_1\rangle_{ABC}$. To accomplish this task, firstly, we need to calculate the matrix M_1 [228], which is given by

$$M_1 = \begin{pmatrix} 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0.82 \end{pmatrix} \tag{3.5.27}$$

where $a = 2\lambda_0\sqrt{0.91-\lambda_0^2}, b = -0.6\sqrt{0.91-\lambda_0^2}$ *and* $c = 0.6\lambda_0$.

The maximum singular value of M_1 is given by

$$\mu_1 = 0.707107\sqrt{1+3.64\lambda_0^2-4\lambda_0^4+\sqrt{1-7.28\lambda_0^2+21.2496\lambda_0^4-29.12\lambda_0^6+16\lambda_0^8}} \tag{3.5.28}$$

Using the result (3.5.4) and (3.5.28), we get

$$\langle S_v \rangle_{\rho_{ABC}^{(1)}} \leq 4(0.707107\sqrt{1+3.64\lambda_0^2-4\lambda_0^4+\sqrt{J}}) \tag{3.5.29}$$

where $\rho_{ABC}^{(1)} = |\psi_1\rangle_{ABC}\langle\psi_1|$ and $J = 1-7.28\lambda_0^2+21.2496\lambda_0^4-29.12\lambda_0^6+16\lambda_0^8$

When the state parameter λ_0 is given by (3.5.26), then the relation between $|\langle S_v \rangle_{\rho_{ABC}^{(1)}}|$ and $S_{NL}(\rho_{AB}^{(t1)})$ may be written as

$$|\langle S_v(\rho_{ABC}^{(1)})\rangle| \leq 4(0.707107\sqrt{1+3.64\lambda_0^2-4\lambda_0^4+\sqrt{J}}) \tag{3.5.30}$$

One can now easily verify that the pure three-qubit state $|\psi_1\rangle_{ABC}$ satisfies the Svetlichny inequality when $S_{NL}(\rho_{AB}^{(t1)}) \in [0, 0.06]$.

3.5.1.3 A family of pure three-qubit states: W-class of Type-II

Consider a family of pure three-qubit state

$$|\psi_2\rangle_{ABC} = \lambda_0|000\rangle_{ABC} + 0.7|100\rangle_{ABC} + \sqrt{0.51-\lambda_0^2}|110\rangle_{ABC}, \ \lambda_0 \in [0.1, 0.7]$$

The state $|\psi_2\rangle_{ABC}$ belongs to W- class of states. The two-qubit state $\rho_{AB}^{(t2)}$ can be purified to $|\psi_2\rangle_{ABC}$. The density matrix $\rho_{AB}^{(t2)}$ is given by [231]

$$\rho_{AB}^{(t2)} = \begin{pmatrix} \lambda_0^2 & 0 & 0.7\lambda_0 & \lambda_0 s \\ 0 & 0 & 0 & 0 \\ 0.7\lambda_0 & 0 & 0.49 & 0.7s \\ \lambda_0 s & 0 & 0.7s & s \end{pmatrix}, \quad \lambda_0 \in [0.1, 0.7] \tag{3.5.31}$$

where $s = \sqrt{0.51 - \lambda_0^2}$. In the given interval of λ_0, the state $\rho_{AB}^{(t2)}$ is an entangled state, but it is not detected by any of the CHSH witness operators W_{CHSH}^{xy}, W_{CHSH}^{yz}, and W_{CHSH}^{xz}. Therefore, we can proceed with any one of the CHSH witness operators. Let us choose the witness operator W_{CHSH}^{xy}. In the $xy-$ plane, the expectation value of CHSH witness operator W_{CHSH}^{xy} with respect to the state $\rho_{AB}^{(t2)}$ is given by

$$Tr[W_{CHSH}^{xy}\rho_{AB}^{(t2)}] = 2 > 0, \quad \lambda_0 \in [0.1, 0.7] \tag{3.5.32}$$

where $W_{CHSH}^{xy} = 2I - B_{xy}$ and $B_{xy} = \sqrt{2}[\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y]$.
Therefore, in this case, the non-locality of a two-qubit entangled state $\rho_{AB}^{(t2)}$ can be calculated as

$$S_{NL} = q(P_{max}^{xy} - \frac{3}{4}) + (1-q)k \tag{3.5.33}$$

where $k = 2 - 2.04\lambda_0^2 + 4\lambda_0^4 - 1.38593\lambda_0\sqrt{0.51 - \lambda_0^2}$ and $P_{max}^{xy} = \frac{1}{2}$.
The parameter q satisfies the inequality

$$q < [0.73, 1] \tag{3.5.34}$$

Considering $q = 0.6$, the strength of the non-locality of $\rho_{AB}^{(t2)}$ is given by

$$S_{NL}(\rho_{AB}^{(t2)}) = \frac{1 + 2\lambda_0^4 - K}{\lambda_0\sqrt{51 - 100\lambda_0^2}} \tag{3.5.35}$$

where $K = [1.02\lambda_0^2 + 0.15\lambda_0\sqrt{51 - 100\lambda_0^2} + 0.692965\lambda_0\sqrt{0.51 - \lambda_0^2}]$.
It can be easily seen that the value of $S_{NL}(\rho_{AB}^{(t2)}) \in [0.1219, 1.18077]$ for $\lambda_0 \in [0.1, 0.7]$.

For the state $|\psi_2\rangle_{ABC}$, the matrix M_2 is given by [228]

$$M_2 = \begin{pmatrix} 0 & 0 & a_1 & 0 & 0 & 0 & b_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_1 & 0 & b_1 & 0 \\ c_1 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.5.36)$$

where $a_1 = 2\lambda_0\sqrt{0.51-\lambda_0^2}, b_1 = -1.4\sqrt{0.51-\lambda_0^2}$ and $c_1 = 1.4\lambda_0$.
The maximum singular value of M_2 is given by

$$\mu_2 = \sqrt{1+3.92\lambda_0^2} \quad (3.5.37)$$

Using the result (3.5.4) and (3.5.37), we get

$$\langle S_v\rangle_{\rho_{ABC}^{(2)}} \leq 4\sqrt{1+3.92\lambda_0^2} \quad (3.5.38)$$

where $\rho_{ABC}^{(2)} = |\psi_2\rangle_{ABC}\langle\psi_2|$.
The relation between $|\langle S_v\rangle_{\rho_{ABC}^{(2)}}|$ and $S_{NL}(\rho_{AB}^{(2)})$ may be given by

$$4 < \langle S_v\rangle_{\rho_{ABC}^{(2)}} \leq \sqrt{16+\frac{33.1546}{S_{NL}(\rho_{AB}^{(t2)})}}$$

One can now find that the pure three-qubit state $|\psi_2\rangle_{ABC}$ violates the Svetlichny inequality, when $S_{NL}(\rho_{AB}^{(t2)}) \in [0.1219, 1.18077]$.

3.5.2 Upper bound of the power of the controller in controlled quantum teleportation in terms of S_{NL}

Controlled quantum teleportation [137] is a variant of quantum teleportation protocol [20], where a party controls the fidelity of the quantum teleportation. To explain the controlled quantum teleportation, let us consider a three-qubit state described by the density operator ρ_{CAB}, which is shared between three distant parties Alice, Bob, and Charlie. Alice and Bob possess the qubit A and B, while the qubit C is with Charlie. In the controlled quantum teleportation, Charlie performs measurement on his qubit C, and as a result, Alice and Bob share a two-qubit state described by the density operator ρ_{AB}. Alice and Bob then use the state ρ_{AB} as a resource state to teleport a qubit. The state ρ_{AB} contains Charlie's measurement parameter, and this parameter is also visible in the expression of the fidelity of teleportation. Thus, Charlie may control the teleportation fidelity by choosing the measurement parameter, and hence he may act as a controller in the teleportation protocol. To quantify Charlie's strength, one may define the power of the controller. To study the controller's power in controlled teleportation, we need to consider the two quantities: (i) Conditioned fidelity denoted by f_C, which is assumed to be greater than $\frac{2}{3}$, and (ii) Non-conditioned fidelity denoted by f_{NC}, which is assumed to be less than $\frac{2}{3}$. Therefore, the power denoted by P may be defined as [167, 169, 170]

$$P = f_C - f_{NC} \tag{3.5.39}$$

In this section, we will show that the controller's power in the controlled quantum teleportation is upper bounded by the quantity $M(\rho_{AB})$ and hence the quantity $S_{NL}(\rho_{AB})$. To obtain the required results, we need to state two lemmas which are given below:

Lemma 3.5.1. If τ denotes the tangle of the three-qubit pure state described by the density matrix ρ_{CAB} and $N(\rho_{AB})$ denotes the negativity of the two-qubit state $\rho_{AB} = Tr_C(\rho_{CAB})$, then the conditioned fidelity f_C is given by

$$\frac{2}{3} < f_C \leq \frac{2}{3} + \frac{\sqrt{\tau + (\sqrt{2}\sqrt{N^2(\rho_{AB}) + N(\rho_{AB})} - N(\rho_{AB}))^2}}{3} \tag{3.5.40}$$

Proof: The conditioned fidelity f_C is given by [47]

$$f_C = \frac{2 + \tau_{AB}}{3} = \frac{2 + \sqrt{\tau + (C(\rho_{AB}))^2}}{3} \tag{3.5.41}$$

where τ_{AB} denotes the partial tangle and it can be expressed in terms of τ as $\tau_{AB} = \sqrt{\tau + (C(\rho_{AB}))^2}$ [47].

Verstraete et. al. [232] proved that the lower bound of the negativity $(N(\rho_{AB}))$ of any two-qubit state ρ_{AB} can be expressed as a function of the concurrence $C(\rho_{AB})$, and it is given by

$$N(\rho_{AB}) \geq \sqrt{(1 - (C(\rho_{AB}))^2 + (C(\rho_{AB}))^2} - 1 + C(\rho_{AB}) \tag{3.5.42}$$

Simplifying (3.5.42) and writing $C(\rho_{AB})$ in terms of $N(\rho_{AB})$, we get

$$0 \leq C(\rho_{AB}) \leq -N(\rho_{AB}) + \sqrt{2}\sqrt{N^2(\rho_{AB}) + N(\rho_{AB})} \tag{3.5.43}$$

Using (3.5.43) in (3.5.41), we get the upper bound of f_C in terms of $N(\rho_{AB})$. Furthermore, from (3.5.41), it is clear that $f_C > \frac{2}{3}$. Hence the lemma. ■

Lemma 3.5.2. If ρ_{CAB} denotes the three-qubit pure state, then the non-conditioned fidelity f_{NC} is given by [127]

$$f_{NC} \geq \frac{3+M(\rho_{AB})}{6} \tag{3.5.44}$$

where $\rho_{AB} = Tr_C(\rho_{CAB})$ is the two-qubit mixed state shared between two distant parties as a resource state to execute the teleportation protocol. ■

Result 3.5.1. If τ denotes the tangle of a three-qubit pure state described by the density matrix ρ_{CAB} and if P denotes the power of the controller in controlled teleportation, then the upper bound of the power is given by

$$P \leq (\frac{1-M(\rho_{AB})}{6}) + \frac{\sqrt{\tau + (\sqrt{2}\sqrt{N^2(\rho_{AB})+N(\rho_{AB})} - N(\rho_{AB}))^2}}{3} \tag{3.5.45}$$

Proof: The power P of the controller can be re-written as

$$P = f_C - f_{NC} \tag{3.5.46}$$

Using Lemma 3.5.1 and Lemma 3.5.2, the power P given in (3.5.46) reduces to the following inequality

$$\begin{aligned} P &\leq (\frac{2}{3} + \frac{\sqrt{\tau + (\sqrt{2}\sqrt{N^2(\rho_{AB})+N(\rho_{AB})} - N(\rho_{AB}))^2}}{3} - (\frac{3+M(\rho_{AB})}{6})) \\ &= ((\frac{1-M(\rho_{AB})}{6}) + \frac{\sqrt{\tau + (\sqrt{2}\sqrt{N^2(\rho_{AB})+N(\rho_{AB})} - N(\rho_{AB}))^2}}{3}) \end{aligned} \tag{3.5.47}$$

■

Since it is assumed that $f_C > \frac{2}{3}$ and $f_{NC} < \frac{2}{3}$, so the power P of the controller cannot be negative [167, 170]. Thus, we may note the following:

Note 1: If the two-qubit reduced state ρ_{AB} does not violate the CHSH inequality, then $M(\rho_{AB}) \leq 1$, and thus the non-conditioned fidelity f_{NC} will be less than $\frac{2}{3}$. Hence, the power P is always positive.

Note 2: If the two-qubit reduced state ρ_{AB} does violate the CHSH inequality, then $M(\rho_{AB}) > 1$ and, in this case, the non-conditioned fidelity $f_{NC} > \frac{2}{3}$. Thus there may be a chance to get the negative power, which is not acceptable. But if we impose restriction on $M(\rho_{AB})$, then we can make the power positive. Hence, the power P is positive, only when the following conditions hold

$$1 < M(\rho_{AB}) < 1+2\sqrt{L} \tag{3.5.48}$$

where L=$\tau+(\sqrt{2}\sqrt{N^2(\rho_{AB})+N(\rho_{AB})}-N(\rho_{AB}))^2$.

Result 3.5.2. If the reduced entangled state ρ_{AB} violates the CHSH inequality and is detected by the witness operator W_{CHSH} then the connection between the non-locality of ρ_{AB} determined by $S_{NL}(\rho_{AB})$ and the three-qubit tangle τ is given by

$$S_{NL}(\rho_{AB}) < \frac{\sqrt{1+2\sqrt{L}}-1}{4} \tag{3.5.49}$$

Now we are in a position to express the controller's power in terms of $S_{NL}(\rho_{AB})$.

Result 3.5.3. Let us consider a three-qubit state ρ_{CAB} shared between three parties, Alice, Bob, and Charlie. If the reduced entangled state $\rho_{AB} = Tr_C(\rho_{CAB})$ violates the CHSH inequality and is detected by the witness operator W_{CHSH}, then the controller's power P can be determined by $S_{NL}(\rho_{AB})$, which is given by the following inequality

$$P < \frac{1}{6} - \frac{4}{3}(S_{NL}(\rho_{AB})(1+2S_{NL}(\rho_{AB})) \tag{3.5.50}$$

Proof: Recalling and re-write (3.5.47) as

$$P < \frac{\sqrt{L}}{3} + \frac{1-M(\rho_{AB})}{6} \tag{3.5.51}$$

where $L = \tau+(\sqrt{2}\sqrt{N^2(\rho_{AB})+N(\rho_{AB})}-N(\rho_{AB}))^2$, τ and $N(\rho_{AB})$ denotes tangle of the three-qubit pure state and negativity of the reduced two-qubit state respectively.
Further, since $S_{NL}(\rho_{AB}) \geq 0$ so, the upper limit of $S_{NL}(\rho_{AB})$ given in (3.5.49) must be positive. Therefore, the expression $\frac{\sqrt{1+2\sqrt{L}}-1}{4}$ in the R.H.S of (3.5.49) reduces to

$$\frac{\sqrt{1+2\sqrt{L}}-1}{4} \geq 0 \implies L < \frac{1}{4} \tag{3.5.52}$$

Simplifying (3.2.43), we get

$$\frac{1-M(\rho_{AB})}{6} < -\frac{4S_{NL}(\rho_{AB})(2S_{NL}(\rho_{AB})+1)}{3} \tag{3.5.53}$$

Using (3.5.52) and (3.5.53) in the inequality (3.5.51), we get the required result. ■

3.6 Conclusion

To summarize, we have considered the problem of detection of non-locality of a given two-qubit state. It is now an accepted fact that non-locality and entanglement are two different concepts, and thus if a two-qubit state is entangled, then it is not necessary that it also depicts the non-local feature. Therefore, one can find many entangled states in the literature that may satisfy Bell's inequality. In the context of the detection of non-local property of a two-qubit entangled state, we consider a Bell game where the maximum probability P^{max} of winning the game is related to the expectation value of the Bell operator. We have defined the strength of non-locality S_{NL} in terms of P^{max} and, later on, re-expressed the expression of S_{NL} in terms of witness operator. First, we made a connection between the strength of the non-locality and the CHSH witness operator and then discussed the estimation of the non-locality of the given entangled state in both cases when (i) the CHSH witness operator detects the entangled state and (ii) CHSH witness operator does not detect the entangled state. Also, we construct an inequality that gives the upper bound of the strength of the non-locality in terms of the expectation value of the optimal witness operator with respect to the two-qubit entangled state. By doing this, we are able to detect the non-locality in the given two-qubit entangled state, which was undetected earlier by the Bell-CHSH operator. Furthermore, we also developed an interconnection between the strength of the non-locality of the two-qubit state and the expectation value of the Svetlichney operator with respect to a pure three-qubit state. This link paves the way to study the non-locality of a pure three-qubit state in terms of the non-locality of a two-qubit system.

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Chapter 4

State dependent bounds of the expectation value of the Svetlichny Operator

"When you change the way you look at things, the things you look at change."

- Max Planck

In this chapter [1]*, we study the problem of the detection of the genuine non-locality of any three-qubit state. It is known that the violation of Svetlichny inequality by any three-qubit state described by the density operator* ρ_{ABC} *witness the genuine non-locality of* ρ_{ABC}*. Further, it may be noted that the bounds of the Svetlichny inequality are state independent. However it is not an easy task to show the violation of Svetlichny inequality as the problem reduces to a complicated optimization problem. Thus, the detection of genuine non-locality of any three-qubit state may be considered a challenging task. Therefore, we have taken a different approach and derived the lower and upper bound of the expectation value of the Svetlichny operator with respect to any three-qubit state to study this problem. We have cited a few examples of three-qubit states whose non-locality was neither detected by the Svetlichny inequality nor by any other method but it is detected by the violation of the bounds obtained here. This is due to the fact that the obtained bounds are state dependent. The expression of the obtained bounds depends on whether the reduced two-qubit entangled state is detected by the CHSH witness operator or not. It may be expressed in*

[1]This chapter is based on a published research paper "Detection of the genuine non-locality of any three-qubit state, *Annals of Physics* **455**, 169400 (2023)".

terms of the following quantities such as (i) the eigenvalues of the product of the given three-qubit state and the composite system of single qubit maximally mixed state and reduced two-qubit state and (ii) the non-locality of reduced two-qubit state. Moreover, we also discuss its possible implementation in the laboratory.

4.1 Introduction

The correlation statistics between the subsystems obtained after performing a local measurement on the entangled system [1, 3] may be incompatible with the principle of local realism. Since Bell's inequality [17] has been derived using the principle of local realism so the generated correlation may violate Bell's inequality. This type of correlation may be called a non-local correlation [110, 233, 234]. The generalized form of Bell's inequality that may be realizable in an experiment was given by Clauser et. al. [48] and it is popularly known as Bell-CHSH inequality. Freedman and Clauser also have provided strong experimental evidence, using a generalized form of Bell's inequality, against the existence of local hidden-variable theories [49]. B. S. Cirelson [235] proved that quantum mechanics allow up to $2\sqrt{2}$ as an upper bound of generalized Bell's inequality. The upper bound of $2\sqrt{2}$ has been achieved by the two-qubit maximally entangled state. In 1982, A. Aspect et. al. [50] showed that maximum violation of generalized Bell's inequality can be achieved in an experiment. Later, Horodecki et. al. [97] also studied the problem of non-locality for two-qubit states and provided a criterion to check the non-locality of ρ_{AB} in terms of the quantity $M(\rho_{AB})$ which is given by (1.8.1). The criterion states that any two-qubit state violates Bell's inequality if and only if M($\rho_{AB}) > 1$.

The study of the non-locality of the multipartite system is a difficult problem but in spite of that, some progress has been achieved. In particular, the non-locality of the three-qubit system is relatively easier to handle. Non-locality of a three-qubit state can be tested by various inequalities such as Svetlichny inequality [86], Mermin inequality [62], and logical inequality based on GHZ type event probabilities [236]. The experimental verification of the non-locality of the three-qubit GHZ state is reported in [237]. The non-locality of a three-qubit pure symmetric state has been explored in [238]. The standard non-locality and genuine non-locality of GHZ symmetric state have been studied in [239].

Mermin inequality [62] can be violated by not only genuine entangled three-qubit states but also by biseparable states. Thus, the discrimination of the classes of three-qubit entangled states is not possible by merely observing the violation of Mermin inequality. But fortunately, there exists another inequality known as Svetlchny inequality [86], a violation of which guarantees the fact that the three-qubit state under investigation is a genuine entangled state. Therefore, the genuine tripartite non-local correlation that may exist in the three-qubit state ρ_{ABC} may be detected by Svetlichny inequality, which is given by [86]

$$|\langle S_v \rangle_{\rho_{ABC}}| \leq 4 \tag{4.1.1}$$

where S_v denote the Svetlichny operator, which may be defined as

$$\begin{aligned} S_v &= \vec{a}.\vec{\sigma}_1 \otimes [\vec{b}.\vec{\sigma}_2 \otimes (\vec{c}+\vec{c'}).\vec{\sigma}_3 + \vec{b'}.\vec{\sigma}_2 \otimes (\vec{c}-\vec{c'}).\vec{\sigma}_3] \\ &+ \vec{a'}.\vec{\sigma}_1 \otimes [\vec{b}.\vec{\sigma}_2 \otimes (\vec{c}-\vec{c'}).\vec{\sigma}_3 - \vec{b'}.\vec{\sigma}_2 \otimes (\vec{c}+\vec{c'}).\vec{\sigma}_3] \end{aligned} \tag{4.1.2}$$

Here $\vec{a},\vec{a'}$; $\vec{b},\vec{b'}$ and $\vec{c},\vec{c'}$ are the unit vectors and the $\vec{\sigma}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$ denote the spin projection operators. It may be noted that the bounds of the inequality (4.1.1) are state independent. The violation of Svetlichny inequality by three-qubit generalized GHZ state, maximal slice state, and W class state has been studied in [87, 88], and it has been found that the maximal violation $4\sqrt{2}$ may be obtained for GHZ state. The theoretical result of Ghose et. al. has been demonstrated experimentally in [240]. An operational method to detect the genuine multipartite non-locality for three-qubit mixed states has been investigated in [228]. Also, the genuine non-locality of three-qubit pure and mixed states has been extensively studied in [229].

In order to obtain the violation of the Svetlichny inequality, one has to calculate the expectation of the Svetlichny operator by maximizing overall measurements of spin in the directions $\vec{a},\vec{a'},\vec{b},\vec{b'},\vec{c},\vec{c'}$. Consequently, the problem of the violation of the Svetlichny inequality reduces to an optimization problem, which is not a very easy task to solve for any arbitrary three-qubit state. This motivates us to find a way by which we can overcome this problem. To achieve our task, we derive the upper and lower bound of the expectation value of the Svetlichny operator with respect to any three-qubit state. These newly obtained upper and lower bounds depend on the non-locality of the reduced two-qubit state of the three-qubit system and we have shown that this may pave the way to study the genuine non-locality of any three-qubit state.

4.2 Lower and upper bound of the expectation value of the Svetlichny operator

In this section, we construct the Hermitian operators to derive a connection between the two-qubit non-locality determined by the strength of the non-locality S_{NL} and the non-locality of an arbitrary (either pure or mixed) three-qubit state determined by the Svetlichny operator S_v. The construction of the Hermitian operator makes us enable to derive the lower and upper bound of the expectation value of the Svetlichny operator with respect to an arbitrary three-qubit state. The derived bound of the expectation value of the Svetlichny operator provides us with a new way to discriminate the genuine three-qubit entangled state.

To proceed forward, let us consider a three-qubit state (pure or mixed) described by the density operator ρ_{ABC} and its reduced two-qubit entangled state $\rho_{ij},\ i,j = A,B,C\ and\ i \neq j$, which can be related by the following way:

$$\rho_{ij} = Tr_k[\rho_{ABC}],\ i,j,k = A,B,C\ and\ i \neq j \neq k \tag{4.2.1}$$

The two operators may be constructed as

$$A_l = pS_v + (1-p)(I_2 \otimes W_{CHSH}) \tag{4.2.2}$$

$$B_l = \rho_{ABC}(I_2 \otimes \rho_{ij}),\ i,j = A,B,C,\ i \neq j \tag{4.2.3}$$

where $p \in [0,1]$ and $W_{CHSH}(= 2I_2 - B_{CHSH})$ denote the CHSH witness operator. I_2 denotes the identity matrix of order 2. Now, in the subsequent subsections, we derive the lower and upper bound of the expectation value of the Svetlichny operator in terms of two-qubit non-locality determined by $S_{NL}(\rho_{ij})$.

4.2.1 Lower bound of the expectation value of Svetlichny operator in terms of two-qubit non-locality determined by S_{NL}

To derive the lower bound of the expectation value of Svetlichny operator S_v, let us start with the quantity $R(Tr[A_lB_l])$. It can be expressed as

$$\begin{aligned} R(Tr[A_lB_l]) &= R(Tr[(pS_v + (1-p)(I_2 \otimes W_{CHSH})) \times \rho_{ABC}(I_2 \otimes \rho_{ij})]) \\ &= pR(Tr[S_v\rho_{ABC}(I_2 \otimes \rho_{ij})]) + (1-p) \times R(Tr[(I_2 \otimes \rho_{ij}W_{CHSH})\rho_{ABC}]) \end{aligned} \tag{4.2.4}$$

Since $(I_2 \otimes \rho_{ij})$ and ρ_{ABC} is a Hermitian operator, and $S_v\rho_{ABC}$ and $(I_2 \otimes \rho_{ij}W_{CHSH})$ are complex matrices so after applying Result 1.3 on (4.2.4), we get

$$R(Tr[S_v\rho_{ABC}(I_2\otimes\rho_{ij})])\leq\lambda_{max}(I_2\otimes\rho_{ij})Tr[\overline{S_v\rho_{ABC}}] \quad (4.2.5)$$

$$R(Tr[(I_2\otimes\rho_{ij}W_{CHSH})\rho_{ABC}])\leq Tr[\overline{I_2\otimes\rho_{ij}W_{CHSH}}]\lambda_{max}(\rho_{ABC}) \quad (4.2.6)$$

Using (4.2.5) and (4.2.6) in (4.2.4), we obtain

$$\begin{aligned} R(Tr[A_lB_l]) &= R(pTr[S_v\rho_{ABC}(I_2\otimes\rho_{ij})])+(1-p)\times R(Tr[(I_2\otimes\rho_{ij}W_{CHSH})\rho_{ABC}]) \\ &\leq p\lambda_{max}(I_2\otimes\rho_{ij})Tr[\overline{S_v\rho_{ABC}}]+(1-p)\lambda_{max}(\rho_{ABC})Tr[\overline{I_2\otimes\rho_{ij}W_{CHSH}}] \\ &= p\lambda_{max}(I_2\otimes\rho_{ij})\langle S_v\rangle_{\rho_{ABC}}+2(1-p)Tr[W_{CHSH}\rho_{ij}]\lambda_{max}(\rho_{ABC}) \quad (4.2.7) \end{aligned}$$

In the last step, one can easily check that $Tr[\overline{S_v\rho_{ABC}}]$= $Tr[S_v\rho_{ABC}]$, $Tr[\overline{I_2\otimes\rho_{ij}W_{CHSH}}]=Tr[I_2\otimes\rho_{ij}W_{CHSH}]$, and $Tr[I_2\otimes\rho_{ij}W_{CHSH}]=2Tr[W_{CHSH}\rho_{ij}]$.
Again applying LHS of Result 1.2 on Hermitian operator $pS_v+(1-p)(I_2\otimes W_{CHSH}))$, and $\rho_{ABC}(I_2\otimes\rho_{ij})$ be any complex matrix and using $Tr[S_v]=0$, we get

$$\begin{aligned} &R(Tr[(pS_v+(1-p)(I_2\otimes W_{CHSH}))\rho_{ABC}(I_2\otimes\rho_{ij})]) \\ &\geq Tr[pS_v+(1-p)(I_2\otimes W_{CHSH})]\lambda_{min}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})}) \\ &=8(1-p)\lambda_{min}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})}) \quad (4.2.8) \end{aligned}$$

In the second line of (4.2.8), we have used the linearity property of trace and $Tr(W_{CHSH})=4$, where $W_{CHSH}=2I-B_{CHSH}$. Combining the inequalities (4.2.7) and (4.2.8), we get

$$[p\lambda_{max}(I_2\otimes\rho_{ij})\langle S_v\rangle_{\rho_{ABC}}+2(1-p)Tr[W_{CHSH}\rho_{ij}]\lambda_{max}(\rho_{ABC})]\geq 8(1-p)\lambda_{min}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})}) \quad (4.2.9)$$

After simplification, the inequality (4.2.9) can be re-expressed as

$$\langle S_v\rangle_{\rho_{ABC}}\geq\frac{8(1-p)\lambda_{min}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})})}{p\lambda_{max}(I_2\otimes\rho_{ij})}-\frac{2(1-p)Tr[W_{CHSH}\rho_{ij}]\lambda_{max}(\rho_{ABC})}{p\lambda_{max}(I_2\otimes\rho_{ij})} \quad (4.2.10)$$

Since our aim is to establish the relationship between $\langle S_v\rangle_{\rho_{ABC}}$ and the strength of the non-locality $S_{NL}(\rho_{ij})$ of two-qubit entangled state ρ_{ij} so we shall consider two cases in which we discuss the following: (i) when ρ_{ij} is detected by the witness operator W_{CHSH} and (ii) when W_{CHSH} does not detect the state ρ_{ij}.

4.2.1.1 When the entangled state ρ_{ij} is detected by the witness operator W_{CHSH}

Let us recall the definition (3.2.14) of $S_{NL}(\rho_{ij})$ and can be re-expressed for the entangled state ρ_{ij} as

$$S_{NL}(\rho_{ij}) = \frac{-Tr[W_{CHSH}\rho_{ij}]}{8}$$

Putting this value of $S_{NL}(\rho_{ij})$ in (4.2.10), we get

$$\langle S_v \rangle_{\rho_{ABC}} \geq \frac{8(1-p)\lambda_{min}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})})}{p\lambda_{max}(I_2 \otimes \rho_{ij})} + \frac{16(1-p)\lambda_{max}(\rho_{ABC})S_{NL}(\rho_{ij})}{p\lambda_{max}(I_2 \otimes \rho_{ij})} \quad (4.2.11)$$

4.2.1.2 When W_{CHSH} does not detected the entangled state ρ_{ij}

In this case, $S_{NL}(\rho_{ij})$ is defined in a different way and it is given by (3.2.29)

$$S_{NL}^{New}(\rho_{ij}) = r(P^{max} - \frac{3}{4}) + (1-r)K, \quad i,j = A,B,C; i \neq j \quad (4.2.12)$$

where P^{max} is given in (3.2.6), r and K are given as

$$r < \frac{K}{\frac{3}{4} - P^{max} + K} \quad (4.2.13)$$

$$K = \frac{Tr[W_{CHSH}\rho_{ij}(\rho_{ij}^{T_j})]}{4N(\rho_{ij})} \quad (4.2.14)$$

T_j represent the partial transposition with respect to the qubit "j" and $N(\rho_{ij})$ denote the negativity of the two-qubit entangled state ρ_{ij}.
To derive the lower bound of $\langle S_v \rangle$ for this case, we need a lemma which can be stated as

Lemma 4.2.1. If an entangled state described by the density operator ρ_{ij} and the witness operator W_{CHSH} does not detect it then

$$K \geq \frac{\lambda_{min}[(\rho_{ij}^{T_j})^2]Tr[W_{CHSH}\rho_{ij}]}{4\lambda_{max}[\rho_{ij}^{T_j}]N(\rho_{ij})}, \quad i,j = A,B,C; \ i \neq j \quad (4.2.15)$$

where K is given by (4.2.14).

Proof: Let us start with the quantity $R(Tr[W_{CHSH}\rho_{ij}(\rho_{ij}^{T_j})^2])$. Applying LHS of Result 1.3 on

Hermitian operator $(\rho_{ij}^{T_j})^2$ and $W_{CHSH}\rho_{ij}$ be any complex matrix, we get

$$R(Tr[W_{CHSH}\rho_{ij}(\rho_{ij}^{T_j})^2]) \geq \lambda_{min}[(\rho_{ij}^{T_j})^2] \times Tr[\overline{W_{CHSH}\rho_{ij}}] \tag{4.2.16}$$

where, $Tr[\overline{W_{CHSH}\rho_{ij}}] = Tr[W_{CHSH}\rho_{ij}]$.
Again applying RHS of Result 1.3 on Hermitian operator $\rho_{ij}^{T_j}$ and $W_{CHSH}\rho_{ij}\rho_{ij}^{T_j}$ be any complex matrix, the quantity $R(Tr[W_{CHSH}\rho_{ij}(\rho_{ij}^{T_j})^2])$ can also be expressed as

$$\begin{aligned} R(Tr[W_{CHSH}\rho_{ij}(\rho_{ij}^{T_j})^2]) &\leq \lambda_{max}[\rho_{ij}^{T_j}]Tr[\overline{W_{CHSH}\rho_{ij}\rho_{ij}^{T_j}}] \\ &= 4\lambda_{max}[\rho_{ij}^{T_j}]N(\rho_{ij})K \end{aligned} \tag{4.2.17}$$

Since, $Tr[\overline{W_{CHSH}\rho_{ij}\rho_{ij}^{T_j}}] = Tr[W_{CHSH}\rho_{ij}\rho_{ij}^{T_j}]$. So, in the second line of (4.2.17), we have used the relation (4.2.14) i.e. $Tr[W_{CHSH}\rho_{ij}\rho_{ij}^{T_j}] = 4N(\rho_{ij})K$.
Using (4.2.16), the equation (4.2.17) can be re-expressed as

$$K \geq \frac{\lambda_{min}[(\rho_{ij}^{T_j})^2]Tr[W_{CHSH}\rho_{ij}]}{4\lambda_{max}[\rho_{ij}^{T_j}]N(\rho_{ij})} \tag{4.2.18}$$

■

Now we are in a position to establish the relationship between $S_{NL}^{New}(\rho_{ij})$ and $\langle S_v \rangle_{\rho_{ABC}}$ when the witness operator W_{CHSH} does not detect the entangled state ρ_{ij}.
Using (4.2.15), the expression for the strength of the non-locality $S_{NL}^{New}(\rho_{ij})$ given in (4.2.12) can be written as

$$S_{NL}^{New}(\rho_{ij}) \geq r(P^{max} - \frac{3}{4}) + (1-r)\frac{Tr[W_{CHSH}\rho_{ij}]\lambda_{min}[(\rho_{ij}^{T_j})^2]}{4\lambda_{max}[\rho_{ij}^{T_j}]N(\rho_{ij})} \tag{4.2.19}$$

The above inequality (4.2.19) may be re-expressed as

$$Tr[W_{CHSH}\rho_{ij}] \quad \leq \quad \frac{4Z[S_{NL}^{New}(\rho_{ij}) - r(P^{max} - \frac{3}{4})]}{(1-r)\lambda_{min}[(\rho_{ij}^{T_j})^2]} \tag{4.2.20}$$

where $Z = N(\rho_{ij})\lambda_{max}[\rho_{ij}^{T_j}]$.
Using the inequality (4.2.20) in (4.2.10), we get

$$\langle S_v \rangle_{\rho_{ABC}} \geq 8(1-p)[\frac{\lambda_{min}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})})}{p\lambda_{max}(I_2 \otimes \rho_{ij})} - G] \tag{4.2.21}$$

where $G = \frac{\lambda_{max}(\rho_{ABC})[S_{NL}^{New}(\rho_{ij})-r(P^{max}-\frac{3}{4})]N(\rho_{ij})\lambda_{max}[\rho_{ij}^{T_j}]}{p(1-r)\lambda_{min}[(\rho_{ij}^{T_j})^2]\lambda_{max}(I_2\otimes\rho_{ij})}$.

We are now in a position to collect all the above obtained results in the following theorem:

Theorem 4.2.1. The lower bound of the expectation value of the Svetlichny operator S_v with respect to three-qubit state ρ_{ABC} is given by

$$(i)\ \ \langle S_v\rangle_{\rho_{ABC}} \geq \frac{8(1-p)\lambda_{min}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})})}{p\lambda_{max}(I_2\otimes\rho_{ij})} + \frac{16(1-p)\lambda_{max}(\rho_{ABC})S_{NL}(\rho_{ij})}{p\lambda_{max}(I_2\otimes\rho_{ij})} \quad (4.2.22)$$

and

$$(ii)\ \ \langle S_v\rangle_{\rho_{ABC}} \geq 8(1-p)\left[\frac{\lambda_{min}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})})}{p\lambda_{max}(I_2\otimes\rho_{ij})} - \frac{\left(S_{NL}^{New}(\rho_{ij})-r(P^{max}-\frac{3}{4})\right)\times A_1}{p(1-r)\lambda_{min}[(\rho_{ij}^{T_j})^2]\lambda_{max}(I_2\otimes\rho_{ij})}\right] \quad (4.2.23)$$

where $A_1 = (N(\rho_{ij})\lambda_{max}(\rho_{ij}^{T_j})\lambda_{max}(\rho_{ABC}))$

The results (i) and (ii) can be applied according as when the entangled state ρ_{ij} does or does not detected by the witness operator W_{CHSH}. ■

4.2.2 Upper bound of the expectation value of Svetlichny operator in terms of two-qubit non-locality determined by S_{NL}

Let us consider two operators A_u and B_u, which may be defined as

$$A_u = qS_v + (1-q)(I_2\otimes W_{CHSH}),\ \ 0\leq q\leq 1 \quad (4.2.24)$$

$$B_u = \rho_{ABC}(I_2\otimes\rho_{ij}) \quad (4.2.25)$$

The expression for $R(Tr[A_uB_u])$ is given by

$$\begin{aligned}
& R(Tr[(qS_v+(1-q)(I_2\otimes W_{CHSH}))\rho_{ABC}(I_2\otimes\rho_{ij})]) \\
= \ & qR(Tr[S_v\rho_{ABC}(I_2\otimes\rho_{ij})]) + (1-q)R(Tr[(I_2\otimes W_{CHSH})\times\rho_{ABC}(I_2\otimes\rho_{ij})]) \\
= \ & qR(Tr[S_v\rho_{ABC}(I_2\otimes\rho_{ij})]) + (1-q)R(Tr[(I_2\otimes\rho_{ij}\times W_{CHSH})\rho_{ABC}]) \\
\geq \ & qR(Tr[S_v\rho_{ABC}(I_2\otimes\rho_{ij})]) + (1-q)Tr[\overline{(I_2\otimes\rho_{ij}W_{CHSH})}]\times\lambda_{min}(\rho_{ABC}) \\
= \ & qR(Tr[S_v\rho_{ABC}(I_2\otimes\rho_{ij})]) + 2(1-q)Tr[W_{CHSH}\rho_{ij}]\times\lambda_{min}(\rho_{ABC}) \quad (4.2.26)
\end{aligned}$$

In the second and third lines, we have used the linearity and cyclic property of the

trace. We have used the LHS inequality of Result 1.3 on Hermitian operator ρ_{ABC} and considering $(I_2 \otimes \rho_{ij} W_{CHSH})$ as any complex matrix in the fourth line. In the last line, we have used $Tr[I_2 \otimes \rho_{ij} W_{CHSH}] = Tr[\overline{I_2 \otimes \rho_{ij} W_{CHSH}}]$ and one of the properties of the trace i.e. $Tr[(I_2 \otimes \rho_{ij} W_{CHSH})]$=2$Tr[W_{CHSH}\rho_{ij}]$.
Applying RHS inequality of Result 1.2 on the Hermitian operator $qS_v + (1-q)(I_2 \otimes W_{CHSH})$ and considering $\rho_{ABC}(I_2 \otimes \rho_{ij})$ be any complex matrix, we get

$$\begin{aligned} & R(Tr[(qS_v + (1-q)(I_2 \otimes W_{CHSH}))\rho_{ABC}(I_2 \otimes \rho_{ij})]) \\ \leq \quad & Tr[qS_v + (1-q)(I_2 \otimes W_{CHSH})]\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) \\ = \quad & 8(1-q)\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) \end{aligned} \tag{4.2.27}$$

In the third line, we find $Tr[S_v] = 0$ and $Tr[I_2 \otimes W_{CHSH}] = 8$.
Combining (4.2.26) and (4.2.27), we get

$$\begin{aligned} & qR(Tr[S_v \rho_{ABC}(I_2 \otimes \rho_{ij})]) + 2(1-q)Tr[W_{CHSH}\rho_{ij}] \times \lambda_{min}(\rho_{ABC}) \\ \leq \quad & 8(1-q)\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) \end{aligned} \tag{4.2.28}$$

Again using Result 1.4 on Hermitian operators $I_2 \otimes \rho_{ij}$ and $S_v\rho_{ABC}$ be any complex matrix, we get

$$\begin{aligned} & Tr[\overline{S_v \rho_{ABC}}]\lambda_k((I_2 \otimes \rho_{ij})) \leq R(Tr[S_v \rho_{ABC}(I_2 \otimes \rho_{ij})]) \\ \Longrightarrow \quad & Tr[S_v \rho_{ABC}]\lambda_k((I_2 \otimes \rho_{ij})) \leq R(Tr[S_v \rho_{ABC}(I_2 \otimes \rho_{ij})]) \end{aligned} \tag{4.2.29}$$

where $Tr[\overline{S_v \rho_{ABC}}] = Tr[S_v \rho_{ABC}]$.
Using (4.2.29), the inequality (4.2.28) may be re-expressed as

$$\langle S_v \rangle_{\rho_{ABC}} \leq \frac{8(1-q)\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})})}{q\lambda_k(I_2 \otimes \rho_{ij})} - \frac{2(1-q)Tr[W_{CHSH}\rho_{ij}]\lambda_{min}(\rho_{ABC})}{q\lambda_k(I_2 \otimes \rho_{ij})} \tag{4.2.30}$$

where $\lambda_k(I_2 \otimes \rho_{ij})$ is the first non-zero eigenvalue of $(I_2 \otimes \rho_{ij})$.
The upper bound (4.2.30) of the expectation value of the operator S_v with respect to any three-qubit state ρ_{ABC} can be further studied in terms of the non-locality $S_{NL}(\rho_{ij})$ of two-qubit state by considering the following two cases: (i) when the state ρ_{ij} is detected by W_{CHSH} and (ii) when the state ρ_{ij} is not detected by W_{CHSH}.

4.2.2.1 When the state ρ_{ij} is detected by W_{CHSH}

In this case, we are considering the two-qubit entangled state ρ_{ij}, which is detected by the witness operator W_{CHSH}. Therefore, using the definition of $S_{NL}(\rho_{ij})$ given in (3.2.14), the inequality (4.2.30) reduces to

$$\langle S_v \rangle_{\rho_{ABC}} \leq \frac{8(1-q)\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})})}{q\lambda_k(I_2 \otimes \rho_{ij})} + \frac{16(1-q)S_{NL}(\rho_{ij})\lambda_{min}(\rho_{ABC})}{q\lambda_k(I_2 \otimes \rho_{ij})} \tag{4.2.31}$$

4.2.2.2 When ρ_{ij} is not detected by W_{CHSH}

When the entangled state ρ_{ij} is not detected by W_{CHSH}, the expression of the strength of the non-locality is given by $S_{NL}^{New}(\rho_{ij})$. Therefore, we can re-write (3.2.29) for the entangled state ρ_{ij} as

$$S_{NL}^{New}(\rho_{ij}) = r(P^{max} - \frac{3}{4}) + (1-r)K, \ 0 \leq r \leq 1 \tag{4.2.32}$$

where P^{max}, r and K are given by (3.2.6), (4.2.13), and (4.2.14). Now, our task is to find out the upper bound of K, which is given by the following lemma.

Lemma 4.2.2. If ρ_{ABC} denote an arbitrary three-qubit state and $\rho_{ij}, i,j = A,B,C, i \neq j$ be its reduced two-qubit entangled state, which is not detected by CHSH witness operator W_{CHSH} then the non-locality of ρ_{ij} may be determined by $S_{NL}^{New}(\rho_{ij})$ given in (3.2.29). The quantity K involved in the expression of $S_{NL}^{New}(\rho_{ij})$ is bounded above and its upper bound is given by

$$K \leq \frac{\lambda_{max}(W_{CHSH})\lambda_{max}(\rho_{ij})Tr[W_{CHSH}\rho_{ij}] + Tr[(\rho_{ij}^{T_j})^2]}{8N(\rho_{ij})} \tag{4.2.33}$$

Proof: Let us consider the two operators given by

$$A_2 = W_{CHSH}\rho_{ij}, \ \ B_2 = \rho_{ij}^{T_j} \tag{4.2.34}$$

For the two operators A_2 and B_2 defined in (4.2.34), we have

$$\begin{aligned} &(A_2 - B_2)^2 \geq 0 \\ &\implies A_2^2 - A_2B_2 - B_2A_2 + B_2^2 \geq 0 \end{aligned} \tag{4.2.35}$$

Taking trace both sides of (4.2.35) and simplifying, we get

$$2Tr(A_2B_2) \leq Tr(A_2^2) + Tr(B_2^2) \tag{4.2.36}$$

Using (4.2.34) and (4.2.36), we get

$$2Tr(W_{CHSH}\rho_{ij}\rho_{ij}^{T_j}) \leq Tr((W_{CHSH}\rho_{ij})^2) + Tr((\rho_{ij}^{T_j})^2) \tag{4.2.37}$$

Also, applying Result 1.3 on Hermitian operator W_{CHSH} and considering $W_{CHSH}(\rho_{ij})^2$ be any complex matrix, and using the fact that $Tr[W_{CHSH}(\rho_{ij})^2] = Tr[\overline{W_{CHSH}(\rho_{ij})^2}]$ we get

$$Tr((W_{CHSH}\rho_{ij})^2) \leq \lambda_{max}(W_{CHSH}) \times Tr[W_{CHSH}(\rho_{ij})^2] \tag{4.2.38}$$

Again applying Result 1.3 on Hermitian operator ρ_{ij} and $W_{CHSH}\rho_{ij}$ be any complex matrix, and using the fact that $Tr[W_{CHSH}\rho_{ij}] = Tr[\overline{W_{CHSH}\rho_{ij}}]$

$$Tr[W_{CHSH}(\rho_{ij})^2] \quad \leq \quad \lambda_{max}(\rho_{ij})Tr[W_{CHSH}\rho_{ij}] \tag{4.2.39}$$

Using (4.2.38) and (4.2.39), we get

$$Tr((W_{CHSH}\rho_{ij})^2) \leq \lambda_{max}(W_{CHSH}) \times \lambda_{max}(\rho_{ij})Tr[W_{CHSH}\rho_{ij}] \tag{4.2.40}$$

Using (4.2.37) and (4.2.40), we get

$$2Tr(W_{CHSH}\rho_{ij}\rho_{ij}^{T_j}) \leq \lambda_{max}(W_{CHSH})\lambda_{max}(\rho_{ij}) \times Tr[W_{CHSH}\rho_{ij}] + Tr[(\rho_{ij}^{T_j})^2] \tag{4.2.41}$$

Putting $Tr[W_{CHSH}\rho_{ij}(\rho_{ij})^{T_j}] = 4N(\rho_{ij})K$ in (4.2.41), we get

$$K \leq \frac{\lambda_{max}(W_{CHSH})\lambda_{max}(\rho_{ij})Tr[W_{CHSH}\rho_{ij}] + Tr[(\rho_{ij}^{T_j})^2]}{8N(\rho_{ij})} \qquad \blacksquare$$

Now, we are in a position to estimate $Tr(W_{CHSH}\rho_{ij})$. Using (4.2.33) in (4.2.32), $Tr(W_{CHSH}\rho_{ij})$ may be estimated as

$$Tr[W_{CHSH}\rho_{ij}] \geq \frac{1}{\lambda_{max}(W_{CHSH})\lambda_{max}(\rho_{ij})} \times [\frac{8N(\rho_{ij})(S_{NL}^{New}(\rho_{ij}) - r(P^{max} - \frac{3}{4}))}{1-r} - Tr[(\rho_{ij}^{T_j})^2]] \tag{4.2.42}$$

Using (4.2.42), the inequality (4.2.30) for the upper bound of $\langle S_v \rangle$ reduces to

$$\langle S_v\rangle_{\rho_{ABC}} \leq \frac{2(1-q)}{q\lambda_k(I_2\otimes\rho_{ij})}\left[4\lambda_{max}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})})-\frac{\lambda_{min}(\rho_{ABC})\times A_2}{\lambda_{max}(W_{CHSH})\lambda_{max}(\rho_{ij})}\right] \quad (4.2.43)$$

where $A_2 = \frac{8N(\rho_{ij})(S_{NL}^{New}(\rho_{ij})-r(P^{max}-\frac{3}{4}))}{1-r} - Tr[(\rho_{ij}^{T_j})^2]$.

The results given by (4.2.31) and (4.2.43) can be collectively given by the following theorem:

Theorem 4.2.2. The upper bound of the expectation value of the Svetlichny operator S_v with respect to any three-qubit state ρ_{ABC} can be expressed in terms of $S_{NL}(\rho_{ij})$ and $S_{NL}^{New}(\rho_{ij})$ as

$$(i)\ \ \langle S_v\rangle_{\rho_{ABC}} \leq \frac{8(1-q)\lambda_{max}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})})}{q\lambda_k(I_2\otimes\rho_{ij})}+\frac{16(1-q)S_{NL}(\rho_{ij})\lambda_{min}(\rho_{ABC})}{q\lambda_k(I_2\otimes\rho_{ij})} \quad (4.2.44)$$

and

$$(ii)\ \ \langle S_v\rangle_{\rho_{ABC}} \leq \frac{2(1-q)}{q\lambda_k(I_2\otimes\rho_{ij})}\left[4\lambda_{max}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})})-\frac{\lambda_{min}(\rho_{ABC})\times A_2}{\lambda_{max}(W_{CHSH})\lambda_{max}(\rho_{ij})}\right] \quad (4.2.45)$$

where $A_2 = \frac{8N(\rho_{ij})(S_{NL}^{New}(\rho_{ij})-r(P^{max}-\frac{3}{4}))}{1-r} - Tr[(\rho_{ij}^{T_j})^2]$.

The results given in (4.2.44) and (4.2.45) holds according as when the entangled state ρ_{ij} does or does not detected by the witness operator W_{CHSH}. ■

4.3 Detection of genuine three-qubit non-local states

In this section, we will derive conditions to identify whether the given three-qubit state (pure or mixed) is a genuine non-local state. We will use the Svetlichny inequality and the lower and upper bound given in Theorem 4.3.1 and Theorem 4.3.2 stated in the previous section, to derive much simpler conditions for the detection of genuine non-locality of the three-qubit state. We will show that the genuine non-locality of the three-qubit state depends on the non-locality of the two-qubit reduced entangled state. The non-locality of two-qubit reduced entangled state ρ_{ij} may be determined by $S_{NL}(\rho_{ij})$ or $S_{NL}^{New}(\rho_{ij})$ accordingly the entangled state ρ_{ij} detected or not detected by the CHSH witness operator W_{CHSH}.

4.3.1 When ρ_{ij} is detected by the witness operator W_{CHSH}

In this section, we will derive the condition of non-locality of the three-qubit state described by the density operator ρ_{ABC} when its reduced two-qubit entangled state ρ_{ij} is detected by the witness operator W_{CHSH}.

Theorem 4.3.1. If any three-qubit state (either pure or mixed) satisfies the Svetlichny inequality and if the reduced two-qubit state of it is detected by the CHSH witness operator then the operators A_l and B_l given in (4.2.2) and (4.2.3) must be chosen in such a way that the parameter p given by (4.2.2) satisfies the following inequality

(i) If $\lambda_{min}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) + 2S_{NL}(\rho_{ij})\lambda_{max}(\rho_{ABC}) > 0$, then

$$0 \leq p \leq u_1, \text{when } d_1^{(-)} > 0 \tag{4.3.1}$$

OR,

$$l_1 \leq p \leq 1, \text{when } d_1^{(+)} > 0 \tag{4.3.2}$$

(ii) If $\lambda_{min}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) + 2S_{NL}(\rho_{ij})\lambda_{max}(\rho_{ABC}) < 0$, then

$$u_1 \leq p \leq 1, \text{when } d_1^{(-)} < 0 \tag{4.3.3}$$

OR,

$$0 \leq p \leq l_1, \text{when } d_1^{(+)} < 0 \tag{4.3.4}$$

The lower bound l_1 and upper bound u_1 are given by

$$l_1 = \frac{2}{d_1^{(+)}} \times [\lambda_{min}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) + 2S_{NL}(\rho_{ij}) \times \lambda_{max}(\rho_{ABC})] \tag{4.3.5}$$

$$u_1 = \frac{2}{d_1^{(-)}} \times [\lambda_{min}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) + 2S_{NL}(\rho_{ij}) \times \lambda_{max}(\rho_{ABC})] \tag{4.3.6}$$

where $d_1^{(+)} = 2[\lambda_{min}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) + 2S_{NL}(\rho_{ij})\lambda_{max}(\rho_{ABC})] + \lambda_{max}(I_2 \otimes \rho_{ij})$ and
$d_1^{(-)} = 2[\lambda_{min}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) + 2S_{NL}(\rho_{ij})\lambda_{max}(\rho_{ABC})] - \lambda_{max}(I_2 \otimes \rho_{ij})$.

Proof: Let us consider a three-qubit state ρ_{ABC} which satisfies the Svetlichny inequality. Therefore, we have

$$-4 \leq \langle S_v \rangle_{\rho_{ABC}} \leq 4 \tag{4.3.7}$$

Now, if a three-qubit state ρ_{ABC} satisfies the Svetlichny inequality then our task is to construct the operator A_l. To accomplish this task, we need to specify the parameter p. Thus, recalling the lower bound of the expectation value of the Svetlichny operator S_v given in (4.2.11) and using (4.3.7), the restriction on p may be obtained by solving the inequality

$$-4 \leq \frac{8(1-p)\lambda_{min}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})})}{p\lambda_{max}(I_2 \otimes \rho_{ij})} + \frac{16(1-p)\lambda_{max}(\rho_{ABC})S_{NL}(\rho_{ij})}{p\lambda_{max}(I_2 \otimes \rho_{ij})} \leq 4 \tag{4.3.8}$$

Solving the inequality (4.3.8) for the parameter p while considering all the cases when $\lambda_{min}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})})+2S_{NL}(\rho_{ij})\lambda_{max}(\rho_{ABC})>0$, and $\lambda_{min}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})})+2S_{NL}(\rho_{ij})\lambda_{max}(\rho_{ABC})<0$, we get the required result. ■

Corollary 4.3.1. Let us define the quantity $U_n^{(1)}=\sqrt{2}[\lambda_{min}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})})+2S_{NL}(\rho_{ij})\lambda_{max}(\rho_{ABC})]$, $U_-^{(1)}=\sqrt{2}[\lambda_{min}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})})+2S_{NL}(\rho_{ij})\lambda_{max}(\rho_{ABC})]-\lambda_{max}(I_2\otimes\rho_{ij})$, and $U_+^{(1)}=\sqrt{2}[\lambda_{min}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})})+2S_{NL}(\rho_{ij})\lambda_{max}(\rho_{ABC})]+\lambda_{max}(I_2\otimes\rho_{ij})$. If the parameter p violate (4.3.2) and (4.3.3) for some three-qubit (pure or mixed) state ρ_{ABC} i.e. if it satisfies the inequality

$$\frac{U_n^{(1)}}{U_+^{(1)}}<p<l_1 \tag{4.3.9}$$

when, $\lambda_{min}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})})+2S_{NL}(\rho_{ij})\lambda_{max}(\rho_{ABC})>0$

OR,
$$\frac{U_n^{(1)}}{U_-^{(1)}}<p<u_1 \tag{4.3.10}$$

when $\lambda_{min}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})})+2S_{NL}(\rho_{ij})\lambda_{max}(\rho_{ABC})<0$,
then the state ρ_{ABC} violates the Svetlichny inequality and thus exhibits the genuine non-locality. ■

Note 1: We should note here that the expression of $\frac{U_n^{(1)}}{U_+^{(1)}}$ and $\frac{U_n^{(1)}}{U_-^{(1)}}$ has been obtained by using the upper limit of $\langle S_v\rangle_{\rho_{ABC}}$ i.e. $\langle S_v\rangle_{\rho_{ABC}}\leq 4\sqrt{2}$.

Theorem 4.3.2. If any three-qubit state (either pure or mixed) satisfies the Svetlichny inequality and if the reduced two-qubit state of it is detected by the CHSH witness operator then the operators A_u and B_u given in (4.2.24) and (4.2.25) must be chosen in such a way that the parameter q given by (4.2.24) satisfies the inequality

$$l_2\leq q\leq 1 \tag{4.3.11}$$

The lower bound l_2 is given by

$$l_2=\frac{2}{d_2^{(+)}}\times[\lambda_{max}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})})+2S_{NL}(\rho_{ij})\lambda_{min}(\rho_{ABC})] \tag{4.3.12}$$

where $d_2^{(+)}=2[\lambda_{max}(\overline{\rho_{ABC}(I_2\otimes\rho_{ij})})+2S_{NL}(\rho_{ij})\lambda_{min}(\rho_{ABC})]+\lambda_k(I_2\otimes\rho_{ij})$.

Proof: Let us consider a three-qubit state ρ_{ABC} which satisfies the Svetlichny inequality given by (4.3.7). Now, if a three-qubit state ρ_{ABC} satisfies the Svetlichny inequality then our task is to construct the operator A_u. To accomplish this task, we need to specify the parameter q. Thus, recalling the upper bound of the expectation value of the Svetlichny operator S_v given in (4.2.31) and using (4.3.7), the restriction on q may be obtained by solving the inequality

$$-4 \leq \frac{8(1-q)\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})})}{q\lambda_k(I_2 \otimes \rho_{ij})} + \frac{16(1-q)S_{NL}(\rho_{ij})\lambda_{min}(\rho_{ABC})}{q\lambda_k(I_2 \otimes \rho_{ij})} \leq 4 \qquad (4.3.13)$$

Solving the L.H.S. of inequality (4.3.13) for the parameter q and simplifying, we get

$$q \geq l_2 = \frac{2}{d_2^{(+)}} \times [\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) + 2S_{NL}(\rho_{ij})\lambda_{min}(\rho_{ABC})]$$

where $d_2^{(+)} = 2[\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) + 2S_{NL}(\rho_{ij})\lambda_{min}(\rho_{ABC})] + \lambda_k(I_2 \otimes \rho_{ij})$.
Then, by solving the L.H.S. of the inequality (4.3.13), we get $q \geq 1$ which is not possible. Thus, considering $q \leq min\{\frac{2}{d_2^{(-)}} \times [\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) + 2S_{NL}(\rho_{ij})\lambda_{min}(\rho_{ABC})], 1\}$, where $d_2^{(-)} = 2[\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) + 2S_{NL}(\rho_{ij})\lambda_{min}(\rho_{ABC})] - \lambda_k(I_2 \otimes \rho_{ij})$, we get the required result. Hence proved. ■

Corollary 4.3.2. Let us define the quantity $U_n^{(2)} = \sqrt{2}[\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) + 2S_{NL}(\rho_{ij})\lambda_{min}(\rho_{ABC})]$ and $U_d^{(2)} = \sqrt{2}[\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) + 2S_{NL}(\rho_{ij})\lambda_{min}(\rho_{ABC})] + \lambda_k(I_2 \otimes \rho_{ij})$. If the parameter q violates the inequality given in (4.3.11) for some three-qubit (pure or mixed) state ρ_{ABC} i.e. if it satisfies the inequality

$$U^{(2)} \equiv \frac{U_n^{(2)}}{U_d^{(2)}} < q < l_2 \qquad (4.3.14)$$

then the state ρ_{ABC} violates the Svetlichny inequality and thus exhibits the genuine nonlocality. ■

Result 4.3.1. If any three-qubit state (either pure or mixed) satisfies the Svetlichny inequality then the Svetlichny operator also satisfies the inequality

$$S_v^{(1)} \leq \langle S_v \rangle_{\rho_{ABC}} \leq S_v^{(2)} \qquad (4.3.15)$$

where $S_v^{(1)}$ and $S_v^{(2)}$ are given by

$$S_v^{(1)} = \frac{8(1-p)}{p\lambda_{max}(I_2 \otimes \rho_{ij})}[\lambda_{min}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) + 2\lambda_{max}(\rho_{ABC})S_{NL}(\rho_{ij})] \qquad (4.3.16)$$

$$S_v^{(2)} = \frac{8(1-q)}{q\lambda_k(I_2 \otimes \rho_{ij})}[\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) + 2\lambda_{min}(\rho_{ABC})S_{NL}(\rho_{ij})] \tag{4.3.17}$$

The two parameters p and q satisfies the inequality (4.3.1), (4.3.2), (4.3.3), (4.3.4) and (4.3.11).

■

Corollary 4.3.3. If any three-qubit state (either pure or mixed) violates the inequality (4.3.15) and if p and q satisfy the inequality (4.3.9), (4.3.10) and (4.3.14) then the given three-qubit state exhibit genuine non-locality. In other words, for any three-qubit state (either pure or mixed) described by the density operator ρ_{ABC} if

$$\langle S_v \rangle_{\rho_{ABC}} < S_v^{(1)}, \ \ \langle S_v \rangle_{\rho_{ABC}} > S_v^{(2)} \tag{4.3.18}$$

then ρ_{ABC} exhibit genuine non-locality. ■

4.3.2 When ρ_{ij} is not detected by the witness operator W_{CHSH}

In this section, we will derive the condition of the non-locality of the three-qubit state described by the density operator ρ_{ABC} when its reduced two-qubit entangled state ρ_{ij} is not detected by the Witness operator W_{CHSH}.

Theorem 4.3.3. If any three-qubit state (either pure or mixed) satisfies the Svetlichny inequality and if the reduced two-qubit state of it is not detected by the CHSH witness operator then the operators A_l and B_l given in (4.2.2) and (4.2.3) must be chosen in such a way that the parameter p given by (4.2.2) satisfies the following inequality

$$l_3 \leq p \leq 1 \tag{4.3.19}$$

The bound l_3 is given by

$$l_3 = \frac{2H}{2H - \lambda_{max}(I_2 \otimes \rho_{ij})} \tag{4.3.20}$$

where $H = \lambda_{min}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) - (\frac{S_{NL}^{New}(\rho_{ij}) - r(P^{max} - \frac{3}{4})}{(1-r)\lambda_{min}[(\rho_{ij}^{T_j})^2]}) \times (N(\rho_{ij})\lambda_{max}(\rho_{ij}^{T_j})\lambda_{max}(\rho_{ABC}))$.

Proof: Let us consider a three-qubit state ρ_{ABC} which satisfies the Svetlichny inequality (4.3.7). Now, if a three-qubit state ρ_{ABC} satisfies the Svetlichny inequality then our task is to construct the operator A_l. To accomplish this task, we need to specify the parameter p. Thus, recalling the lower bound of the expectation value of the Svetlichny operator S_v given in

(4.2.23) and using (4.3.7), the restriction on p may be obtained by solving the inequality

$$-4 \leq 8(1-p)\left[\frac{\lambda_{min}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})})}{p\lambda_{max}(I_2 \otimes \rho_{ij})} - \frac{\left(S_{NL}^{New}(\rho_{ij}) - r(P^{max} - \frac{3}{4}) \times A_1\right)}{p(1-r)\lambda_{min}[(\rho_{ij}^{T_j})^2]\lambda_{max}(I_2 \otimes \rho_{ij})}\right] \leq 4 \quad (4.3.21)$$

where $A_1 = N(\rho_{ij})\lambda_{max}(\rho_{ij}^{T_j})\lambda_{max}(\rho_{ABC})$.
Solving the L.H.S. of inequality (4.3.21) for the parameter p and simplifying, we get

$$p \geq l_3 = \frac{2H}{2H - \lambda_{max}(I_2 \otimes \rho_{ij})} \quad (4.3.22)$$

where $H = \lambda_{min}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) - (\frac{S_{NL}^{New}(\rho_{ij}) - r(P^{max} - \frac{3}{4})}{(1-r)\lambda_{min}[(\rho_{ij}^{T_j})^2]}) \times (N(\rho_{ij})\lambda_{max}(\rho_{ij}^{T_j})\lambda_{max}(\rho_{ABC}))$.
Then, by solving the R.H.S. of the inequality (4.3.21), we get $p \geq 1$ which is not possible. Thus, considering $p \leq min\{\frac{2H}{2H - \lambda_{max}(I_2 \otimes \rho_{ij})}, 1\}$, we get the required result. ■

Corollary 4.3.4. If the parameter p violate the inequality given in (4.3.19) for some three-qubit (pure or mixed) state ρ_{ABC} i.e. if p satisfies the inequality

$$U^{(3)} \equiv \frac{\sqrt{2}H}{\sqrt{2}H - \lambda_{max}(I_2 \otimes \rho_{ij})} < p < l_3 \quad (4.3.23)$$

then the three-qubit state ρ_{ABC} violates the Svetlichny inequality and thus exhibits the genuine non-locality. ■

Theorem 4.3.4. If any three-qubit state (either pure or mixed) satisfies the Svetlichny inequality and if the reduced two-qubit state of it is not detected by the CHSH witness operator then the operators A_u and B_u given in (4.2.24) and (4.2.25) must be chosen in such a way that the parameter q given by (4.2.24) satisfies the inequality

$$l_4 \leq q \leq 1 \quad (4.3.24)$$

The lower bound l_4 is given by

$$l_4 = \frac{F}{F + 2\lambda_k(I_2 \otimes \rho_{ij})} \quad (4.3.25)$$

where $F = (4\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) - \frac{\lambda_{min}(\rho_{ABC})}{\lambda_{max}(W_{CHSH})\lambda_{max}(\rho_{ij})}(\frac{8N(\rho_{ij})(S_{NL}^{New}(\rho_{ij}) - r(P^{max} - \frac{3}{4}))}{1-r} - Tr[(\rho_{ij}^{T_j})^2]))$.

Proof: Let us consider a three-qubit state ρ_{ABC} which satisfies the Svetlichny inequality (4.3.7). Now, if a three-qubit state ρ_{ABC} satisfies the Svetlichny inequality then our task is to construct the operator A_u. To accomplish this task, we need to specify the parameter q. Thus, recalling the upper bound of the expectation value of the Svetlichny operator S_v given in

(4.2.43) and using (4.3.7), the restriction on q may be obtained by solving the inequality

$$-4 \leq \frac{2(1-q)}{q\lambda_k(I_2 \otimes \rho_{ij})}\left[4\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) - \frac{\lambda_{min}(\rho_{ABC}) \times A_2}{\lambda_{max}(W_{CHSH})\lambda_{max}(\rho_{ij})}\right] \leq 4 \quad (4.3.26)$$

where $A_2 = \frac{8N(\rho_{ij})(S_{NL}^{New}(\rho_{ij})-r(P^{max}-\frac{3}{4}))}{1-r} - Tr[(\rho_{ij}^{T_j})^2]$.

Solving the R.H.S. of inequality (4.3.26) for the parameter q and simplifying, we get

$$q \geq l_4 = \frac{F}{F + 2\lambda_k(I_2 \otimes \rho_{ij})} \quad (4.3.27)$$

where $F = (4\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) - \frac{\lambda_{min}(\rho_{ABC})}{\lambda_{max}(W_{CHSH})\lambda_{max}(\rho_{ij})}(\frac{8N(\rho_{ij})(S_{NL}^{New}(\rho_{ij})-r(P^{max}-\frac{3}{4}))}{1-r} - Tr[(\rho_{ij}^{T_j})^2]))$.

Then, by solving the L.H.S. of the inequality (4.3.26), we get $q \geq 1$ which is not possible. Thus, considering $q \leq min\{\frac{F}{F-2\lambda_k(I_2 \otimes \rho_{ij})}, 1\}$, we get the required result. ■

Corollary 4.3.5. If the parameter q violates the inequality given in (4.3.24) for some three-qubit (pure or mixed) state ρ_{ABC} i.e. if it satisfies the inequality

$$U^{(4)} \equiv \frac{F}{F + 2\sqrt{2}\lambda_k(I_2 \otimes \rho_{ij})} < q < l_4 \quad (4.3.28)$$

then the three-qubit state violates the Svetlichny inequality and thus exhibits the genuine non-locality. ■

Result 4.3.2. If any three-qubit state (either pure or mixed) satisfies the Svetlichny inequality and if p and q are given by (4.3.19) and (4.3.24) then the Svetlichny operator also satisfies the inequality

$$S_v^{(3)} \leq \langle S_v \rangle_{\rho_{ABC}} \leq S_v^{(4)} \quad (4.3.29)$$

where $S_v^{(3)}$ and $S_v^{(4)}$ are given by

$$S_v^{(3)} = \frac{8(1-p)}{p\lambda_{max}(I_2 \otimes \rho_{ij})}\left[\lambda_{min}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) - \frac{\left(S_{NL}^{New}(\rho_{ij}) - r(P^{max} - \frac{3}{4}) \times A_1\right)}{(1-r)\lambda_{min}[(\rho_{ij}^{T_j})^2]}\right] \quad (4.3.30)$$

$$S_v^{(4)} = \frac{2(1-q)}{q\lambda_k(I_2 \otimes \rho_{ij})}\left[4\lambda_{max}(\overline{\rho_{ABC}(I_2 \otimes \rho_{ij})}) - \frac{\lambda_{min}(\rho_{ABC}) \times A_2}{\lambda_{max}(W_{CHSH})\lambda_{max}(\rho_{ij})}\right] \quad (4.3.31)$$

where $A_1 = N(\rho_{ij})\lambda_{max}(\rho_{ij}^{T_j})\lambda_{max}(\rho_{ABC})$ and $A_2 = \frac{8N(\rho_{ij})(S_{NL}^{New}(\rho_{ij})-r(P^{max}-\frac{3}{4}))}{1-r} - Tr[(\rho_{ij}^{T_j})^2]$. ■

Corollary 4.3.6. If any three-qubit state (either pure or mixed) violates the inequality (4.3.29) and if p and q satisfy the inequality given by (4.3.23) and (4.3.28) then the given three-qubit state exhibit genuine non-locality. In other words, for any three-qubit state (either pure or mixed) described by the density operator ρ_{ABC} if

$$\langle S_v \rangle_{\rho_{ABC}} < S_v^{(3)}, \ \ \langle S_v \rangle_{\rho_{ABC}} > S_v^{(4)} \tag{4.3.32}$$

then ρ_{ABC} exhibit genuine non-locality. ■

4.4 Illustrations

We are now in a position to illustrate our scheme of detecting the genuine non-locality of a given three-qubit state (pure or mixed) with a few examples.

4.4.1 Examples of three-qubit states for which the reduced two-qubit state detected by the CHSH witness operator

In this section, we will illustrate our results given in (4.3.18) with the help of the following two examples of three-qubit states for which the reduced two-qubit state is detected by the CHSH witness operator. Examples of three-qubit states fall under the following two categories: (i) A pure three-qubit state belongs to the W class and (ii) A mixed three-qubit state which may be taken as a convex combination of GHZ state and two other states belongs to W class.

Example 4.1: A pure three-qubit W class of state

Let us consider a pure three-qubit state of the form

$$|\psi^{(1)}\rangle_{ABC} = \lambda_0|000\rangle + 0.3|101\rangle + \sqrt{0.91-\lambda_0^2}|110\rangle, \ \lambda_0 \in [0, 0.953939] \tag{4.4.1}$$

The pure state described by the density operator $\rho_{ABC}^{(1)} = |\psi^{(1)}\rangle_{ABC}\langle\psi^{(1)}|$ is an entangled state and also we have

$$\lambda_{max}(\rho_{ABC}^{(1)}) = 1, \ \ \lambda_{min}(\rho_{ABC}^{(1)}) = 0 \tag{4.4.2}$$

Tracing out system B from the three-qubit state $\rho_{ABC}^{(1)}$, the reduced state $\rho_{AC}^{(1)}$ is given by

$$\rho_{AC}^{(1)} = \begin{pmatrix} \lambda_0^2 & 0 & 0 & 0.3\lambda_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.91-\lambda_0^2 & 0 \\ 0.3\lambda_0 & 0 & 0 & 0.09 \end{pmatrix} \tag{4.4.3}$$

The state $\rho_{AC}^{(1)}$ is an entangled state as there exist a witness operator $W_{CHSH}^{(xz)}(=2I - B_{CHSH}^{(xz)})$ that detect it. The CHSH witness operator $B_{CHSH}^{(xz)}$ is given by (3.2.7). This is clear from the following fact

$$\begin{aligned} Tr[W_{CHSH}^{(xz)}\rho_{AC}^{(1)}] &= 3.15966 - 0.848528\lambda_0 - 2.82843\lambda_0^2 \\ &< 0, \text{ for } \lambda_0 \in [0.91753, 0.953939] \end{aligned} \tag{4.4.4}$$

Since the two-qubit state $\rho_{AC}^{(1)}$ is an entangled state and it is detected by $W_{CHSH}^{(xz)}$ so the strength of its non-locality may be measured by $S_{NL}(\rho_{AC}^{(1)})$. It is then given by

$$S_{NL}(\rho_{AC}^{(1)}) = \frac{-Tr[W_{CHSH}^{(xz)}\rho_{AC}^{(1)}]}{8} \in \ [0, 0.030], \text{ for } \lambda_0 \in [0.917, 0.953] \tag{4.4.5}$$

Further, we can calculate the following using the three-qubit state $\rho_{ABC}^{(1)}$ and the reduced two-qubit state $\rho_{AC}^{(1)}$

$$\begin{aligned} &\lambda_{max}(I_2 \otimes \rho_{AC}^{(1)}) = 0.09 + \lambda_0^2, \ \ \lambda_k(I_2 \otimes \rho_{AC}^{(1)}) = 0.91 - \lambda_0^2 \\ &\lambda_{min}(\overline{\rho_{ABC}^{(1)}(I_2 \otimes \rho_{AC}^{(1)})}) = \frac{\lambda_0^4 - \lambda_0^3}{2} + \frac{9(\lambda_0^2 - \lambda_0)}{200} \\ &\lambda_{max}(\overline{\rho_{ABC}^{(1)}(I_2 \otimes \rho_{AC}^{(1)})}) = \frac{\lambda_0^4 + \lambda_0^3}{2} + \frac{9(\lambda_0^2 + \lambda_0)}{200} \end{aligned} \tag{4.4.6}$$

Also, the range of p and q are given by

$$0 < p < 0.07 \tag{4.4.7}$$

$$0.93 < q < 1 \tag{4.4.8}$$

Using the information given in (4.4.2), (4.4.5), (4.4.6), (4.4.7), and (4.4.8), the value of the expression of $S_v^{(1)}$ and $S_v^{(2)}$ can be calculated for the three-qubit state $\rho_{ABC}^{(1)}$ and they are tabulated in the Table 4.1.

State parameter (λ_0)	Operator parameter (p)	Operator parameter (q)	$\langle S_v^{(1)}\rangle_{\rho_{ABC}^{(1)}}$	$\langle S_v^{(2)}\rangle_{\rho_{ABC}^{(1)}}$
0.92	0.05	0.95	-4.98604	5.4752
0.93	0.019	0.97	-5.2497	4.70144
0.94	0.012	0.98	4.62363	5.48997
0.95	0.04	0.9943	5.06912	5.62141

Table 4.1: We have chosen different values of the three-qubit state parameter λ_0 for which its reduced two-qubit state is entangled. Then we get a value of $S_{NL}(\rho_{AC}^{(1)})$ and corresponding to it, we have chosen a value of the parameters p and q given in (4.4.7) and (4.4.8) respectively. Using the information given in (4.4.6) and considering few values of p, q and λ_0, Table 4.1 is prepared. It depicts the values of $\langle S_v^{(1)}\rangle_{\rho_{ABC}^{(1)}}$ & $\langle S_v^{(2)}\rangle_{\rho_{ABC}^{(1)}}$ given in (4.3.16) and (4.3.17) indicating the fact that the state $|\psi^{(1)}\rangle_{ABC}$ exhibit genuine non-locality.

Example 4.2: A mixed three-qubit state: Combination of GHZ state and two W class of states

Let us consider a mixed three-qubit state of the form [241]

$$\rho_{ABC}^{(2)} = 0.2|GHZ\rangle\langle GHZ| + t|W_1\rangle\langle W_1| + (0.8-t)|W_2\rangle\langle W_2|, \quad t \in [0, 0.8] \tag{4.4.9}$$

where $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), |W_1\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), |W_2\rangle = \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle)$.

The mixed three-qubit state described by the density operator $\rho_{ABC}^{(2)}$ is an entangled state when $t \in [0, 0.8]$ and also we have

$$\lambda_{max}(\rho_{ABC}^{(2)}) = t, \quad \lambda_{min}(\rho_{ABC}^{(2)}) = 0 \tag{4.4.10}$$

Tracing out system A from the three-qubit state $\rho_{ABC}^{(2)}$, the reduced state $\rho_{BC}^{(2)}$ is given by

$$\rho_{BC}^{(2)} = \begin{pmatrix} \frac{0.6+2t}{6} & 0 & 0 & 0 \\ 0 & \frac{0.8}{3} & \frac{0.8}{3} & 0 \\ 0 & \frac{0.8}{3} & \frac{0.8}{3} & 0 \\ 0 & 0 & 0 & \frac{2.2-2t}{6} \end{pmatrix} \tag{4.4.11}$$

The state $\rho_{BC}^{(2)}$ is an entangled state for $t \in [0.5, 0.8]$.

Let us now consider the witness operator W_{CHSH}, which is given by

$$W_{CHSH} = 2I \otimes I - A_0 \otimes B_0 + A_0 \otimes B_1 - A_1 \otimes B_0 - A_1 \otimes B_1 \tag{4.4.12}$$

where the Hermitian operators A_0, A_1, B_0, B_1 are given by

$$A_0 = \sigma_x,\ A_1 = \sigma_y,\ B_0 = 0.95\sigma_x + 0.95\sigma_y + 0.447\sigma_z,\ B_1 = -0.95\sigma_x + 0.95\sigma_y + 0.447\sigma_z \tag{4.4.13}$$

The expectation value of W_{CHSH} with respect to the two-qubit state $\rho_{BC}^{(2)}$ can be calculated as

$$Tr[W_{CHSH}\rho_{BC}^{(2)}] = -0.0266667 < 0 \tag{4.4.14}$$

Therefore, the two-qubit state $\rho_{BC}^{(2)}$ is detected by witness operator W_{CHSH}. Thus, the strength of its non-locality may be measured by $S_{NL}(\rho_{BC}^{(2)})$, which is given by

$$S_{NL}(\rho_{BC}^{(2)}) = \frac{-Tr[W_{CHSH}\rho_{BC}^{(2)}]}{8} = 0.00333,\ \text{for } t \in [0.5, 0.8] \tag{4.4.15}$$

Further, we are now in a position to calculate the value of the following expressions involving the three-qubit state $\rho_{ABC}^{(2)}$ and the reduced two-qubit state $\rho_{BC}^{(2)}$

$$\lambda_{max}(I_2 \otimes \rho_{BC}^{(2)}) = 0.5333,\ \ \lambda_k(I_2 \otimes \rho_{BC}^{(2)}) = 0.333(1.1 - t) \tag{4.4.16}$$

Also, the range of p and q are given by

$$0 < p < 0.05 \tag{4.4.17}$$

$$0.34 < q < 0.37 \tag{4.4.18}$$

Using the information given in (4.4.10), (4.4.15), (4.4.16), (4.4.17), and (4.4.18), the value of the expression of $S_v^{(1)}$ and $S_v^{(2)}$ can be tabulated for the three-qubit state $\rho_{ABC}^{(2)}$ in Table 4.2.

4.4.2 Examples of three-qubit states for which the reduced two-qubit state is not detected by W_{CHSH}

In this section, we have considered three examples of three-qubit states in which the reduced two-qubit states are not detected by CHSH witness operator W_{CHSH}. The three examples are given in the following form: (i) A pure three-qubit state which belong to the GHZ class (ii) A mixed state which may be taken as a convex combination

State Parameter (t)	Operator parameter (p)	Operator parameter (q)	$\langle S_v^{(1)}\rangle_{\rho_{ABC}^{(2)}}$	$\langle S_v^{(2)}\rangle_{\rho_{ABC}^{(2)}}$
0.55	0.006	0.48	-4.96125	5.40786
0.65	0.019	0.44	-4.5171	4.75185
0.72	0.025	0.38	-5.32538	5.04819
0.79	0.039	0.46	-4.77032	4.54815

Table 4.2: We have chosen different values of the three-qubit state parameter t for which its reduced two-qubit state is entangled. Then we get a value of $S_{NL}(\rho_{BC}^{(2)})$ and corresponding to it, we have chosen a value of the operator parameters p and q given in (4.4.17) and (4.4.18) respectively. Using the information given in (4.4.16) and considering few values of p, q and t, Table 4.2 is prepared. It depicts the values of $\langle S_v^{(1)}\rangle_{\rho_{ABC}^{(2)}}$ & $\langle S_v^{(2)}\rangle_{\rho_{ABC}^{(2)}}$ given in (4.3.16) and (4.3.17) indicating the fact that the state $\rho_{ABC}^{(2)}$ exhibit genuine non-locality.

of three-qubit GHZ and W state and (iii) A mixed state which may be taken as a convex combination of three-qubit maximally mixed state and W state.

Example 4.3: A pure three-qubit GHZ class of state: Maximal Slice State

Let us consider a pure three-qubit GHZ class of state, which can be taken in the form [242]

$$|\psi^{(3)}\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle_{ABC} + Cos\theta|110\rangle_{ABC} + Sin\theta|111\rangle_{ABC}),\ \theta \in [0, \frac{\pi}{2}] \tag{4.4.19}$$

The pure state described by the density operator $\rho_{ABC}^{(3)} = |\psi^{(3)}\rangle_{ABC}\langle\psi^{(3)}|$ is an entangled state for $\theta \in (0, \frac{\pi}{2})$.

Also, for the state $\rho_{ABC}^{(3)}$, we have

$$\lambda_{max}(\rho_{ABC}^{(3)}) = 1,\ \ \lambda_{min}(\rho_{ABC}^{(3)}) = 0 \tag{4.4.20}$$

Tracing out system A from the three-qubit state $\rho_{ABC}^{(3)}$, the reduced two-qubit state $\rho_{BC}^{(3)}$ is given by

$$\rho_{BC}^{(3)} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & Cos\theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Cos\theta & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{4.4.21}$$

It can be easily verified that $\rho_{BC}^{(3)}$ is an entangled state for the state parameter $\theta \in [1.05, \frac{\Pi}{2}]$. Thus there must exist a witness operator that may detect $\rho_{BC}^{(3)}$ as an entangled state. But, in this example, our task is to show that even if some witness operator

does not detect the reduced two-qubit entangled state then also we are able to detect the non-locality of the three-qubit state described by the density operator $\rho_{BC}^{(3)}$.
To serve our purpose, we find here a witness operator $W_{CHSH}^{(xy)} = 2I - B_{CHSH}^{(xy)}$, whose expectation value with respect to the state $\rho_{BC}^{(3)}$ is given by $Tr[W_{CHSH}^{(xy)}\rho_{BC}^{(3)}] = 2 > 0$. Thus, the CHSH witness operator $W_{CHSH}^{(xy)}$ does not detect $\rho_{BC}^{(3)}$ as an entangled state. Since the two-qubit state $\rho_{BC}^{(3)}$ is an entangled state and it is not detected by $W_{CHSH}^{(xy)}$ so the strength of its non-locality may be measured by $S_{NL}^{New}(\rho_{BC}^{(3)})$. Using (3.2.29) and (4.2.13), we can calculate the range of $S_{NL}^{New}(\rho_{BC}^{(3)})$ and r. Therefore, we have

$$S_{NL}^{New}(\rho_{BC}^{(3)}) \in [0.05, 1.5], \ \theta \in [\frac{147\pi}{440}, \frac{\pi}{2}] \tag{4.4.22}$$

$$\text{and,} \quad r < [0.5, 1], \ \theta \in [\frac{147\pi}{440}, \frac{\pi}{2}] \tag{4.4.23}$$

Further, we can calculate the following using the three-qubit state $\rho_{ABC}^{(3)}$ and the reduced two-qubit state $\rho_{BC}^{(3)}$

$$\begin{aligned}
&\lambda_{max}(I_2 \otimes \rho_{BC}^{(3)}) = \frac{1+2Cos\theta}{2}, \ \ \lambda_k(I_2 \otimes \rho_{BC}^{(3)}) = \frac{1-2Cos\theta}{2}, \ \ \lambda_{max}((\rho_{BC}^{(3)})^{T_C}) = 0.5 \\
&\lambda_{min}[\overline{\rho_{ABC}^{(3)}(I_2 \otimes \rho_{BC}^{(3)})}] = \frac{3 - Cos2\theta - 2\sqrt{8 + 3Cos2\theta - Cos4\theta}}{16} \\
&\lambda_{max}[\overline{\rho_{ABC}^{(3)}(I_2 \otimes \rho_{BC}^{(3)})}] = \frac{3 - Cos2\theta + 2\sqrt{8 + 3Cos2\theta - Cos4\theta}}{16} \\
&Tr[W_{CHSH}^{(xy)}\rho_{BC}^{(3)}(\rho_{BC}^{(3)})^{T_C}] = 1, \ \ \lambda_{min}[((\rho_{BC}^{(3)})^{T_C})^2] = Cos^2\theta
\end{aligned} \tag{4.4.24}$$

Moreover, the range of p and q are given by

$$0.75 < p < 1 \tag{4.4.25}$$

$$0.59 < q < 1 \tag{4.4.26}$$

Using the information given in (4.4.20), (4.4.24), (4.4.25), and (4.4.26), the value of the expression of $S_v^{(3)}$ and $S_v^{(4)}$ can be calculated for the three-qubit state $\rho_{ABC}^{(3)}$ and they are tabulated in the Table 4.3.

Example 4.4: A three-qubit mixed state: A convex combination of three-qubit W state and a state belong to GHZ class

Let us take a mixed three-qubit state of the form

$$\rho_{ABC}^{(4)} = p_s|GHZ\rangle\langle GHZ| + (1-p_s)|W\rangle\langle W|, \ \ p_s \in [0,1]$$

State Parameter (θ(in radian))	Operator parameter (p)	Operator parameter (q)	$\langle S_v^{(3)} \rangle_{\rho_{ABC}^{(3)}}$	$\langle S_v^{(4)} \rangle_{\rho_{ABC}^{(3)}}$
1.2	0.67	0.86	-4.64621	5.03388
1.3	0.78	0.79	-5.24561	4.75133
1.4	0.915	0.69	-4.81717	5.53806
1.5	0.985	0.64	-5.333	5.25777

Table 4.3: We have chosen different values of the three-qubit state parameter θ for which its reduced two-qubit state is entangled. Then we get a value of $S_{NL}^{New}(\rho_{BC}^{(3)})$ and corresponding to it, we have chosen a value of the operator parameters p and q given in (4.4.25) and (4.4.26) respectively. Using the information given in (4.4.24) and considering a few values of p, q, and θ, Table 4.3 is prepared. It depicts the values of $\langle S_v^{(3)} \rangle_{\rho_{ABC}^{(3)}}$ & $\langle S_v^{(4)} \rangle_{\rho_{ABC}^{(3)}}$ given in (4.3.30) and (4.3.31) indicating the fact that the state $\rho_{ABC}^{(3)}$ exhibit genuine non-locality.

where $|GHZ\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |101\rangle), |W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$.
The mixed three-qubit state described by the density operator $\rho_{ABC}^{(4)}$ is an entangled state when $p_s \in [0.4, 0.9]$ and also we have

$$\lambda_{max}(\rho_{ABC}^{(4)}) = \frac{3+\sqrt{3}\sqrt{3-10p_s+10p_s^2}}{6},\ \lambda_{min}(\rho_{ABC}^{(4)}) = 0 \tag{4.4.27}$$

Tracing out system A from the three-qubit state $\rho_{ABC}^{(4)}$, the reduced state $\rho_{BC}^{(4)}$ is given by

$$\rho_{BC}^{(4)} = \begin{pmatrix} \frac{1-p_s}{3} & 0 & 0 & 0 \\ 0 & \frac{2+p_s}{6} & \frac{1-p_s}{3} & 0 \\ 0 & \frac{1-p_s}{3} & \frac{2+p_s}{6} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.4.28}$$

$\rho_{BC}^{(4)}$ is an entangled state for $p_s \in [0.4, 0.9]$. Also, we have

$$Tr[W_{CHSH}^{(xy)}\rho_{BC}^{(4)}] \quad = \quad \frac{2(3-2\sqrt{2}+2\sqrt{2}p_s)}{3} > 0,\ \ 0.4 \le p_s \le 0.9 \tag{4.4.29}$$

In this example also, we find that the same CHSH witness operator $W_{CHSH}^{(xy)}$ given in the previous example, is not able to detect the entangled state $\rho_{BC}^{(4)}$. The strength of the non-locality of $\rho_{BC}^{(4)}$ may be measured by $S_{NL}^{New}(\rho_{BC}^{(4)})$ using (3.2.29). Therefore, $S_{NL}^{New}(\rho_{BC}^{(4)})$ may be calculated as

$$S_{NL}^{New}(\rho_{BC}^{(4)}) \ \in \ [0.04, 1.91628],\ p_s \ \in \ [0.4, 0.9] \tag{4.4.30}$$

and the parameter r is given by

$$r < [0.59, 1],\ p_s \in [0.4, 0.9] \tag{4.4.31}$$

Further, we can calculate the following using the three-qubit state $\rho_{ABC}^{(4)}$ and the reduced two-qubit state $\rho_{BC}^{(4)}$

$$\begin{aligned}
&\lambda_{max}(I_2 \otimes \rho_{BC}^{(4)}) = \frac{4-p_s}{6},\ \lambda_k(I_2 \otimes \rho_{BC}^{(4)}) = \frac{1-p_s}{3},\ \lambda_{max}((\rho_{BC}^{(4)})^{T_C}) = \frac{2+p_s}{6}\\
&Tr[W_{CHSH}^{(xy)}\rho_{BC}^{(4)}(\rho_{BC}^{(4)})^{T_C}] = \frac{6-4\sqrt{2}+2\sqrt{2}p_s+(3+2\sqrt{2})p_s^2}{9}\\
&\lambda_{min}[((\rho_{BC}^{(4)})^{T_C})^2] = \frac{3-6p_s+3p_s^2-\sqrt{5}\sqrt{1-4p_s+6p_s^2-4p_s^3+p_s^4}}{18}
\end{aligned} \tag{4.4.32}$$

Also, the range of p in terms of state parameter p_s is given by

$$\frac{\sqrt{2}H}{\sqrt{2}H - \frac{4-p_s}{6}} < p < \frac{2H}{2H - \frac{4-p_s}{6}} \tag{4.4.33}$$

The range of q in terms of state parameter p_s is given by

$$\frac{F}{F + 2\sqrt{2}\frac{1-p_s}{3}} < q < \frac{F}{F - 2\sqrt{2}\frac{1-p_s}{3}} \tag{4.4.34}$$

where F and H given in the previous section can be calculated using the information given in (4.4.32).

Using the information given in (4.4.27), (4.4.32), (4.4.33), and (4.4.34), the value of the expression of $S_v^{(3)}$ and $S_v^{(4)}$ can be calculated for the three-qubit state $\rho_{ABC}^{(4)}$ and they are tabulated in the Table 4.4.

State Parameter (p_s)	Operator parameter (p)	Operator parameter (q)	$\langle S_v^{(3)} \rangle_{\rho_{ABC}^{(4)}}$	$\langle S_v^{(4)} \rangle_{\rho_{ABC}^{(4)}}$
0.5	0.9	0.75	-5.3768	4.99623
0.6	0.95	0.8	-5.46426	4.39912
0.7	0.98	0.82	-5.03592	4.96321
0.8	0.993	0.86	-5.11832	5.48576

Table 4.4: We have chosen different values of the three-qubit state parameter p_s for which its reduced two-qubit state is entangled. Then we get a value of $S_{NL}^{New}(\rho_{BC}^{(4)})$ and corresponding to it, we have chosen a value of the operator parameters p and q given in (4.4.33) and (4.4.34) respectively. Using the information given in (4.4.32) and considering a few values of p, q, and p_s, Table 4.4 is prepared. It depicts the values of $\langle S_v^{(3)} \rangle_{\rho_{ABC}^{(4)}}$ & $\langle S_v^{(4)} \rangle_{\rho_{ABC}^{(4)}}$ given in (4.3.30) and (4.3.31) indicating the fact that the state $\rho_{ABC}^{(4)}$ exhibit genuine non-locality.

Example 4.5: A three-qubit mixed State: A convex combination of maximally mixed state and W state

Let us consider a mixed three-qubit state of the form [243]

$$\rho^{(5)}_{ABC} = \frac{1-p_s}{8} I_8 + p_s |W\rangle_{ABC}\langle W|, p_s \in (0.816, 1] \tag{4.4.35}$$

where I_8 denotes the maximally mixed state represented by the Identity matrix and $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$.

The mixed three-qubit state described by the density operator $\rho^{(5)}_{ABC}$ is an entangled state when $p_s \in (0.816, 1]$ and also we have

$$\lambda_{max}(\rho^{(5)}_{ABC}) = \frac{1+7p_s}{8},\ \lambda_{min}(\rho^{(5)}_{ABC}) = \frac{1-p_s}{8} \tag{4.4.36}$$

Taking partial trace over the system A, the three-qubit state $\rho^{(5)}_{ABC}$ reduces to the two-qubit state described by the density operator $\rho^{(5)}_{BC}$, which is given by

$$\rho^{(5)}_{BC} = \begin{pmatrix} \frac{p_s+3}{12} & 0 & 0 & 0 \\ 0 & \frac{p_s+3}{12} & \frac{p_s}{3} & 0 \\ 0 & \frac{p_s}{3} & \frac{p_s+3}{12} & 0 \\ 0 & 0 & 0 & \frac{1-p_s}{4} \end{pmatrix},\ 0.816 < p_s \leq 1 \tag{4.4.37}$$

$\rho^{(5)}_{BC}$ is an entangled state for $p_s \in (0.816, 1]$ but we find that

$$Tr[W^{(xy)}_{CHSH}\rho^{(5)}_{BC}] \ = \ 2 - \frac{4\sqrt{2}p_s}{3} > 0,\ \ 0.816 < p_s \leq 1 \tag{4.4.38}$$

(4.4.38) implies that the CHSH witness operator does not detect the entangled state $\rho^{(5)}_{BC}$. The strength of the non-locality of the two-qubit reduced state may be measured by $S^{New}_{NL}(\rho^{(5)}_{BC})$. The strength $S^{New}_{NL}(\rho^{(5)}_{BC})$ and the parameter r is given by

$$S^{New}_{NL}(\rho^{(5)}_{BC}) \in [0.54124, 0.5484],\ 0.816 < p_s \leq 1,\ r \in [0.61, 0.69] \tag{4.4.39}$$

Further, we can now calculate the following eigenvalues and traces, which are given by

$$
\begin{aligned}
&\lambda_{max}(I_2\otimes\rho_{BC}^{(5)}) = \frac{3+5p_s}{12},\ \ \lambda_k(I_2\otimes\rho_{BC}^{(5)}) = \frac{1-p_s}{4},\ \ Tr[((\rho_{BC}^{(5)})^{T_C})^2] = \frac{9+11p_s^2}{36}\\
&\lambda_{min}(\overline{\rho_{ABC}^{(5)}(I_2\otimes\rho_{BC}^{(5)})}) = \frac{9+30p_s+25p_s^2-8\sqrt{3}\sqrt{9p_s^2+14p_s^3+9p_s^4}}{288}\\
&\lambda_{max}(\overline{\rho_{ABC}^{(5)}(I_2\otimes\rho_{BC}^{(5)})}) = \frac{9+30p_s+25p_s^2+8\sqrt{3}\sqrt{9p_s^2+14p_s^3+9p_s^4}}{288}\\
&Tr[W_{CHSH}^{(xy)}\rho_{BC}^{(5)}(\rho_{BC}^{(5)})^{T_C}] = \frac{9-6\sqrt{2}p_s+(3-2\sqrt{2})p_s^2}{18}\\
&\lambda_{min}[((\rho_{BC}^{(5)})^{T_C})^2] = \frac{9-6p_s+21p_s^2-4\sqrt{5}\sqrt{9p_s^2-6p_s^3+p_s^4}}{144}\\
&\lambda_{max}((\rho_{BC}^{(5)})^{T_C}) = \frac{3-p_s+2\sqrt{5}p_s}{12},\ \ \lambda_{max}(W_{CHSH}^{(xy)}) = 2(1+\sqrt{2}) \qquad (4.4.40)
\end{aligned}
$$

Also, the range of p and q in terms of state parameter p_s are given by

$$\frac{\sqrt{2}H}{\sqrt{2}H-\frac{3+p_s}{12}} < p < \frac{2H}{2H-\frac{3+p_s}{12}} \qquad (4.4.41)$$

and,

$$\frac{F}{F+2\sqrt{2}\frac{3+p_s}{12}} < q < \frac{F}{F-2\sqrt{2}\frac{3+p_s}{12}} \qquad (4.4.42)$$

where F and H given in the previous section can be calculated using the information given in (4.4.40).

Therefore, using the information given in (4.4.36), (4.4.40), (4.4.41), and (4.4.42), the value of the expression of $S_v^{(3)}$ and $S_v^{(4)}$ can be calculated for the three-qubit state $\rho_{ABC}^{(5)}$ and they are tabulated in the Table 4.5.

State Parameter (p_s)	Operator parameter (p)	Operator parameter (q)	$\langle S_v^{(3)}\rangle_{\rho_{ABC}^{(5)}}$	$\langle S_v^{(4)}\rangle_{\rho_{ABC}^{(5)}}$
0.82	0.72	0.93	-4.35959	5.06538
0.87	0.6	0.95	-4.6602	5.30281
0.92	0.45	0.97	-5.4101	5.45763
0.97	0.35	0.99	-5.147	5.10813

Table 4.5: We have chosen different values of the three-qubit state parameter p_s for which its reduced two-qubit state is entangled. Then we get a value of $S_{NL}^{New}(\rho_{BC}^{(5)})$ and corresponding to it, we have chosen a value of the operator parameters p and q given in (4.4.41) and (4.4.42) respectively. Using the information given in (4.4.40) and considering a few values of p, q, and p_s, Table 4.5 is prepared. It depicts the values of $\langle S_v^{(3)}\rangle_{\rho_{ABC}^{(5)}}$ & $\langle S_v^{(4)}\rangle_{\rho_{ABC}^{(5)}}$ given in (4.3.30) and (4.3.31) indicating the fact that the state $\rho_{ABC}^{(5)}$ exhibit genuine non-locality.

4.5 Comparing our criterion with other existing criteria

In this section, we have compared our results with other pre-existing criteria such as (i) M. Li's criterion [228] (ii) Different types of Svetlichny inequality [80], for the detection of genuine non-locality of pure or mixed three-qubit states.
We may re-state M. Li's criterion as [228]: If S_v denote the Svetlichny operator and if any pure or mixed three-qubit states described by the density operator ρ violate the inequality

$$max|\langle S_v\rangle_\rho| \leq 4\lambda_1 \tag{4.5.1}$$

then the state ρ may possess genuine non-local property.
Here maximum is taken over all measurement settings and λ_1 denoting the maximum singular value of the matrix $M = [M_{j,ik}]$ with $M_{ijk} = Tr[\rho(\sigma_i \otimes \sigma_j \otimes \sigma_k)]$. We note that the upper bound given in (4.5.1) is state dependent.

Example 4.6: Let us consider a mixed three-qubit state of the form [228]

$$\rho^{(6)}_{ABC} \quad = \quad t|\phi_{gs}\rangle\langle\phi_{gs}| + \frac{1-t}{8}I, \ \ t \in [0,1] \tag{4.5.2}$$

where I denote an identity matrix of order 8. The state $|\phi_{gs}\rangle$ is given by

$$|\phi_{gs}\rangle = \frac{1}{2}|000\rangle + \frac{\sqrt{3}}{2}|11\rangle(Cos\theta_3|0\rangle + Sin\theta_3|1\rangle), \ \ \theta_3 \in [0, \frac{\pi}{2}] \tag{4.5.3}$$

The maximum and minimum eigenvalue of $\rho^{(6)}_{ABC}$ is given by

$$\lambda_{max}(\rho^{(6)}_{ABC}) = \frac{1+7t}{8}, \ \ \lambda_{min}(\rho^{(6)}_{ABC}) = \frac{1-t}{8} \tag{4.5.4}$$

It can be observed that if we trace out either system A or system B then the resulting two-qubit state will become a separable state and thus we cannot apply our result. So, we consider the two-qubit state resulting from tracing out the system C from the three-qubit state $\rho^{(6)}_{ABC}$. The reduced two-qubit state $\rho^{(6)}_{AB}$ is given by

$$\rho_{AB}^{(6)} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{\sqrt{3}t}{4}Cos\theta_3 \\ 0 & \frac{1-t}{4} & 0 & 0 \\ 0 & 0 & \frac{1-t}{4} & 0 \\ \frac{\sqrt{3}t}{4}Cos\theta_3 & 0 & 0 & \frac{1+2t}{4} \end{pmatrix} \quad (4.5.5)$$

The state $\rho_{AB}^{(6)}$ is an entangled state for $t \in [0.83,1]$ and $\theta_3 \in [0.615, 0.6219]$ as there exists a witness operator $W_{CHSH}^{(xz)}(= 2I - B_{CHSH}^{(xz)})$ that detects it. The CHSH witness operator $B_{CHSH}^{(xz)}$ is given by (3.2.7). This is clear from the following fact

$$\begin{aligned} Tr[W_{CHSH}^{(xz)}\rho_{AB}^{(6)}] &= 2 - \frac{t(2+\sqrt{3}Cos\theta_3)}{\sqrt{2}} \\ &< 0, \text{ for } t \in [0.83,1] \ \& \ \theta_3 \in [0.615, 0.6219] \end{aligned} \quad (4.5.6)$$

Since the two-qubit state $\rho_{AB}^{(6)}$ is an entangled state and it is detected by $W_{CHSH}^{(xz)}$ so the strength of its non-locality may be measured by $S_{NL}(\rho_{AB}^{(6)})$. It is then given by

$$S_{NL}(\rho_{AB}^{(6)}) = \frac{-Tr[W_{CHSH}^{(xz)}\rho_{AB}^{(6)}]}{8} \in [0, 0.04], \text{for } t \in [0.83,1] \ \& \ \theta_3 \in [0.615, 0.6219] \quad (4.5.7)$$

Further, we are now in a position to calculate the value of the following quantities, which are given by

$$\lambda_{max}(I_2 \otimes \rho_{AB}^{(6)}) = \frac{2+2t+\sqrt{2}t\sqrt{5+3Cos2\theta_3}}{8}, \quad \lambda_k(I_2 \otimes \rho_{AB}^{(6)}) = \frac{1-t}{4} \quad (4.5.8)$$

Moreover, the range of p and q in terms of state parameter θ are given by

$$\frac{\sqrt{2}A}{\sqrt{2}A - \frac{2+2t+\sqrt{2}t\sqrt{5+3Cos2\theta_3}}{8}} < p < \frac{2A}{2A - \frac{2+2t+\sqrt{2}t\sqrt{5+3Cos2\theta_3}}{8}} \quad (4.5.9)$$

$$\frac{\sqrt{2}B}{\sqrt{2}B + \frac{1-t}{4}} < q < \frac{2B}{2B + \frac{1-t}{4}} \quad (4.5.10)$$

where $A = \lambda_{min}(\overline{\rho_{ABC}^{(6)}(I_2 \otimes \rho_{AB}^{(6)})}) + 2\lambda_{max}(\rho_{ABC}^{(6)})S_{NL}(\rho_{AB}^{(6)})$ and $B = \lambda_{max}(\overline{\rho_{ABC}^{(6)}(I_2 \otimes \rho_{AB}^{(6)})}) + 2\lambda_{min}(\rho_{ABC}^{(6)})S_{NL}(\rho_{AB}^{(6)})$ can be calculated using the information given in (4.5.8).

Further, using the information given in (4.5.4), (4.5.7), (4.5.8), (4.5.9), and (4.5.10), the value of the expression of $S_v^{(1)}$ and $S_v^{(2)}$ can be tabulated for the three-qubit state

$\rho^{(6)}_{ABC}$ in Table 4.6.

Comparison Analysis						
	Our Work				M.Li et. al. Work [228]	
State Parameter t, θ_3	Operator Parameter p, q	$\langle S^{(1)}_v \rangle_{\rho^{(6)}_{ABC}}$	$\langle S^{(2)}_v \rangle_{\rho^{(6)}_{ABC}}$	Whether (4.3.15) satisfied or violated?	Upper Bound of $\langle S_v \rangle_{\rho^{(6)}_{ABC}} = 4\lambda_1$	Whether (4.5.1) satisfied or violated?
$t = 0.84$, $\theta_3 = 0.616$	$p = 0.1$, $q = 0.92$	-5.07155	5.07541	Violated	3.36062	Satisfied
$t = 0.87$, $\theta_3 = 0.618$	$p = 0.09$, $q = 0.93$	-4.76827	5.65134	Violated	3.4831	Satisfied
$t = 0.9$, $\theta_3 = 0.62$	$p = 0.07$, $q = 0.959$	-5.02466	4.35569	Violated	3.60576	Satisfied
$t = 0.95$ $\theta_3 = 0.6205$	$p = 0.04$, $q = 0.979$	-5.19447	4.66488	Violated	3.80675	Satisfied
$t = 0.99$, $\theta_3 = 0.6215$	$p = 0.019$, $q = 0.996$	-4.403	4.59326	Violated	3.96844	Satisfied
$t = 0.998$, $\theta_3 = 0.6216$	$p = 0.012$, $q = 0.9992$	-4.83292	4.62339	Violated	4.00064	May violate
$t = 0.999$, $\theta_3 = 0.6217$	$p = 0.01$, $q = 0.9996$	-5.4911	4.62769	Violated	4.0048	May violate

Table 4.6: We have chosen different values of the three-qubit state parameter (t, θ_3) for which its reduced two-qubit state $\rho^{(6)}_{AB}$ is entangled. Then we get a value of $S_{NL}(\rho^{(6)}_{AB})$ and corresponding to it, we have chosen a value of the operator parameters p and q given in (4.5.9) and (4.5.10) respectively. . Using the information given in (4.5.8) and considering few values of p, q, t and θ_3, Table 4.6 is prepared. It depicts the values of $\langle S^{(1)}_v \rangle_{\rho^{(6)}_{ABC}}$ & $\langle S^{(2)}_v \rangle_{\rho^{(6)}_{ABC}}$ given in (4.3.16) and (4.3.17) indicating the fact that the state $\rho^{(6)}_{ABC}$ exhibit genuine non-locality.

We are now in a position to compare our result with the result given in [228]. We have calculated the maximum singular value λ_1 of the matrix $M = [M_{j,ik}]$, where $M_{ijk} = Tr[\rho^{(6)}_{ABC}(\sigma_i \otimes \sigma_j \otimes \sigma_k)]$ and the values of λ_1 given in Table 4.6. It is clear from Table 4.6 that the state $\rho^{(6)}_{ABC}$ with parameters $t \in [0.83, 1]$ and $\theta_3 \in [0.615, 0.6219]$ violate the bounds given in Result 4.4.1 and thus able to detect the genuine non-locality of $\rho^{(6)}_{ABC}$. On the other hand, the state $\rho^{(6)}_{ABC}$ satisfies (4.5.1) and thus we can say that M. Li et. al.'s criterion is unable to detect the genuine non-locality of the state $\rho^{(6)}_{ABC}$.

Example 4.7: In [80], J.-D. Bancal et. al. have considered a pure state $|\psi^{(7)}\rangle_{ABC}$ of the form

$$|\psi^{(7)}\rangle_{ABC} = \frac{\sqrt{3}}{2}|000\rangle + \frac{\sqrt{3}}{4}|110\rangle + \frac{1}{4}|111\rangle \tag{4.5.11}$$

The state (4.5.11) is peculiar in the sense that it does not violate 1087 types of Svetlichny Inequality, which have been constructed in [80]. Thus, our task is to enquire whether the genuine non-locality of the pure state (4.5.11) is detected by our criterion.

The pure state (4.5.11) described by the density operator $\rho^{(7)}_{ABC} = |\psi^{(7)}\rangle_{ABC}\langle\psi^{(7)}|$, is an entangled state, and also we have

$$\lambda_{max}(\rho_{ABC}^{(7)}) = 1, \ \ \lambda_{min}(\rho_{ABC}^{(7)}) = 0 \tag{4.5.12}$$

Tracing out system C from the three-qubit state $\rho_{ABC}^{(7)}$, the reduced two-qubit state $\rho_{AB}^{(7)}$ is given by

$$\rho_{AB}^{(7)} = \begin{pmatrix} \frac{3}{4} & 0 & 0 & \frac{3}{8} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{3}{8} & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{4.5.13}$$

The state $\rho_{AB}^{(7)}$ is an entangled state and it is detected by the witness operator $W_{CHSH}^{(xz)}(= 2I - B_{CHSH}^{(xz)})$. It is clear from the following fact

$$Tr[W_{CHSH}^{(xz)}\rho_{AB}^{(7)}] = -0.47487 \tag{4.5.14}$$

The strength of the non-locality of two-qubit state $\rho_{AB}^{(7)}$ may be measured by $S_{NL}(\rho_{AB}^{(7)})$ and it is given by

$$S_{NL}(\rho_{AB}^{(7)}) = \frac{-Tr[W_{CHSH}^{(xz)}\rho_{AB}^{(7)}]}{8} = 0.0593588 \tag{4.5.15}$$

Further, the eigenvalues and traces are given by

$$\begin{aligned} &\lambda_{max}(I_2 \otimes \rho_{AB}^{(7)}) = 0.950694, \ \lambda_{min}(\overline{\rho_{ABC}^{(7)}(I_2 \otimes \rho_{AB}^{(7)})}) = -0.0783743 \\ &\lambda_{max}(\overline{\rho_{ABC}^{(7)}(I_2 \otimes \rho_{AB}^{(7)})}) = 0.656499, \ \lambda_k(I_2 \otimes \rho_{AB}^{(7)}) = 0.0493061 \end{aligned} \tag{4.5.16}$$

Moreover, the range of p and q is given by

$$0.00687286 < p < 0.0096921 \tag{4.5.17}$$

$$0.949571 < q < 0.963807 \tag{4.5.18}$$

Using the information given in (4.5.12), (4.5.15), (4.5.16), (4.5.17), and (4.5.18), the value of the expression of $S_v^{(1)}$ and $S_v^{(2)}$ can be calculated for the three-qubit state $\rho_{ABC}^{(7)}$ and they are tabulated in Table 4.7.

Operator parameter (p)	Operator parameter (q)	$\langle S_v^{(1)}\rangle_{\rho_{ABC}^{(7)}}$	$\langle S_v^{(2)}\rangle_{\rho_{ABC}^{(7)}}$
0.007	0.95	-5.5534	5.60622
0.0075	0.955	-5.18056	5.01918
0.008	0.959	-4.85433	4.55396
0.0085	0.96	-4.56648	4.43826
0.009	0.963	-4.31061	4.0926

Table 4.7: We traced out system C from ρ_{ABC} and got a reduced two-qubit state which is entangled. Then we calculated $S_{NL}(\rho_{AB}^{(7)})$ and corresponding to it, we have chosen a value of the parameters p and q given in (4.5.17) and (4.5.18) respectively. Using the information given in (4.5.16) and considering a few values of p, and q, Table 4.7 is prepared. It depicts the values of $\langle S_v^{(1)}\rangle_{\rho_{ABC}^{(7)}}$ & $\langle S_v^{(2)}\rangle_{\rho_{ABC}^{(7)}}$ given in (4.3.16) and (4.3.17) indicating the fact that the state $|\psi^{(7)}\rangle_{ABC}$ exhibit genuine non-locality.

Therefore, we can infer that for the corresponding p and q, the state $|\psi^{(7)}\rangle_{ABC}$ exhibits genuine non-locality. So, by using our approach, we can say that the state $|\psi^{(7)}\rangle_{ABC}$ may exhibit genuine non-locality.

Example 4.8: Let us take a mixed three-qubit state of the form [244]

$$\rho_{ABC}^{(8)} = \frac{1}{8} I\otimes I\otimes I + \sum_{k=x,y,z}\Big(\frac{1}{24}(I\otimes\sigma_k\otimes\sigma_k) - \frac{c}{16}(\sigma_k\otimes I\otimes\sigma_k + \sigma_k\otimes\sigma_k\otimes I)\Big),\ c\in(0,1] \tag{4.5.19}$$

where σ_k are the Pauli matrices $k = x,y,z$. Toth and Acin [244] have shown that the mixed three-qubit state (4.5.19) is a genuine entangled state for $c\in(0.869,1]$ although it admits a local hidden variable model. Now we will show that the state $\rho_{ABC}^{(8)}$ violate the bound (4.3.29). To execute this task, let us calculate the maximum and minimum eigenvalue of $\rho_{ABC}^{(8)}$. They are given by

$$\lambda_{max}(\rho_{ABC}^{(8)}) = \frac{2+3c}{12}, \lambda_{min}(\rho_{ABC}^{(8)}) = 0 \tag{4.5.20}$$

Tracing out system C from the three-qubit state $\rho_{ABC}^{(8)}$, the reduced two-qubit state $\rho_{AB}^{(8)}$ is given by

$$\rho_{AB}^{(8)} = \begin{pmatrix} \frac{2-c}{8} & 0 & 0 & 0 \\ 0 & \frac{2+c}{8} & \frac{-c}{4} & 0 \\ 0 & \frac{-c}{4} & \frac{2+c}{8} & 0 \\ 0 & 0 & 0 & \frac{2-c}{8} \end{pmatrix} \tag{4.5.21}$$

$\rho_{AB}^{(8)}$ is an entangled state for $c\in(0.869,1]$. Also, we have

$$Tr[W^{(xy)}_{CHSH}\rho^{(8)}_{AB}] = 2-\sqrt{2}c \ > 0, \ 0.869 < c \leq 1$$

We find that CHSH witness operator $W^{(xy)}_{CHSH}$ is not able to detect the entangled state $\rho^{(8)}_{AB}$. The strength of the non-locality of $\rho^{(8)}_{AB}$ may be measured by $S^{New}_{NL}(\rho^{(8)}_{AB})$. Therefore, using (3.2.48), the parameter r is given by

$$r < [0.73, 0.815), \ c \ \in \ (0.869, 1] \tag{4.5.22}$$

Hence, using (3.2.29), the strength of the non-locality $S^{New}_{NL}(\rho^{(8)}_{AB})$ may be calculated as

$$S^{New}_{NL}(\rho^{(8)}_{AB}) \ \in \ [0.21, 0.44), \ c \ \in \ (0.869, 1] \tag{4.5.23}$$

Further, we can calculate the following using the three-qubit state $\rho^{(8)}_{ABC}$ and the reduced two-qubit state $\rho^{(8)}_{AB}$

$$\begin{aligned}
&\lambda_{max}(I_2\otimes\rho^{(8)}_{AB}) = \frac{2+3c}{8}, \quad \lambda_k(I_2\otimes\rho^{(8)}_{AB}) = \lambda_{max}((\rho^{(8)}_{AB})^{T_B}) = \frac{2-c}{8}, \\
&\lambda_{max}(\overline{\rho^{(8)}_{ABC}(I_2\otimes\rho^{(8)}_{AB})}) = -\frac{(c-2)(2+3c)}{96}, \ \ \lambda_{min}[((\rho^{(8)}_{AB})^{T_B})^2] = \frac{4-12c+9c^2}{64} \\
&\lambda_{min}(\overline{\rho^{(8)}_{ABC}(I_2\otimes\rho^{(8)}_{AB})}) = 0, \ \ Tr[W^{(xy)}_{CHSH}\rho^{(8)}_{AB}(\rho^{(8)}_{AB})^{T_B}] = \frac{4+2\sqrt{2}c+(1+\sqrt{2})c^2}{8}
\end{aligned} \tag{4.5.24}$$

Also, the range of p and q in terms of state parameter c is given by

$$\frac{\sqrt{2}H}{\sqrt{2}H-\frac{2+3c}{8}} < p < \frac{2H}{2H-\frac{2+3c}{8}} \tag{4.5.25}$$

$$\frac{F}{F+2\sqrt{2}(\frac{2-c}{8})} < q < \frac{F}{F+2(\frac{2-c}{8})} \tag{4.5.26}$$

where $H = \frac{(c-2)(2+3c)}{6(4-12c+9c^2)}$, $F = -\frac{(c-2)(2+3c)}{24}$.

Using the information given in (4.5.20), (4.5.24), (4.5.25), and (4.5.26), the value of the expression of $S^{(3)}_v$ and $S^{(4)}_v$ can be calculated for the three-qubit state $\rho^{(8)}_{ABC}$ and they are tabulated in the Table 4.8.

State Parameter (c)	Operator parameter (p)	Operator parameter (q)	$\langle S_v^{(3)} \rangle_{\rho_{ABC}^{(8)}}$	$\langle S_v^{(4)} \rangle_{\rho_{ABC}^{(8)}}$
0.87	0.89	0.36	-4.14775	5.4637
0.89	0.87	0.37	-4.15282	5.30108
0.92	0.82	0.38	-4.73136	5.17754
0.95	0.8	0.43	-4.29487	4.28605
0.99	0.76	0.44	-4.14294	4.21697

Table 4.8: We have chosen different values of the three-qubit state parameter c for which its reduced two-qubit state is entangled. Since $\rho_{BC}^{(8)}$ is not detected by $W_{CHSH}^{(xy)}$ so we have calculated the value of $S_{NL}^{New}(\rho_{BC}^{(8)})$ and corresponding to it, we have chosen a value of the operator parameters p and q given in (4.5.25) and (4.5.26) respectively. Using the information given in (4.5.24) and considering few values of p, q and c, Table 4.8 is prepared. It depicts the values of $\langle S_v^{(3)} \rangle_{\rho_{ABC}^{(8)}}$ & $\langle S_v^{(4)} \rangle_{\rho_{ABC}^{(8)}}$ given in (4.3.30) and (4.3.31) indicating the fact that the state $\rho_{ABC}^{(8)}$ exhibit genuine non-locality.

4.6 Conclusion

In this chapter, we have considered the problem of detection of non-locality of an arbitrary three-qubit state (pure or mixed). This problem may be handled by the violation of Svetlichny inequality but to do this, we have to maximize the expectation value of the Svetlichny operator overall measurements of unit spin vectors. This optimization problem may not be very simple so we have adopted a new procedure to detect the genuine non-locality of an arbitrary three-qubit state. We have derived a state-dependent lower and upper bound of the expectation value of the Svetlichny operator S_v with respect to any pure or mixed three-qubit state described by the density operator ρ_{ABC}. These bounds established a connection between $\langle S_v \rangle_{\rho_{ABC}}$ and the strength of the non-locality of the reduced two-qubit entangled state $\rho_{ij},\ i \neq j,\ i,j = A,B,C$. We should note here that the considered reduced two-qubit state must be an entangled state. The strength of the non-locality of the reduced two-qubit state may be measured either by $S_{NL}(\rho_{ij})$ or by $S_{NL}^{New}(\rho_{ij})$ depending on whether it is detected or not detected by CHSH witness operator. We have shown that the modified lower and upper bound of the expectation value of the Svetlichny operator may help in getting the violation of the inequality for those three-qubit genuine entangled states which was earlier not detected by Svetlichny inequality or by any other inequalities. To implement our results in an experiment, let us discuss briefly the possible implementation of the partial trace, eigenvalues, and partial transposition in an experiment: (i) Partial Trace- Possible experimental implementation of partial trace has been discussed in [11–14]. (ii) Eigenvalues- It is shown that there exist methods by which one may determine the eigenvalues of a state experimentally in a relatively easier way than full state tomography [186, 200]. (iii) Partial Transposition- Partially transposed density matrices are generically unphysical because it is a positive but not completely positive map. In spite of this limitation, measurement of their moments is possible [245]. Using their moments, one may estimate the values of the trace of a function of partial transposition [246]. Since the inequalities derived here for the purpose of detecting the genuine entanglement in the three-qubit system depends on partial trace, eigenvalues, and partial transposition operation so we may expect that our result may be verified in an experiment also.

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Chapter 5

Controlled Quantum Teleportation: Estimation of Controller's Power through Witness Operator

"Teleportation is not a dream anymore. It's a daily routine, at least for particles."

- Charles Bennett

In this chapter [1], we estimate the controller's power through the witness operator. Controlled quantum teleportation (CQT) can be considered as a variant of quantum teleportation in which three parties are involved where one party acts as the controller. The usability of the CQT scheme depends on two types of fidelities viz. conditioned fidelity and non-conditioned fidelity. The difference between these fidelities may be termed as the power of the controller and it plays a vital role in the CQT scheme. Thus, our aim is to estimate the power of the controller in such a way that its estimated value can be obtained in an experiment. To achieve our goal, we have constructed a witness operator and have shown that its expected value may be used in the estimation of the lower bound of the power of the controller. Furthermore, we have shown that it is possible to make the standard W state useful in the CQT scheme if one of its qubits either passes through the amplitude damping channel or the phase damping channel. We have also shown that the phase damping channel performs better

[1]This chapter is based on a accepted research paper "Estimation of Power in the Controlled Quantum Teleportation through the Witness Operator", *Accepted in The European Physical Journal D (EPJ D) (2024).*

than the amplitude damping channel in the sense of generating more power for the controller in the CQT scheme.

5.1 Introduction

The process of transferring an unknown quantum state between two parties at two distant locations without transferring the physical information about the unknown quantum state itself is known as quantum teleportation [1, 3, 20]. This means that neither any physical information about the state is transferred nor a swap operation between the sender and the receiver is performed. Teleportation protocol makes use of the non-local correlations generated by using an entangled pair between the sender and the receiver, and the exchange of classical information between them. This concept plays a central role in quantum communication using quantum repeaters [22, 115] and can also be used to implement logic gates for universal quantum computation [116]. Quantum teleportation was also demonstrated experimentally [54, 146, 247, 248].
We call the Bennett et. al. [20] protocol of teleporting a single qubit using a two-qubit shared state as standard quantum teleportation protocol and it has already been discussed in Chapter 1. Quantum teleportation using a three-qubit state as a resource state was introduced by Karlsson et. al. [137]. It is a variant of teleportation in which three members such as Alice (A), Bob (B), and Charlie (C) are participating with one qubit each. Later, this type of quantum teleportation protocol is popularly called the controlled quantum teleportation (CQT). A lot of studies on CQT schemes have been studied in the literature [166–181].
In the CQT scheme, we may consider that Alice, Bob, and Charlie share a three-qubit pure/mixed state described by the density operator ρ_{ABC}. We assume throughout the chapter that Charlie acts as a controller who performs single-qubit Von Neumann measurement on his qubit which is given in (1.4.22) and (1.4.23). And the three-qubit state ρ_{ABC} after the measurement is projected onto the two-qubit state which is given in (1.12.1). And by tracing out Charlie's qubit a reduced two-qubit state described by the density operator $\rho_{AB} = Tr_C(\rho_{ABC})$. In the CQT scheme, the faithfulness of the teleportation may be quantified by the conditioned fidelity denoted by $f_C(\rho_{AB}^{(k)})$ and the non-conditioned fidelity $f_{NC}(\rho_{AB})$.
If we are not allowing any filtering operation then we may observe that the two-qubit state obtained either through the Von Neumann measurement or through the application of partial trace operation, may or may not be useful as a resource state in quantum teleportation. In this scenario, the controlled teleportation scheme may be helpful in the sense that by controlling the measurement parameter, the controller may

be able to increase the teleportation fidelity in the conventional teleportation scheme. Therefore, the enhancement of the teleportation fidelity may be measured by a quantity known as the controller's power ($P_{CT}^{(k)}$) of the controlled quantum teleportation. It may be defined as the difference between the conditioned fidelity ($f_C(\rho_{AB}^{(k)})$) and the non-conditioned fidelity ($f_{NC}(\rho_{AB})$)

$$P_{CT}^{(k)} = f_C(\rho_{AB}^{(k)}) - f_{NC}(\rho_{AB}), \quad k = 0, 1 \tag{5.1.1}$$

In the controlled quantum teleportation scheme, there are two basic assumptions: (i) $f_C(\rho_{AB}^{(k)}) > \frac{2}{3}$ and (ii) $f_{NC}(\rho_{AB}) \leq \frac{2}{3}$. If these two conditions are satisfied by any three-qubit states then we say that the given three-qubit state is useful in the CQT scheme.

5.2 Witness operators

In this section, our task is to construct a witness operator and study the relationship between the expected value of the constructed witness operator and the Bell-CHSH inequality. We have shown that the constructed witness operator may detect the two-qubit entangled state even when Bell-CHSH inequality is unable to detect it. Moreover, we find that the two-qubit entangled states, which are not detected by Bell-CHSH inequality but detected by the witness operator, are useful for teleportation. In general, it has already been shown in the literature [97] that any two-qubit state described by the density matrix ρ_{AB} violates CHSH inequality if and only if $M(\rho_{AB}) > 1$, where the quantity $M(\rho_{AB})$ is defined in (1.8.1).

5.2.1 Construction of witness operator $W_{ij}^{(1)}$

To start with, let us first recall different Bell-CHSH operator defined in $xy-$, $yz-$, and $zx-$ plane, which are collectively denoted as $B_{CHSH}^{(ij)}$ $(i, j = x, y, z; i \neq j)$ and it is given by [227]

$$B_{CHSH}^{(ij)} = \sigma_i \otimes \frac{\sigma_i + \sigma_j}{\sqrt{2}} + \sigma_i \otimes \frac{\sigma_i - \sigma_j}{\sqrt{2}} + \sigma_j \otimes \frac{\sigma_i + \sigma_j}{\sqrt{2}} - \sigma_j \otimes \frac{\sigma_i - \sigma_j}{\sqrt{2}} \tag{5.2.1}$$

Afterward, we will use the short form B_{ij} instead of using the long form $B_{CHSH}^{(ij)}$ throughout the chapter. The four Bell states in the computational basis are denoted by $|\phi^{\pm}\rangle_{AB}$, $|\psi^{\pm}\rangle_{AB}$ and can be expressed as (1.10.1), (1.10.2), (1.10.3),and (1.10.4).

Now we are in a position to construct the operator W_{ij} that may be expressed in the form as

$$W_{ij} = (\frac{1}{2} + 2a)I - A - aB_{ij}, i, j = x, y, z \ \& \ i \neq j \tag{5.2.2}$$

where a is a positive real number. The operator A given in (5.2.2) may take any form of two-qubit Bell states and other operators have their usual meaning. In particular, if we take $A = |\phi^+\rangle_{AB}\langle\phi^+|$ then the operator W_{ij} reduces to $W_{ij}^{(\phi^+)}$, where $W_{ij}^{(\phi^+)}$ is given by

$$W_{ij}^{(\phi^+)} = (\frac{1}{2} + 2a)I - |\phi^+\rangle_{AB}\langle\phi^+| - aB_{ij}, i, j = x, y, z \ \& \ i \neq j \tag{5.2.3}$$

Theorem 5.2.1. The operator $W_{ij}^{(\phi^+)}$ given in (5.2.3) is a witness operator.

Proof: We call the operator $W_{ij}^{(\phi^+)}$, a witness operator if it satisfies the conditions $C1$ and $C2$ given in (1.5.26) and (1.5.27) respectively.

(a) To show the validity of condition $C1$, take the operator $W_{ij}^{(\phi^+)}$ and consider an arbitrary two-qubit separable state described by the density operator σ_{sep}. The expectation value of the operator $W_{ij}^{(\phi^+)}$ with respect to σ_{sep} is given by

$$Tr[W_{ij}^{(\phi^+)}\sigma_{sep}] = (\frac{1}{2} + 2a) - \langle\phi^+|\sigma_{sep}|\phi^+\rangle - aTr[B_{ij}\sigma_{sep}] \tag{5.2.4}$$

If $F(\sigma_{sep})$ denote the singlet fraction [249] of the state σ_{sep} then we have

$$F(\sigma_{sep}) \geq \langle\phi^+|\sigma_{sep}|\phi^+\rangle \tag{5.2.5}$$

Using (5.2.5) in (5.2.4), we get

$$Tr[W_{ij}^{(\phi^+)}\sigma_{sep}] \geq (\frac{1}{2} + 2a) - F(\sigma_{sep}) - aTr[B_{ij}\sigma_{sep}] \tag{5.2.6}$$

For any separable state σ_{sep}, we have $-2 \leq Tr[B_{ij}\sigma_{sep}] \leq 2$. Thus, for $a > 0$, the inequality (5.2.6) reduces to

$$Tr[W_{ij}^{(\phi^+)}\sigma_{sep}] \geq \begin{cases} \frac{1}{2} - F(\sigma_{sep}) + 2a, & Tr[B_{ij}\sigma_{sep}] \in [-2, 0] \\ \frac{1}{2} - F(\sigma_{sep}), & Tr[B_{ij}\sigma_{sep}] \in [0, 2] \end{cases} \tag{5.2.7}$$

Since the singlet fraction of any separable state σ_{sep} satisfies the inequality $F(\sigma_{sep}) \leq \frac{1}{2}$ so for any $a > 0$ and for any separable state σ_{sep}, we have $Tr[W_{ij}^{(\phi^+)}\sigma_{sep}] \geq 0$. Hence $C1$ is verified.

(b) To prove the validity of the condition $C2$, it is enough to show that there exists an entangled state σ_{ent} for which $Tr[W_{ij}^{(\phi^+)}\sigma_{ent}] < 0$. For this, let us consider an entangled state $\sigma_{ent}^{(p)}$, which may be defined as [194]

$$\sigma_{ent}^{(p)} = p|\phi^+\rangle_{AB}\langle\phi^+| + \frac{1-p}{4}I, \ \ \frac{1}{3} < p \leq 1 \tag{5.2.8}$$

Let us now consider the operator B_{yz} which is defined in (5.2.1). In the interval $\frac{1}{3} < p \leq 1$, we find that the state $\sigma_{ent}^{(p)}$ satisfy the Bell-CHSH inequality using the operator B_{yz}, i.e.,

$$Tr[B_{yz}\sigma_{ent}^{(p)}] = 0, \ \ \frac{1}{3} < p \leq 1 \tag{5.2.9}$$

Here, we observe that the entangled state $\sigma_{ent}^{(p)}$ is not detected by the operator B_{yz}. The operator $W_{yz}^{(\phi^+)}$ may be expressed as

$$W_{yz}^{(\phi^+)} = (\frac{1}{2}+2a)I - |\phi^+\rangle_{AB}\langle\phi^+| - aB_{yz}, \ \ a > 0 \tag{5.2.10}$$

The expectation value of the operator $W_{yz}^{(\phi^+)}$ with respect to the state $\sigma_{ent}^{(p)}$ is given by

$$\begin{aligned} Tr[W_{yz}^{(\phi^+)}\sigma_{ent}^{(p)}] &= \frac{1}{2}+2a-\langle\phi^+|\sigma_{ent}^{(p)}|\phi^+\rangle - aTr[B_{yz}\sigma_{ent}^{(p)}] \\ &= \frac{1}{2}+2a-\frac{1+3p}{4} \\ &= \frac{1-3p}{4}+2a<0, \ a\in[0,0.001], \ \frac{1}{3}<p\leq 1 \end{aligned}$$

Thus, there exist an entangled state $\sigma_{ent}^{(p)}$ for which $Tr[W_{yz}^{(\phi^+)}\sigma_{ent}^{(p)}] < 0$. Therefore, $C2$ is verified.

Thus, we can now say that the operator $W_{yz}^{(\phi^+)}$ may serve as a valid entanglement witness operator. Similarly, it can be shown that there exists a finite range of the parameter a for which $Tr[W_{xy}^{(\phi^+)}\sigma_{ent}^{(p)}] < 0$ and $Tr[W_{zx}^{(\phi^+)}\sigma_{ent}^{(p)}] < 0$. Hence, the operator $W_{ij}^{(\phi^+)}$ for any $i,j = x,y,z; i \neq j$ is a witness operator. ■

Moreover, if we replace the operator A by other Bell states like $|\phi^-\rangle_{AB}\langle\phi^-|$ or $|\psi^\pm\rangle_{AB}\langle\psi^\pm|$ then it can be shown that the corresponding operators $W_{ij}^{(\phi^-)}$ or $W_{ij}^{(\psi^\pm)}$ may serve as witness operator for any $i,j = x,y,z; i \neq j$. Therefore, the witness operators $W_{ij}^{(\phi^-)}$, $W_{ij}^{(\psi^\pm)}$ may be expressed in the following way:

$$W_{ij}^{(\phi^-)} = (\frac{1}{2}+2a)I - |\phi^-\rangle_{AB}\langle\phi^-| - aB_{ij}, i,j = x,y,z \ \ i \neq j \tag{5.2.11}$$

$$W_{ij}^{(\psi^\pm)} = (\frac{1}{2}+2a)I - |\psi^\pm\rangle_{AB}\langle\psi^\pm| - aB_{ij}, i,j = x,y,z \ \ i \neq j \tag{5.2.12}$$

5.2.2 Characteristic of the introduced witness operator

In this section, we may take into account the Bell state $|\phi^+\rangle_{AB}$ and then discuss the relation between the three quantities such as (i) $M(\rho_{AB})$, which determine whether the quantum state violating the Bell-CHSH inequality (ii) $F(\rho_{AB})$ denoting the singlet fraction of the state ρ_{AB} that determine whether the state is useful as a resource state in quantum teleportation [127] and (iii) the expectation value of the witness operator $W_{ij}^{(\phi^+)}$ that detect the signature of the entanglement. Specifically, we derived here the lower and upper bound of the expectation value of the witness operator $W_{ij}^{(\phi^+)}$. Using these bounds, we have obtained a few results that focus on the condition for which the witness operator may or may not detect the entangled state. Furthermore, we note that all the results obtained by considering the operator $|\phi^+\rangle_{AB}\langle\phi^+|$ may also be obtained by considering the other three Bell operators such as $|\phi^-\rangle_{AB}\langle\phi^-|$, $|\psi^\pm\rangle_{AB}\langle\psi^\pm|$.

Result 5.2.1. Consider an entangled state ρ_{ent} such that $M(\rho_{ent}) \leq 1$. Then the lower and upper bound of the expectation value of the witness operator $W_{ij}^{(\phi^+)}$ with respect to an entangled state ρ_{ent} is given by

$$U(a) \leq Tr[W_{ij}^{(\phi^+)}\rho_{ent}] \leq L(a), \quad a > 0 \tag{5.2.13}$$

where $U(a) = \frac{1}{2} - F(\rho_{ent}) + 2a(1 - \sqrt{M(\rho_{ent})})$ and $L(a) = \frac{1}{2} - \langle\phi^+|\rho_{ent}|\phi^+\rangle + 4a$.

Proof: To derive the required lower bound of the expectation value of the witness operator $W_{ij}^{(\phi^+)}$, let us recall the witness operator defined in (5.2.3). The expectation value of $W_{ij}^{(\phi^+)}$ with respect to an entangled state ρ_{ent}, is given by

$$\begin{aligned} Tr[W_{ij}^{(\phi^+)}\rho_{ent}] &= (\frac{1}{2} + 2a) - \langle\phi^+|\rho_{ent}|\phi^+\rangle - aTr[B_{ij}\rho_{ent}] \\ &\geq (\frac{1}{2} + 2a) - F(\rho_{ent}) - aTr[B_{ij}\rho_{ent}] \\ &\geq (\frac{1}{2} - F(\rho_{ent})) + 2a(1 - \sqrt{M(\rho_{ent})}) \end{aligned} \tag{5.2.14}$$

In the second step, we have used $\langle\phi^+|\rho_{ent}|\phi^+\rangle \leq F(\rho_{ent})$. In the third step, we use the following $Tr[B_{ij}\rho_{ent}] = \langle B_{ij}\rangle_{\rho_{ent}} \leq max_{B_{ij}}\langle B_{ij}\rangle_{\rho_{ent}} = 2\sqrt{M(\rho_{ent})}$ for any (i, j), where $i, j = x, y, z; i \neq j$ [97].

Let us now derive the upper bound of the expectation value of the witness operator $W_{ij}^{(\phi^+)}$. Again, the expectation value of $Tr[W_{ij}^{(\phi^+)}\rho_{ent}]$ can be expressed as

$$Tr[W_{ij}^{(\phi^+)}\rho_{ent}] = \frac{1}{2} - \langle\phi^+|\rho_{ent}|\phi^+\rangle + a(2 - Tr[B_{ij}\rho_{ent}]) \quad (5.2.15)$$

Let us assume that the two-qubit entangled state ρ_{ent} satisfies the Bell-CHSH inequality, i.e., $Tr[B_{ij}\rho_{ent}] \in [-2,2]$ for any $i, j = x, y, z; i \neq j$. If we split the interval $[-2,2]$ into two subintervals $[-2,0)$ and $[0,2]$, then we have the following two cases:
(i) If $Tr[B_{ij}\rho_{ent}] \in [0,2]$ then we get

$$Tr[W_{ij}^{(\phi^+)}\rho_{ent}] \leq \frac{1}{2} - \langle\phi^+|\rho_{ent}|\phi^+\rangle + 2a \quad (5.2.16)$$

(ii) If $Tr[B_{ij}\rho_{ent}] \in [-2,0]$, we get

$$Tr[W_{ij}^{(\phi^+)}\rho_{ent}] \leq \frac{1}{2} - \langle\phi^+|\rho_{ent}|\phi^+\rangle + 4a \quad (5.2.17)$$

Thus, combining (5.2.16) and (5.2.17) and since $a > 0$, we get

$$Tr[W_{ij}^{(\phi^+)}\rho_{ent}] \leq \frac{1}{2} - \langle\phi^+|\rho_{ent}|\phi^+\rangle + 4a \quad (5.2.18)$$

Hence, if $M(\rho_{ent}) \leq 1$ then the lower and upper bound of the expectation value of the witness operator $W_{ij}^{(\phi^+)}$ is given by

$$(\frac{1}{2} - F(\rho_{ent})) + 2a(1 - \sqrt{M(\rho_{ent})}) \leq Tr[W_{ij}^{(\phi^+)}\rho_{ent}] \leq \frac{1}{2} - \langle\phi^+|\rho_{ent}|\phi^+\rangle + 4a, \ \ a > 0 \quad \blacksquare$$

The inequality (5.2.13) estimates the lower and upper bound of the expectation value of the witness operator $W_{ij}^{(\phi^+)}$ with respect to any two-qubit entangled state. Further, we note that the lower bound of $Tr[W_{ij}^{(\phi^+)}\rho_{ent}]$ depends on two quantities such as (i) $F(\rho_{ent})$ and (ii) $M(\rho_{ent})$. Based on these two quantities, we can make the following observations from the inequality (5.2.13):
Observation 1: If there exist any two-qubit entangled state ρ_{ent} such that $M(\rho_{ent}) \leq 1$ and $F(\rho_{ent}) \leq \frac{1}{2}$ then it is clear from (5.2.13) that the witness operator $W_{ij}^{(\phi^+)}$ cannot detect the entangled state ρ_{ent}.
This observation may be illustrated by the following example: Let us consider the two-qubit state [129]

$$\rho_F = F|\phi^+\rangle\langle\phi^+| + (1-F)|01\rangle\langle 01|, \ \ \frac{1}{3} < F \leq \frac{1}{2} \quad (5.2.19)$$

where F denotes the singlet fraction of the state. One may easily verify that the state ρ_F is an entangled state when $\frac{1}{3} < F \leq \frac{1}{2}$.
The expectation value of the Bell operators B_{xy}, B_{yz} and B_{xz} in different setting with respect to the state ρ_F is given by

$$\langle B_{xy}\rangle_{\rho_F} = 0,\ \langle B_{yz}\rangle_{\rho_F} \in (-0.9428, -0.7071),\ \langle B_{xz}\rangle_{\rho_F} \in (0, 0.707107) \qquad (5.2.20)$$

Therefore, using (5.2.20), we can find that the state ρ_F satisfies the Bell-CHSH inequality i.e. $M(\rho_F) \leq 1$. Let us now calculate the expectation value of the corresponding witness operators $W_{xy}^{(\phi^+)}$, $W_{yz}^{(\phi^+)}$, and $W_{zx}^{(\phi^+)}$ with respect to the state ρ_F. For positive 'a', the expectation values are given by

$$Tr[W_{xy}^{(\phi^+)}\rho_F] = \frac{1}{2} + 2a - F > 0 \qquad (5.2.21)$$

$$Tr[W_{yz}^{(\phi^+)}\rho_F] = (\frac{1}{2} - F) + a[2 + \sqrt{2}(1 - F)] > 0 \qquad (5.2.22)$$

$$Tr[W_{zx}^{(\phi^+)}\rho_F] = (\frac{1}{2} - F) + a[2 + \sqrt{2}(1 - 3F)] > 0 \qquad (5.2.23)$$

Thus, it is clear from (5.2.21), (5.2.22) and (5.2.23) that the entangled state ρ_F,is not detected by the witness operator $W_{xy}^{(\phi^+)}$. The observation-1 is now verified for a particular quantum state described by the density operator ρ_F.
But, in general, from the inequality (5.2.13) we can conclude that if any quantum entangled state ρ_{ent} satisfies $M(\rho_{ent}) \leq 1$ and $F(\rho_{ent}) \leq \frac{1}{2}$ then the witness operator $W_{ij}^{(\phi^+)}$, $i, j = x, y, z,\ i \neq j$ does not detect the entangled state ρ_{ent}.
Observation 2: If there exist any two-qubit entangled state ρ_{ent} which is useful in teleportation i.e. $F(\rho_{ent}) > \frac{1}{2}$ then the witness operator $W_{ij}^{(\phi^+)}$ may detect the entangled state ρ_{ent} when the parameter a lies in some specific range. This observation may be written in the form of another result, which is stated below:

Result 5.2.2. Let us consider a two-qubit entangled state described by a density operator ρ_{ent}. If $F(\rho_{ent}) > \frac{1}{2}$ and if the parameter a lies in the range $0 < a \leq \frac{\langle\phi^+|\rho_{ent}|\phi^+\rangle - \frac{1}{2}}{4}$ then the witness operator $W_{ij}^{(\phi^+)}$ detect the entangled state ρ_{ent}.
Proof: The expectation value of the witness operator $W_{ij}^{(\phi^+)}$ $(i, j = x, y, z; i \neq j)$ with respect to the entangled state ρ_{ent} can be written as

$$Tr[W_{ij}^{(\phi^+)}\rho_{ent}] = \frac{1}{2} - \langle\phi^+|\rho_{ent}|\phi^+\rangle + a(2 - Tr[B_{ij}\rho_{ent}]) \qquad (5.2.24)$$

Using (5.2.13), it can be easily shown that if $F(\rho_{ent}) > \frac{1}{2}$ and if whether the state ρ_{ent} satisfies the Bell-CHSH inequality or violate it, the upper and lower bound of $Tr[W_{ij}^{(\phi^+)}\rho_{ent}]$ will be a negative quantity. Thus, we have

$$Tr[W_{ij}^{(\phi^+)}\rho_{ent}] = \text{a negative quantity} \quad (5.2.25)$$

Therefore, (5.2.25) clearly indicate the fact that the witness operator $W_{ij}^{(\phi^+)}$ $(i,j=x,y,z;i\neq j)$ detect the entangled state ρ_{ent}. Hence proved. ■

We will now verify Result 5.2.2 by considering the Bell state $|\psi^-\rangle_{AB}$ instead of taking $|\phi^+\rangle_{AB}$. To verify Result 5.2.2, let us consider the two-qubit state described by the density operator $\rho(\theta)$

$$\rho(\theta) = \frac{1}{2}\begin{pmatrix} a(\theta) & 0 & 0 & 0 \\ 0 & b(\theta) & c(\theta) & 0 \\ 0 & c(\theta) & d(\theta) & 0 \\ 0 & 0 & 0 & e(\theta) \end{pmatrix}, \; 0 \leq \theta \leq 0.4175\pi \quad (5.2.26)$$

where $a(\theta) = (3-2\sqrt{2})Sin^2\theta,\; b(\theta) = (3-2\sqrt{2})Cos^2\theta,\; c(\theta) = (1-\sqrt{2})Cos\theta,\; d(\theta) = 1+(2\sqrt{2}-2)Sin^2\theta$ and $e(\theta) = (2\sqrt{2}-2)Cos^2\theta$.

It can be easily verified that $\rho(\theta)$ is an entangled state and $M(\rho(\theta)) < 1$ for $\theta \in [0, 0.4175\pi]$. Thus, the entangled state $\rho(\theta)$ will satisfy Bell-CHSH inequality for $\theta \in [0, 0.4175\pi]$, and thus it is undetected by the Bell-CHSH operator. Further, the singlet fraction of $\rho(\theta)$, i.e., $F(\rho(\theta))$ can be calculated as

$$F(\rho(\theta)) = \frac{1}{8}(3+4(-1+\sqrt{2})Cos\theta + (5-4\sqrt{2})Cos(2\theta))$$

We can verify that $F(\rho(\theta)) > \frac{1}{2}$ when $\theta \in [0, 0.4175\pi]$ and $a \in (0, 0.00560188]$.

By direct calculation, we obtain the value of the following expressions in terms of the state parameter θ as

$$\begin{aligned} \langle B_{xy}\rangle_{\rho(\theta)} &= 2(-2+\sqrt{2})Cos\theta \\ \langle B_{yz}\rangle_{\rho(\theta)} = \langle B_{xz}\rangle_{\rho(\theta)} &= \frac{1}{\sqrt{2}}(-1-2(-1+\sqrt{2})Cos\theta + (-5+4\sqrt{2})Cos2\theta) \\ Tr[\rho(\theta)|\psi^-\rangle\langle\psi^-|] &= \frac{1}{8}(3+4(-1+\sqrt{2})Cos\theta + (5-4\sqrt{2})Cos(2\theta)) \end{aligned} \quad (5.2.27)$$

We are now in a position to calculate the expectation value of the witness operator $W_{ij}^{(\psi^-)}$ with respect to the state $\rho(\theta)$. It is given by

$$Tr[W_{xy}^{(\psi^-)}\rho(\theta)] = \frac{1}{8}(1+16a-4(-1+\sqrt{2}+4(-2+\sqrt{2})a)Cos\theta+(-5+4\sqrt{2})Cos2\theta)$$
$$Tr[W_{yz}^{(\psi^-)}\rho(\theta)] = \frac{1}{8}(1+16a+4\sqrt{2}a-4(-1+\sqrt{2}+2(-2+\sqrt{2})a)Cos\theta+(-5+4\sqrt{2}+4(-8+5\sqrt{2})a)Cos2\theta)$$
$$Tr[W_{xz}^{(\psi^-)}\rho(\theta)] = Tr[W_{yz}^{(\psi^-)}\rho(\theta)] \quad (5.2.28)$$

We find that the witness operator $W_{xy}^{(\psi^-)}$ detect the state $\rho(\theta)$ when $a \in (0, 0.00032]$ & $\theta \in [0, 0.4175\pi]$. We also find that witness operator $W_{yz}^{(\psi^-)}$ & $W_{xz}^{(\psi^-)}$ detect the state $\rho(\theta)$ when $a \in (0, 0.00016]$ & $0 \leq \theta < 0.4175\pi$. Therefore, there exist a range of the parameter a for which the entangled state $\rho(\theta)$ is detected by the witness operator $W_{ij}^{(\psi^-)}$ when $i, j = x, y, z; i \neq j$.

Now, we are in a position to derive the non-trivial lower bound of the teleportation fidelity when ρ_{ent} is used as a resource state in quantum teleportation. It may be expressed in terms of the expectation value of the witness operator and M(ρ_{ent}).

Result 5.2.3. If there exists an entangled state described by the density operator ρ_{ent}, which satisfies the Bell-CHSH inequality but detected by the witness operator $W_{ij}^{(\phi^+)}$, then the entangled state ρ_{ent} is useful in teleportation with teleportation fidelity $f(\rho_{ent})$, which satisfies the inequality

$$f(\rho_{ent}) \geq \frac{2}{3}\{1 - Tr[W_{ij}^{(\phi^+)}\rho_{ent}] + 2a(1-\sqrt{M(\rho_{ent})})\} \quad (5.2.29)$$

where $a \in (0, \frac{\frac{1}{2}+Tr[W_{ij}^{(\phi^+)}\rho_{ent}]}{2(1-\sqrt{M(\rho_{ent})})}]$.

Proof: Let us start with the lower bound of the expectation value of the witness operator $W_{ij}^{(\phi^+)}$ $(i, j = x, y, z; i \neq j)$. Therefore, the inequality (5.2.14) can be re-expressed as

$$F(\rho_{ent}) \geq \frac{1}{2} - Tr[W_{ij}^{(\phi^+)}\rho_{ent}] + 2a(1-\sqrt{M(\rho_{ent})}) \quad (5.2.30)$$

The relation between the teleportation fidelity $f(\rho_{ent})$ and singlet fraction $F(\rho_{ent})$ of an entangled state ρ_{ent} is given by [126]

$$f(\rho_{ent}) = \frac{2F(\rho_{ent})+1}{3} \quad (5.2.31)$$

Using (5.2.30) and (5.2.31), we get

$$f(\rho_{ent}) \geq \frac{2}{3}\{1 - Tr[W_{ij}^{(\phi^+)}\rho_{ent}] + 2a(1 - \sqrt{M(\rho_{ent})})\} \tag{5.2.32}$$

Using the fact that $M(\rho_{ent}) \leq 1$ and the witness operator $W_{ij}^{(\phi^+)}$ detect the entangled state ρ_{ent}, it can be easily verified that $f(\rho_{ent}) > \frac{2}{3}$. Further, imposing the condition that $f(\rho_{ent}) \leq 1$, we can obtain the upper bound of the parameter a, which is given by

$$a \leq \frac{\frac{1}{2} + Tr[W_{ij}^{(\phi^+)}(\rho_{ent})]}{2(1 - \sqrt{M(\rho_{ent})})} \tag{5.2.33}$$

Therefore, the interval of the parameter a for which the entangled state ρ_{ent} satisfies the inequality $M(\rho_{ent}) \leq 1$ and useful for teleportation is given by

$$a \in \left(0, \frac{\frac{1}{2} + Tr[W_{ij}^{(\phi^+)}(\rho_{ent})]}{2(1 - \sqrt{M(\rho_{ent})})}\right] \qquad \blacksquare$$

5.3 Estimation of controller's power

We have assumed here that the controlled teleportation scheme involves three parties namely Alice (A), Bob (B), and Charlie (C), who have shared a three-qubit state. In this protocol, the measurement is performed by Charlie (acting as a controller) on his qubit. As a result of the measurement, the two-qubit state will be shared between Alice and Bob described by the density operator ρ_{AB}. The shared state ρ_{AB} may or may not violate the Bell-CHSH inequality and accordingly the state may or may not be useful in the conventional teleportation scheme [127]. Therefore, the study of the violation of Bell-CHSH inequality is important in this scenario and thus we consider it here as the CHSH game [224]. In the CHSH game, we assume that the two distant players, Alice (A) and Bob (B) receive binary questions $s,t \in \{0,1\}$ respectively, and similarly their answers $a,b \in \{0,1\}$ are single bits. Alice and Bob win the CHSH game if their answers satisfy $a \oplus b = st$. Thus, the CHSH game can be considered as a particular example of XOR games. In this game, the non-locality of the shared state ρ_{AB} may be determined when Alice and Bob perform measurements on their respective qubit and the outcomes of their measurements are correlated. Therefore, the maximum probability P_{ij} of winning the game overall strategy is given by (3.2.6). Since the maximum probability of winning the game depends on the expectation value of the

Bell operator B_{ij}, so P^{max} is somehow related to the non-locality of the state ρ_{AB}. Adding the known fact that the state ρ_{AB} violate the Bell-CHSH inequality if $\langle B_{ij}\rangle_{\rho_{AB}} > 2$ and thus, we find that the state ρ_{AB} is non-local when $P^{max} > \frac{3}{4}$. Hence, the shared state ρ_{AB} may be useful for teleportation when $P^{max} > \frac{3}{4}$.

It may be easily shown that the winning probability P_{ij} may be estimated in terms of the expectation value of the witness operator W_{ij} with respect to the state ρ_{AB}. The result may be stated as

Lemma 5.3.1. The probability P_{ij} of the CHSH game may be estimated as

(i) When $F(\rho_{AB}) \leq \frac{1}{2}$

$$\frac{3}{4} - \frac{Tr[W_{ij}^{(\phi^+)}\rho_{AB}]}{8a} \leq P_{ij} \leq 1 \tag{5.3.1}$$

(i) When $F(\rho_{AB}) > \frac{1}{2}$

$$0 \leq P_{ij} < \frac{3}{4} - \frac{Tr[W_{ij}^{(\phi^+)}\rho_{AB}]}{8a} \tag{5.3.2}$$

where $F(\rho_{AB})$ denote the singlet fraction of the state ρ_{AB}.

Proof: Let us recall the witness operator $W_{ij}^{(\phi^+)}$ given in (5.2.3). Therefore, Using (5.2.3) and (3.2.6), the expression for P_{ij} may be re-written as

$$P_{ij} = \frac{3}{4} + \frac{1}{8a}[\frac{1}{2} - \langle\phi^+|\rho_{AB}|\phi^+\rangle - Tr[W_{ij}^{(\phi^+)}\rho_{AB}]] \tag{5.3.3}$$

Using the fact that $\langle\phi^+|\rho_{AB}|\phi^+\rangle \leq F(\rho_{AB})$ and considering the two different cases i) $F(\rho_{AB}) \leq \frac{1}{2}$ and ii) $F(\rho_{AB}) > \frac{1}{2}$ separately, we can easily obtain the above estimation given in (5.3.1) and (5.3.2). The above result may be proved in the same way for $W_{ij}^{(\phi^-)}$, $W_{ij}^{(\psi^+)}$ and $W_{ij}^{(\psi^-)}$. ■

5.3.1 Estimation of non-conditioned teleportation fidelity

Let us suppose that the three-qubit state shared between Alice (A), Bob (B), and Charlie (C) is described by the density operator ρ_{ABC}. The reduced two-qubit state shared between Alice and Bob is described by the density operator $\rho_{AB} = Tr_C(\rho_{ABC})$. If ρ_{AB} is used as a resource state in quantum teleportation then the faithfulness of the teleportation is determined by the non-conditioned teleportation fidelity which is denoted by $f_{NC}(\rho_{AB})$. The non-conditioned fidelity can be expressed in terms of the correlation tensor T_{AB} as [127]

$$f_{NC}(\rho_{AB}) = \frac{3+||T_{AB}||_1}{6} \tag{5.3.4}$$

where $||.||_1$ denote the trace norm.
To express $f_{NC}(\rho_{AB})$ in terms of witness operator, we recall the expression of P_{ij} given in (5.3.3). It may be re-written as

$$P_{ij} = \frac{3}{4} + \frac{1}{8a}(\frac{1}{2} - \langle\phi^+|\rho_{AB}|\phi^+\rangle - Tr[W_{ij}^{(\phi^+)}\rho_{AB}])$$

Using $\langle\phi^+|\rho_{AB}|\phi^+\rangle \leq F(\rho_{AB})$ in the expression of P_{ij}, we get the inequality as

$$Tr[W_{ij}^{(\phi^+)}\rho_{AB}] \geq 8a(\frac{3}{4} - P_{ij}) + \frac{1}{2} - F(\rho_{AB}) \tag{5.3.5}$$

One of the assumptions to execute the controlled quantum teleportation scheme is that the non-conditioned teleportation fidelity must be less than $\frac{2}{3}$. Thus, considering $F(\rho_{AB}) \leq \frac{1}{2}$ and $P_{ij} \leq \frac{3}{4}$, we get

$$\frac{1}{2} - Tr[W_{ij}^{(\phi^+)}\rho_{AB}] \leq F(\rho_{AB}) \leq \frac{1}{2} \tag{5.3.6}$$

Using the relation between singlet fraction $(F(\rho_{AB}))$ and non-conditioned teleportation fidelity $(f_{NC}(\rho_{AB}))$, the inequality (5.3.6) may be expressed in terms of $f_{NC}(\rho_{AB})$. Therefore, the inequality (5.3.6) may be re-expressed as

$$\frac{2}{3}(1 - Tr[W_{ij}^{(\phi^+)}\rho_{AB}]) \leq f_{NC}(\rho_{AB}) \leq \frac{2}{3} \tag{5.3.7}$$

While constructing the witness operator $W_{ij}^{(\phi^+)}$, we should be careful in choosing the positive value of the parameter a. The value of a is choosen in such a way that $Tr[W_{ij}^{(\phi^+)}\rho_{AB}] \geq 0$.

5.3.2 Estimation of the conditioned teleportation fidelity

In the controlled teleportation protocol, when the controller Charlie measures on his qubit, the three-qubit state ρ_{ABC} reduces to $\rho_{AB}^{(k)}$ according to the measurement outcome $k = 0, 1$. If Alice and Bob use the shared state $\rho_{AB}^{(k)}$ as a resource state in the teleportation protocol then the fidelity of the teleportation may be termed as conditioned teleportation fidelity and it is denoted by $f_C(\rho_{AB}^{(k)})$. There is an interesting relationship between the conditioned teleportation fidelity and the partial tangle τ_{AB} and it is given by [47]

$$f_C(\rho_{AB}^{(k)}) = \frac{2+\tau_{AB}^{(k)}}{3}, \quad k=0,1 \tag{5.3.8}$$

To implement the controlled quantum teleportation, it is assumed that $f_C(\rho_{AB}^{(k)}) > \frac{2}{3}$ [167, 169–171]. Therefore, the conditioned teleportation fidelity $f_C(\rho_{AB}^{(k)})$ may be estimated by using the Result 5.2.3

$$\frac{2}{3}\{1 - Tr[W_{ij}^{(\phi^+)}\rho_{AB}^{(k)}] + 2a(1-\sqrt{M(\rho_{AB}^{(k)})})\} \leq f_C(\rho_{AB}^{(k)}) \tag{5.3.9}$$

The condition of controlled teleportation will be met when the witness operator detects the entangled state $\rho_{AB}^{(k)}$ that satisfies the Bell-CHSH inequality. The value of the parameter a involved in the witness operator will be chosen in such a way that the witness operator detects $\rho_{AB}^{(k)}$.

5.3.3 Lower and upper bound of the controller's power

The power of the controlled quantum teleportation for the k^{th} measurement outcome may be defined as

$$P_{CT}^{(k)} = f_C(\rho_{AB}^{(k)}) - f_{NC}(\rho_{AB}), \quad k=0,1 \tag{5.3.10}$$

Using (5.3.4) and (5.3.8), the expression of the power given in (5.3.10) reduces to

$$P_{CT}^{(k)} = \frac{1}{6} + \frac{1}{6}(2\tau_{AB}^{(k)} - ||T_{AB}||_1) \tag{5.3.11}$$

Our task is now to estimate the value of $||T_{AB}||_1$ and $\tau_{AB}^{(k)}$.

(i) Estimation of $||T_{AB}||_1$: Let us recall (5.3.7) and using (5.3.4) in it, we get the estimation of $||T_{AB}||_1$ which is given by

$$1 - 4Tr[W_{ij}^{NC}\rho_{AB}] \leq ||T_{AB}||_1 \leq 1 \tag{5.3.12}$$

In this case, the parameter a is chosen in such a way that the witness operator W_{ij}^{NC} does not detect the state ρ_{AB}. Therefore, we can put the restriction on $Tr[W_{ij}^{NC}\rho_{AB}]$ as

$$0 \leq Tr[W_{ij}^{NC}\rho_{AB}] \leq \frac{1}{4} \tag{5.3.13}$$

The upper bound of $Tr[W_{ij}^{NC}\rho_{AB}]$ is obtained by using the condition $||T_{AB}||_1 \geq 0$.

(ii) Estimation of $\tau_{AB}^{(k)}$: Using (5.3.8) and (5.3.9) and simplifying the inequality, we get

$$4a(1-\sqrt{M(\rho_{AB}^{(k)})})-2Tr[W_{ij}^{C}\rho_{AB}^{(k)}] \leq \tau_{AB}^{(k)} \leq 1 \tag{5.3.14}$$

The parameter a in the LHS of the above inequality (5.3.14) can be chosen in such a way that the witness operator W_{ij}^{C} detects the state ρ_{AB}^{k}.

Now, we are in a position to derive the lower and upper bound of the power $P_{CT}^{(k)}$. To start with, let us use the upper bound of $||T_{AB}||_1$ and the lower bound of $\tau_{AB}^{(k)}$ in the expression (5.3.11) of the power of the controlled teleportation. Therefore, it reduces the power given in (5.3.11) to the inequality that gives the lower bound as

$$\frac{4a}{3}(1-\sqrt{M(\rho_{AB}^{(k)})})-\frac{2}{3}Tr[W_{ij}^{(C)}\rho_{AB}^{(k)}] \leq P_{CT}^{(k)} \tag{5.3.15}$$

Similarly, using the lower bound of $||T_{AB}||_1$ and the upper bound of $\tau_{AB}^{(k)}$ in the expression of the power of the controlled teleportation, we get the upper bound of the power which is given by

$$P_{CT}^{(k)} \leq \frac{1}{3}+\frac{2}{3}Tr[W_{ij}^{NC}\rho_{AB}] \tag{5.3.16}$$

Further, if we use the restriction given in (5.3.13) then the inequality (5.3.16) reduces to

$$P_{CT}^{(k)} \leq \frac{1}{2} \tag{5.3.17}$$

Combining (5.3.15) and (5.3.17), we get

$$\frac{4a}{3}(1-\sqrt{M(\rho_{AB}^{(k)})})-\frac{2}{3}Tr[W_{ij}^{(C)}\rho_{AB}^{(k)}] \leq P_{CT}^{(k)} \leq \frac{1}{2} \tag{5.3.18}$$

5.3.4 Estimation of the lower bound of the power for pure three-qubit states

In this section, we study the controlled quantum teleportation protocol by considering the pure three-qubit states such as standard GHZ state, Maximally Slice State (MSS), and a W class of states. Then we estimate the lower bound of the power of the controller for all the above mentioned states.
Let us consider a three-qubit standard GHZ state of the form

$$|\psi^{(1)}\rangle_{CAB} = \lambda_0|000\rangle+\lambda_4|111\rangle, \;\; \lambda_0^2+\lambda_4^2=1 \tag{5.3.19}$$

Now, to execute the controlled teleportation scheme with the three-qubit state described by the density operator $\rho_{CAB}^{(1)} = |\psi^{(1)}\rangle_{CAB}\langle\psi^{(1)}|$, the assumptions on the non-conditioned fidelity and conditioned fidelity must be fulfilled. Therefore, we need to calculate the non-conditioned fidelity and conditioned fidelity and thus the power of the controller.
(i) Non-conditioned fidelity: We trace out system C from the three-qubit state $\rho_{CAB}^{(1)}$. The resulting two-qubit state $\rho_{AB}^{(1)}$ is given by

$$\rho_{AB}^{(1)} = Tr_C(\rho_{CAB}^{(1)}) = \lambda_0^2|00\rangle\langle 00|+\lambda_4^2|11\rangle\langle 11| \tag{5.3.20}$$

Using $\rho_{AB}^{(1)}$ as a resource state in quantum teleportation, the non-conditioned fidelity can be calculated as

$$f_{NC}(\rho_{AB}^{(1)}) = \frac{2}{3} \tag{5.3.21}$$

(ii) Conditioned fidelity: Charlie, performed measurement on his qubit in the single qubit generalized basis $\{B_0, B_1\}$. After the measurement, the state collapses either to $\rho_{AB}^{G(0)}$ or $\rho_{AB}^{G(1)}$, where

$$\begin{aligned}
\rho_{AB}^{G(0)} &= \frac{1}{p_0}\Big((t^2+y_3^2)\lambda_0^2|00\rangle\langle 00|+\lambda_0\lambda_4(-ty_2+y_1y_3+\iota(ty_1+y_2y_3))|00\rangle\langle 11| \\
&+ \lambda_0\lambda_4(-ty_2+y_1y_3-\iota(ty_1+y_2y_3))|11\rangle\langle 00|+\lambda_4^2(y_1^2+y_2^2)|11\rangle\langle 11|\Big), \\
&\text{where } \; p_0 = (t^2+y_3^2)\lambda_0^2+(y_1^2+y_2^2)\lambda_4^2
\end{aligned} \tag{5.3.22}$$

$$\rho_{AB}^{G(1)} = \frac{1}{p_1}\Big((y_1^2+y_2^2)\lambda_0^2|00\rangle\langle 00| + \lambda_0\lambda_4(ty_2-y_1y_3-i(ty_1+y_2y_3))|00\rangle\langle 11|$$
$$+ \lambda_0\lambda_4(ty_2-y_1y_3+i(ty_1+y_2y_3))|11\rangle\langle 00| + \lambda_4^2(t^2+y_3^2)|11\rangle\langle 11|\Big),$$
$$\text{where } p_1 = (y_1^2+y_2^2)\lambda_0^2+(y_3^2+t^2)\lambda_4^2 \quad (5.3.23)$$

Let us now use the state $\rho_{AB}^{G(0)}$ for the teleportation of a single qubit. We choose the normalized measurement parameters (y_1,y_2,y_3,t) in such a way that the conditioned fidelity is greater than $\frac{2}{3}$ i.e. $f_C(\rho_{AB}^{G(0)}) > \frac{2}{3}$ and also the normalization condition (1.4.23) holds. Therefore, choosing the measurement parameters $y_1 = -0.25$, $y_2 = -0.49$, $y_3 = 0.39$ and $t = -0.74$, we can calculate the conditioned fidelity in terms of the state parameter λ_4 as

$$f_C(\rho_{AB}^{G(0)}) = \frac{1}{1.79958-\lambda_4^2}\left(1.19972-0.666667\lambda_4^2+0.799676\lambda_4\sqrt{1-\lambda_4^2}\right) \quad (5.3.24)$$

We may observe that $f_C(\rho_{AB}^{G(0)})$ varies from $[0.66667, 0.99997]$ when λ_4 varies from $[0,1]$. Thus, the assumptions on non-conditioned fidelity and conditioned fidelity are met. It can be easily verified that these assumptions still hold if we consider the state $\rho_{AB}^{G(1)}$. This means that the GHZ state described by the density operator $\rho_{CAB}^{(1)}$ is useful for controlled quantum teleportation.

Now, our task is to calculate the power of the controller when a three-qubit GHZ state (5.3.19) is shared between Alice, Bob, and Charlie (controller). To estimate the power of the controller, we again consider the state $\rho_{AB}^{G(0)}$ and proceed toward the calculation of the lower bound of the power that needs the following information:

(i) $M(\rho_{AB}^{G(0)}) > 1$ for $\lambda_4 \in [0,1]$. This indicates that the state $\rho_{AB}^{G(0)}$ violates the Bell-CHSH inequality and therefore, the state is useful in conventional quantum teleportation [127].

(ii) The expectation value of the witness operator $W_{xy}^{(\phi^+)}$ with respect to the state $\rho_{AB}^{G(0)}$ is given by

$$Tr[W_{xy}^{(\phi^+)}(\rho_{AB}^{G(0)})] = \frac{1}{2} - 2a - \frac{0.90-0.5\lambda_4^2+1.2\lambda_4\sqrt{1-\lambda_4^2}}{1.8-\lambda_4^2} \quad (5.3.25)$$

The value of a (>0) is chosen in such a way that the witness operator $W_{xy}^{(\phi^+)}$ detects the state $\rho_{AB}^{G(0)}$. Thus, we find that when $a \in (0, 0.232)$ & $0.598 \le \lambda_4 \le 0.95$, the witness operator detect the state $\rho_{AB}^{G(0)}$. Therefore, the lower bound of the controller's power can be estimated using the formula given below:

$$P_{CT}^{G(0)} \geq \frac{4a}{3}(1-\sqrt{M(\rho_{AB}^{G(0)})})-\frac{2}{3}Tr[W_{ij}^{(\phi^+)}\rho_{AB}^{G(0)}] \quad (5.3.26)$$

It can be easily verified that the lower bound of power lies in the interval (0.001472,0.333) for $a \in (0, 0.1555)$ & $0.598 \leq \lambda_4 \leq 0.95$.

Also, we note that the calculation of the power of the controller for the state $\rho_{AB}^{G(1)}$ may be done in a similar way.

Moreover, we may consider other pure three-qubit states such as maximally slice state $|\psi^{(2)}\rangle_{ABC} = \lambda_0|000\rangle + \lambda_1|100\rangle + \frac{1}{\sqrt{2}}|111\rangle$ given in [167] and W class states $|W_n\rangle = \frac{1}{\sqrt{2+2n}}(|100\rangle + \sqrt{n}|010\rangle + \sqrt{n+1}|001\rangle)$ introduced in [140]. We have analyzed the power of the controller for these classes of states, which is given in Table 5.1.

Further, we find that the pure three-qubit W class of state described by $|W_1\rangle$ is more useful in the CQT scheme than all other W class of states such as $|W_2\rangle$, $|W_3\rangle$ etc. $|W_1\rangle$ is more useful in the CQT scheme in the sense that when $|W_1\rangle$ is used, the power of the controller is greater than all the power calculated over the states $|W_2\rangle$, $|W_3\rangle$ etc. This finding is shown in Figure 5.1.

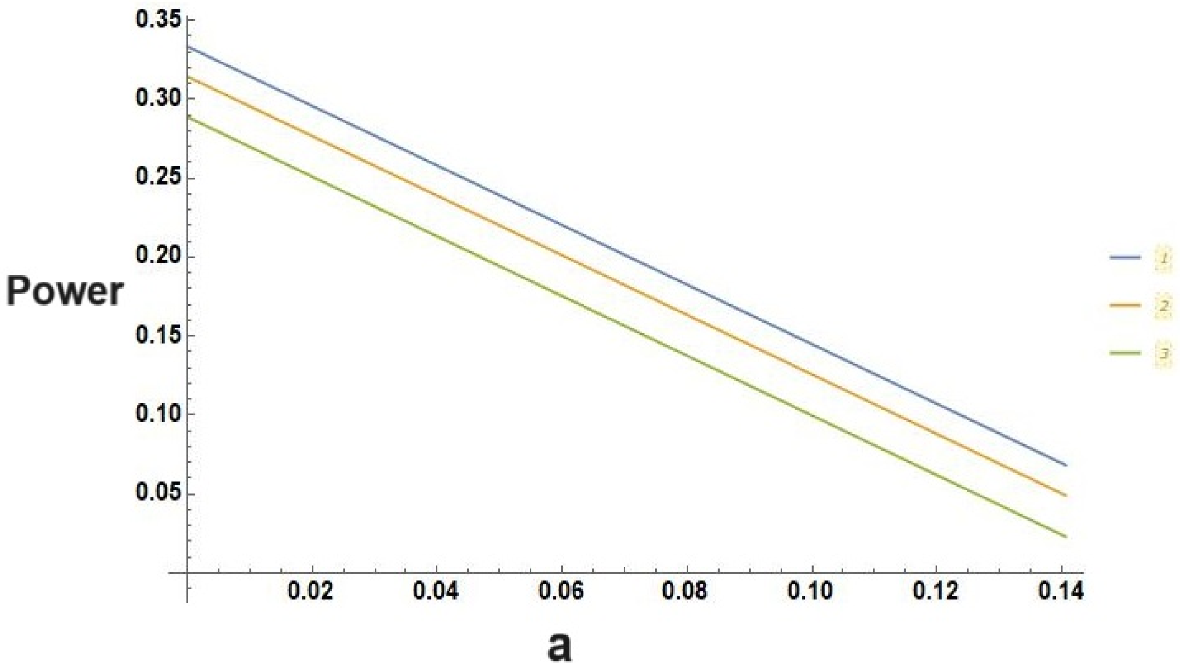

Figure 5.1: This graph shows the relation between parameter (a) of the witness operator and the lower limit of controller's power $P_{CT}^{W_1(0)}$, $P_{CT}^{W_2(0)}$, $P_{CT}^{W_3(0)}$ when the measurement is performed on one of the qubit of W_1, W_2, W_3 and the measurement outcome is $k=0$. Blue line denotes the power of W_1 state, Yellow line denotes the power of W_2 state, and Green line denotes the power of W_3 state

Three-qubit State	Non-Conditioned Fidelity	Conditioned Fidelity	Estimated lower bound of the power
$\|\psi^{(2)}\rangle$	[0.5,0.6667]	$f_C(\rho_{AB}^{MSS(0)})= f_C(\rho_{AB}^{MSS(1)})$= [0.667538,1] $\lambda_4 \in [0,0.643]$	(0,0.3333), $a \in (0,0.0402]$ $\lambda_4 \in [0,0.6]$
$\|W_1\rangle$	$\frac{2}{3}$	$f_C(\rho_{AB}^{W_1(0)})$= 0.9999999999968732	[0.00146455,0.33333), a $\in$ (0,0.1767]
		$f_C(\rho_{AB}^{W_1(1)})$=0.99999999976277	[0.00146455,0.33333), a $\in$ (0,0.1767]
$\|W_2\rangle$	0.657135	$f_C(\rho_{AB}^{W_2(0)})= f_C(\rho_{AB}^{W_2(1)})$=0.980937	[0.0001815,0.31427), a $\in$ (0,0.1665]
$\|W_3\rangle$	0.644337	$f_C(\rho_{AB}^{W_3(0)})$=$f_C(\rho_{AB}^{W_3(1)})$=0.955342	[0.000427,0.288675), a $\in$ (0,0.1525]

Table 5.1: In this table, we have estimated the lower bound of the controller's power using various three-qubit pure states such as maximally slice state $|\psi^{(2)}\rangle$, and $|W_n\rangle$, $n = 1,2,3$ states. We have found that all the three-qubit states are useful for controlled teleportation and furthermore, we obtain that $|W_1\rangle$ is more useful in controlled teleportation in comparison to $|W_2\rangle$ and $|W_3\rangle$ state.

5.4 Controlled teleportation in noisy environment

In this section, we analyze the power of controlled quantum teleportation when one of the qubit of the shared state interacts with the noisy environment. We have considered here amplitude damping channel and phase damping channel as a noisy channel for our study. As it is known that the standard W state is not useful in controlled quantum teleportation [170] so we study controlled quantum teleportation using the standard W state. Therefore, we investigate the possibility of using the standard W state in CQT protocol when one of the qubit passes through the noisy environment.
To start with, let us consider the standard W state, which is given by

$$|\psi^{(W)}\rangle_{BAC} = \frac{1}{\sqrt{3}}(|000\rangle + |101\rangle + |110\rangle) \quad (5.4.1)$$

To execute our protocol, we assume that a source generates three-qubit entangled state $\rho_{BAC}^{(W)} = |\psi^{(W)}\rangle_{BAC}\langle\psi^{(W)}|$, where $|\psi^{(W)}\rangle_{BAC}$ is given in the form (5.4.1). In this protocol, let us further assume that the two parties Alice and Charlie are in one place while Bob is residing in some distant place. Alice possesses the two qubits A and B respectively. On the other hand, Charlie has the qubit C. Since Alice would like to send some information to Bob via a shared quantum state so she needs to construct an entangled channel between them. Thus, Alice has to send a qubit (suppose, a qubit B) involved in the three-qubit entangled state $\rho_{BAC}^{(W)}$ to Bob through the noisy environment. The noisy environment may be described either as (i) Amplitude Damping Channel or (ii) Phase Damping Channel. The qubit B then interacts with the noisy environment while travelling to Bob's place and assuming that finally, it reaches to Bob. In this way, a channel is constructed between Alice and Bob through which Alice can send her information to Bob using quantum teleportation protocol. Since the qubit B has interacted with the noisy environment so there may be a possibility of the degradation of the entanglement of the established channel between Alice and Bob. Thus, the teleportation fidelity may become less than $\frac{2}{3}$. In this scenario, Charlie may play a major role as a controller to enhance the teleportation fidelity. Hence, we can calculate the power of the controller in this version of controlled teleportation.

5.4.1 Amplitude damping channel

Recalling the standard W state given in (5.4.1) and follows the above described protocol where the qubit B is interacting with the noisy environment. Let us consider first the amplitude damping channel as the noisy environment through which the qubit B is passing. Amplitude damping channel is described by the Kraus operators defined as [250]

$$K_1 = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1|, K_2 = \sqrt{p}|0\rangle\langle 1|, \quad 0 \leq p \leq 1 \tag{5.4.2}$$

The Kraus operator satisfies $K_1^\dagger K_1 + K_2^\dagger K_2 = I$. I denote the identity operator.
When the qubit B passes through the amplitude damping channel, the state $\rho_{BAC}^{(W)}$ reduces to

$$\begin{aligned} \rho_{BAC}^{(W1)} &= (K_1 \otimes I \otimes I)\rho_{BAC}^{(W)}(K_1^\dagger \otimes I \otimes I) + (K_2 \otimes I \otimes I)\rho_{BAC}^{(W)}(K_2^\dagger \otimes I \otimes I) \\ &= \frac{1}{3}(|000\rangle\langle 000| + p(|001\rangle\langle 001| + |001\rangle\langle 010| + |010\rangle\langle 001| + |010\rangle\langle 010|) \\ &+ \sqrt{1-p}(|000\rangle\langle 101| + |000\rangle\langle 110| + |101\rangle\langle 000| + |110\rangle\langle 000|) \\ &+ (1-p)(|101\rangle\langle 101| + |101\rangle\langle 110| + |110\rangle\langle 101| + |110\rangle\langle 110|)) \end{aligned} \tag{5.4.3}$$

Now, our task is to see whether the channel generated between Alice and Bob is useful for conventional teleportation. To verify this, we trace out the system C from the state described by the density operator $\rho_{BAC}^{(W1)}$. The resulting two-qubit state is then given by

$$\begin{aligned} \rho_{BA}^{(W1)} &= \frac{1}{3}(|00\rangle\langle 00| + p(|00\rangle\langle 00| + |01\rangle\langle 01|) + \sqrt{1-p}(|00\rangle\langle 11| + |11\rangle\langle 00|) \\ &+ (1-p)(|10\rangle\langle 10| + |11\rangle\langle 11|)) \end{aligned} \tag{5.4.4}$$

The non-conditioned fidelity of teleportation when $\rho_{BA}^{(W1)}$ is used as a resource state, is given by

$$f_{NC}(\rho_{BA}^{(W1)}) = \frac{5+2\sqrt{1-p}}{9} \leq \frac{2}{3}, \text{ for } \frac{3}{4} \leq p \leq 1 \quad (5.4.5)$$

Therefore, we find here that there exists a range of the noisy parameter p ($\frac{3}{4} \leq p \leq 1$) for which $f_{NC}(\rho_{BA}^{(W1)}) \leq \frac{2}{3}$. Let us recall again the three-qubit state $\rho_{BAC}^{(W1)}$. Now, Charlie performs Von Neumann measurement $\{B_k,\ k=0,1\}$ on his qubit C. According to the measurement result, the resulting two-qubit states are given by

$$\begin{aligned}
\rho_{BA}^{W1(0)} &= \frac{1}{3N_0}\Big((t^2+y_3^2)(|00\rangle\langle 00|+p|01\rangle\langle 01|+\sqrt{1-p}(|00\rangle\langle 11|+|11\rangle\langle 00|)+(1-p)|11\rangle\langle 11|) \\
&+ (-ty_2+y_1y_3+i(ty_1+y_2y_3))(p|01\rangle\langle 00|+\sqrt{1-p}|00\rangle\langle 10|+(1-p)|11\rangle\langle 10|) \\
&+ (-ty_2+y_1y_3-i(ty_1+y_2y_3))(p|00\rangle\langle 01|+\sqrt{1-p}|10\rangle\langle 00|+(1-p)|10\rangle\langle 11|) \\
&+ (y_1^2+y_2^2)(p|00\rangle\langle 00|+(1-p)|10\rangle\langle 10|)\Big) \quad (5.4.6)
\end{aligned}$$

$$\begin{aligned}
\rho_{BA}^{W1(1)} &= \frac{1}{3N_1}\Big((y_1^2+y_2^2)(|00\rangle\langle 00|+p|01\rangle\langle 01|+\sqrt{1-p}(|00\rangle\langle 11|+|11\rangle\langle 00|)+(1-p)|11\rangle\langle 11|) \\
&+ (ty_2-y_1y_3-i(ty_1+y_2y_3))(p|01\rangle\langle 00|+\sqrt{1-p}|00\rangle\langle 10|+(1-p)|11\rangle\langle 10|) \\
&+ (ty_2-y_1y_3+i(ty_1+y_2y_3))(p|00\rangle\langle 01|+\sqrt{1-p}|10\rangle\langle 00|+(1-p)|10\rangle\langle 11|) \\
&+ (t^2+y_3^2)(p|00\rangle\langle 00|+(1-p)|10\rangle\langle 10|)\Big) \quad (5.4.7)
\end{aligned}$$

where $N_0 = \frac{2(t^2+y_3^2)+(y_1^2+y_2^2)}{3}$ and $N_1 = \frac{(t^2+y_3^2)+2(y_1^2+y_2^2)}{3}$.

In the first case, we consider the two-qubit state $\rho_{BA}^{W1(0)}$ shared between Alice and Bob. We now choose the measurement parameter (t, y_1, y_2, y_3) in such a way that the conditioned fidelity of teleportation would be greater than $\frac{2}{3}$. Therefore, the measurement parameters may be chosen as

$$\begin{aligned}
&t = 0.9615239544277027,\ y_1 = -0.0000000645028702137500 4 \\
&y_2 = -0.000000029154369318260298,\ y_3 = 0.2747211110648374 \quad (5.4.8)
\end{aligned}$$

The conditioned fidelity of teleportation is then given by

$$f_C(\rho_{BA}^{W1(0)}) = 0.666667+0.333333\sqrt{1-p}-0.166667p,\ \ 0.75 \leq p \leq 0.82842 \quad (5.4.9)$$

We may observe that the conditioned fidelity $f_C(\rho_{BA}^{W1(0)})$ is greater than $\frac{2}{3}$ when $0.75 \leq p \leq 0.82842$. In all other range of the parameter p, either $f_{NC} > \frac{2}{3}$ or $f_C \leq \frac{2}{3}$. Thus, we will consider $0.75 \leq p \leq 0.82842$ where all conditions of controlled teleportation are met. In a similar way, the condition for the controlled teleportation can be studied by considering the second case when the measurement on the Charlie's qubit generates a two-qubit state described by the density operator $\rho_{AB}^{W1(1)}$. In any case, we find that both the states $\rho_{AB}^{W1(0)}$ and $\rho_{AB}^{W1(1)}$ are useful in the controlled quantum teleportation scheme.

5.4.2 Phase damping channel

The phase damping channel is described by the Kraus Operator, which may be defined as [251]

$$K_1 = \sqrt{1-p}(|0\rangle\langle 0| + |1\rangle\langle 1|),\ K_2 = \sqrt{p}|0\rangle\langle 0|,\ K_3 = \sqrt{p}|1\rangle\langle 1|,\ \ 0 \leq p \leq 1 \quad (5.4.10)$$

Let us recall the standard W state described by the density operator $\rho_{BAC}^{(W)} = |\psi^{(W)}\rangle_{BAC}\langle\psi^{(W)}|$ where $|\psi^{(W)}\rangle_{BAC}$ is given in (5.4.1) and follow the same protocol, as we did for amplitude damping channel. When a qubit B interacted with the phase damping channel, the state $\rho_{BAC}^{(W)}$ reduces to

$$\begin{aligned}
\rho_{BAC}^{(W2)} &= (K_1 \otimes I \otimes I)\rho_{BAC}^{(W)}(K_1^{\dagger} \otimes I \otimes I) + (K_2 \otimes I \otimes I)\rho_{BAC}^{(W)}(K_2^{\dagger} \otimes I \otimes I) \\
&+ (K_3 \otimes I \otimes I)\rho_{BAC}^{(W)}(K_3^{\dagger} \otimes I \otimes I) \\
&= \frac{1}{3}\big(|000\rangle\langle 000| + |101\rangle\langle 101| + |110\rangle\langle 101| + |101\rangle\langle 110| + |110\rangle\langle 110| \\
&+ |101\rangle\langle 000| + |110\rangle\langle 000| + |000\rangle\langle 101| + |000\rangle\langle 110| - p(|101\rangle\langle 000| \\
&+ |110\rangle\langle 000| + |000\rangle\langle 101| + |000\rangle\langle 110|)\big) \quad (5.4.11)
\end{aligned}$$

To verify whether the controlled teleportation scheme is applicable for the state $\rho_{BAC}^{(W2)}$, we need to calculate non-conditioned fidelity and conditioned fidelity.
(i) Non-conditioned fidelity: The non-conditioned fidelity can be calculated as

$$f_{NC}(\rho_{BA}^{(W2)}) = \frac{7-2p}{9} \leq \frac{2}{3}, \ \text{for} \ \frac{1}{2} < p \leq 1 \quad (5.4.12)$$

where $\rho_{BA}^{(W2)} = Tr_C(\rho_{BAC}^{(W2)})$.
(ii) Conditioned fidelity: To calculate it, Charlie applies the measurement on his qubit

in the basis $\{B_0, B_1\}$. According to the measurement result, the resulting two-qubit states are given by

$$\begin{aligned}\rho_{BA}^{W2(0)} &= \frac{1}{3N_2}\Big((t^2+y_3^2)(|00\rangle\langle00|+|11\rangle\langle11|+|11\rangle\langle00|+|00\rangle\langle11|-p(|11\rangle\langle00|+|00\rangle\langle11|))\\ &+ +(-ty_2+y_1y_3+\imath(ty_1+y_2y_3))(|11\rangle\langle10|+|00\rangle\langle10|-p|00\rangle\langle10|)+(-ty_2+y_1y_3\\ &- \imath(ty_1+y_2y_3))(|10\rangle\langle11|+|10\rangle\langle00|-p|10\rangle\langle00|)+(y_1^2+y_2^2)|10\rangle\langle10|\Big) \qquad (5.4.13)\end{aligned}$$

$$\begin{aligned}\rho_{BA}^{W2(1)} &= \frac{1}{3N_3}\Big((y_1^2+y_2^2)(|00\rangle\langle00|+|11\rangle\langle11|+|11\rangle\langle00|+|00\rangle\langle11|-p(|11\rangle\langle00|+|00\rangle\langle11|))\\ &+ (ty_2-y_1y_3-i(ty_1+y_2y_3))(|11\rangle\langle10|+|00\rangle\langle10|-p|00\rangle\langle10|)+(ty_2-y_1y_3\\ &+ i(ty_1+y_2y_3))(|10\rangle\langle11|+|10\rangle\langle00|-p|10\rangle\langle00|)+(t^2+y_3^2)|10\rangle\langle10|\Big) \qquad (5.4.14)\end{aligned}$$

where $N_2 = \frac{2(t^2+y_3^2)+(y_1^2+y_2^2)}{3}$ and $N_3 = \frac{(t^2+y_3^2)+2(y_1^2+y_2^2)}{3}$.
If the measurement parameters are given by

$$\begin{aligned}&t = 0.9615239543413954,\ y_1 = 0.000002698965323056848\\ &y_2 = -0.000000004258892841348826,\ y_3 = 0.2747211523762679 \qquad (5.4.15)\end{aligned}$$

then the conditioned fidelity $f_C(\rho_{BA}^{W2(0)})$ is given by

$$f_C(\rho_{BA}^{W2(0)}) = 1-0.333333p,\ \ \frac{1}{2}\leq p\leq 1 \qquad (5.4.16)$$

It may be easily verified that $f_C(\rho_{BA}^{W2(0)}) \in [0.66667, 0.833333]$ for $p \in [0.5, 1]$. Thus, the controlled teleportation protocol may be implemented using the state $\rho_{BA}^{W2(0)}$. In a similar fashion, it may be shown that the state $\rho_{BA}^{W2(1)}$ is useful in controlled teleportation.

5.4.3 Comparision analysis of the power of the controlled teleportation

In this section, we compare the power of the controlled teleportation when the standard W state given by (5.4.1) is evolved under the amplitude damping channel and phase damping channel. We will show here that the power of the controlled teleportation in the case of phase damping channel is greater than the power in the case of amplitude damping channel.

(a) Power of the controlled teleportation when standard W state is evolved under amplitude damping channel: Since we find that both the state $\rho_{AB}^{W1(0)}$ and $\rho_{AB}^{W1(1)}$ are

useful in the controlled teleportation scheme so we can consider any one of the state $\rho_{AB}^{W1(0)}$ or $\rho_{AB}^{W1(1)}$ to calculate the power of the controller. Let us consider the two-qubit state $\rho_{AB}^{W1(0)}$ for the estimation of the power of the controller. To estimate it, we need to calculate the following:
(i) The quantity $M(\rho_{AB}^{W1(0)})$ is calculated and found out to be less than one.
(ii) The expectation value of the constructed witness operator $W_{ij}^{(\phi+)}$ with respect to the state $\rho_{BA}^{W1(0)}$, which is given by

$$Tr[W_{xy}^{(\phi^+)}(\rho_{BA}^{W1(0)})] = \frac{p}{4} - \frac{\sqrt{1-p}}{2} + 2a, \quad 0.75 \leq p \leq 0.82842 \tag{5.4.17}$$

The value of $a > 0$ is chosen in such a way that the witness operator $W_{xy}^{(\phi^+)}$ detect the state $\rho_{BA}^{W1(0)}$. We find that the witness operator $W_{xy}^{(\phi^+)}$ detects the state $\rho_{BA}^{W1(0)}$ when $a \in (0, 0.005]$.
With all the above information, we can estimate the power ($P_{CT}^{W1(0)}$), which is given by

$$\frac{4a}{3}(1 - \sqrt{M(\rho_{BA}^{W1(0)})}) - \frac{2}{3}Tr[W_{ij}^{(\phi^+)}\rho_{BA}^{W1(0)}] \leq P_{CT}^{W1(0)} \leq \frac{1}{2} \tag{5.4.18}$$

We have calculated the lower limit of the power $P_{CT}^{W1(0)}$ and found that the lower limit varies in the interval $[0.0056075, 0.041667)$ when $a \in (0, 0.005]$ & $0.75 \leq p \leq 0.8164$.
(b) Power of the controlled teleportation when standard W state is evolved under phase damping channel: In this scenario also, we find that both the state $\rho_{AB}^{W2(0)}$ and $\rho_{AB}^{W2(1)}$ are useful in the controlled teleportation scheme so we can consider any one of the state $\rho_{AB}^{W2(0)}$ or $\rho_{AB}^{W2(1)}$ to calculate the power of the controller. Let us consider the two-qubit state $\rho_{AB}^{W2(0)}$ for the estimation of the power of the controller. The power of the controller can be estimated by

$$\frac{4a}{3}(1 - \sqrt{M(\rho_{BA}^{W2(0)})}) - \frac{2}{3}Tr[W_{ij}^{(\phi^+)}\rho_{BA}^{W2(0)}] \leq P_{CT}^{W2(0)} \leq \frac{1}{2} \tag{5.4.19}$$

where the expectation value of the witness operator $W_{xy}^{(\phi^+)}$ with respect to the state $\rho_{BA}^{W2(0)}$ is given by

$$Tr[W_{xy}^{(\phi^+)}\rho_{BA}^{W2(0)}] = \frac{1}{2} + 2a - (1 - 0.5p) < 0 \text{ for } a \in (0, 0.035], p \in [0.5, 0.859] \tag{5.4.20}$$

Also, the quantity M($\rho_{BA}^{W2(0)}$) can be easily calculated and found to be greater than 1. Therefore, the lower bound of power $P_{W2}^{(0)}$ lying in the interval $[0, 0.16667)$ for $a \in$

$(0, 0.005]$ & $0.5 \leq p \leq 0.859$.

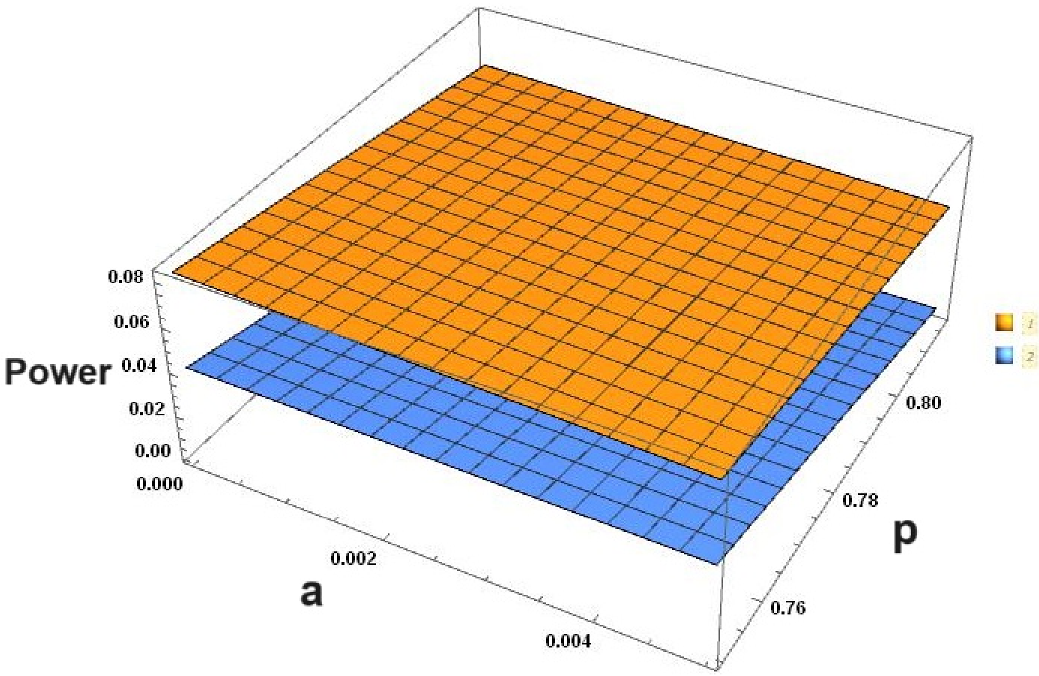

Figure 5.2: This graph shows the relationship between the witness parameter (a), noise parameter (p), and the controller's power. Yellow region indicates the controller's power of standard W state with phase damping channel and Blue region indicates the controller's power of standard W state with amplitude damping channel

Now, we are in a position to compare the estimation of the power of the controlled quantum teleportation when one of the qubit of standard W state is passing through the amplitude damping channel and phase damping channel. In the comparison, it can be clearly seen from Figure 5.2 that the controller's power is more for the standard W state when it is under phase damping channel. Though when one of the qubit of the standard W state is passing either through amplitude damping channel or phase damping channel then the resulting states are useful in controlled quantum channel but phase damping channel is more effective than amplitude damping channel.

5.5 Conclusion

To summarize, we have considered the problem of estimation of the power of the controller in the CQT scheme. To investigate it, we have constructed a witness operator and have shown that the entangled state will be useful for teleportation as a resource state if the same entangled state is detected by the constructed witness operator and if it satisfies the Bell-CHSH inequality. Thus, at least for some cases, we need not have to use the filtering operation [129] to increase the teleportation fidelity. On the other hand, the study of the violation of Bell-CHSH inequality is equally important in the CQT scheme and thus we have considered the CHSH game for the estimation of the probability of success of the game through the constructed witness operator. The estimated probability of success helps in the derivation of the lower bound of the conditioned and non-conditioned fidelity in terms of the expectation value of the witness operator. Therefore, we are now able to estimate the lower and upper bound of the power of the controller in terms of the witness operator. Thus, this can pave a way to estimate the power of the controlled teleportation in an experiment. Moreover, we have found that the state $|W_1\rangle$ is not only useful for conventional teleportation between two parties but also useful in the CQT scheme and performs better than all the other W-class of states described by $|W_n\rangle,\ n = 2,3,...$ [140]. We have also studied the CQT scheme using the standard W state under a noisy environment. We found that when one of the qubits of the standard W state passes either through the amplitude damping channel or the phase damping channel, the resulting state will be a mixed state which will be useful in controlled quantum teleportation protocol. We also observe that the phase damping channel makes the controller power more positive than the amplitude damping channel. Thus, we may conclude that the phase damping channel is more useful than the amplitude damping channel while performing the CQT protocol with the standard W state.

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Conclusion and Future scope

Conclusion

In this thesis, we have derived a different form of criteria, which is based on the maximum eigenvalue, for the detection of entangled state which is useful in quantum teleportation. Secondly, we have extensively studied the non-locality of two-qubit entangled states and also we have connected the non-locality of two-qubit quantum states with controlled quantum teleportation. Moreover, we have studied the non-locality problem for three-qubit system. To achieve the aim, we considered the Svetlichny operator and derived state dependent upper and lower bound of the Svetlichny operator. We have shown that using the derived bounds, one can detect the non-locality of any general three-qubit quantum state. Further, we have detected the non-locality existed in those three-qubit states which were not detected earlier. Finally, in the CQT protocol, we have expressed the lower bound of the controller's power in terms of the defined witness operator. This study may be useful in the estimation of the power of the controller in an experiment.

In Chapter 1, we have given some basic definitions, and a few concepts of linear algebra and quantum mechanics, with some results obtained in the literature. We then provide a brief review of the theory of bipartite and tripartite non-locality. We have reviewed quantum teleportation using bipartite system as a resource state. The discussion of controlled quantum teleportation is also presented, which is helpful for three-party communication.

In Chapter 2, we derive a criterion to detect the usefulness of a two-qudit entangled state in quantum teleportation. The state usefulness in quantum teleportation can be detected through the singlet fraction criterion but it has some drawbacks which can be listed as: i) it is not an easy task to calculate the singlet fraction in higher dimensions,

ii) Maximally entangled states are not known for higher dimensions and iii) Singlet fraction criteria may not serve as a potential candidate for experiment. This motivated us to derive another criterion which is beyond singlet fraction and is applicable for higher dimensions as well. The proposed criteria are based on maximum eigenvalues. We have shown our criteria works for those entangled states as well which were not detected by singlet fraction criteria. Our criteria can in principle be determined in an experiment because the maximum eigenvalue can be determined experimentally.
Chapter 3 basically deals with the non-locality of two-qubit entangled states. The main motivation for this work comes from the fact that there exist $2 \otimes 2$ dimensional entangled states, which satisfies Bell's inequality. Thus, one may think that those states may not possess the non-local properties, which may not be true. Since their non-locality is in hidden mode. So, we have taken a step to fill this loophole in our work by considering the XOR game which is also known as the Bell game. We have investigated this problem and tried to fix it by revisiting the non-locality of the two-qubit entangled state by defining the strength of non-locality denoted by S_{NL}. The strength of non-locality may be expressed in terms of the maximum probability of winning the game P_{max} but we found that the developed relation works only for those states that are detected by B_{CHSH} operator. So, we also derive other criteria to detect the non-locality of those entangled states that are not detected by the B_{CHSH} operator. Moreover, we have studied the relation between S_{NL} and $M(\rho)$, which are considered as the two measures of non-locality of two-qubit entangled state ρ. Then, we have also considered the optimal witness operator to study the strength of the non-locality. Lastly, we have cited two applications where the strength of non-locality S_{NL} may be used for (i) the detection of genuine non-locality in pure three-qubit system and (ii) deriving the upper bound of the controller's power in controlled quantum teleportation.
Chapter 4 deals with the detection of non-locality of the three-qubit state. Non-locality of a three-qubit state can be tested by various inequalities such as Svetlichny inequality, Mermin inequality, and logical inequality based on GHZ-type event probabilities. In order to obtain the expectation value of the Svetlichny inequality, one has to calculate the expectation value of the Svetlichny operator by maximizing over measurements of spin in all directions but it is not an easy task as the problem of showing the genuine non-locality of any three-qubit state reduces to the problem of a complicated optimization problem. Thus, the detection of genuine non-locality of any three-qubit state may be considered a challenging task. This motivate us to find a way by which we can overcome this problem. To do our task, we have taken a different approach to identify

the genuine non-locality of an arbitrary three-qubit state. We have derived a state-dependent upper and lower bound of the expectation value of the Svetlichny operator with respect to any three-qubit state. The derived upper and lower bound depends on the non-locality of the reduced two-qubit state of the three-qubit system. Thus, we have shown how to avoid the complicated optimization problem while verifying the non-locality of the three-qubit system.

In Chapter 5, we have studied the protocol for controlled quantum teleportation. In this work, we have estimated the power of the controller in the controlled quantum teleportation. To achieve this task, we have derived the lower and upper bound of the controller's power. We find that the upper bound assumes the constant value $\frac{1}{2}$ whereas the lower bound depends on the two-qubit state obtained after the controller measures on his qubit. We have shown that the derived lower bound may be expressed in terms of a newly defined witness operator. Thus, it may be considered that the power of the controller may be estimated in an experiment. We also have considered the pure three-qubit states which were not useful in controlled quantum teleportation but we have shown that if one of the qubits undergoes the amplitude damping channel or phase damping channel then the reduced three-qubit mixed state may be useful in the CQT. Further, we find that the controller's power will be larger in the case of the phase damping channel than the amplitude damping channel.

Future Scope

A lot of literature is available regarding the topics of non-locality, quantum teleportation, and control quantum teleportation as well. There are still many open problems related to this, which may be explored in the near future. A few problems are discussed below:

(i) The strength of non-locality may be defined for higher dimensional or multiparty systems. To do this, we have to generalize the idea of the XOR game for the higher dimensional and multiparty system and derive the maximum probability of success of the game. Thereafter, we may verify the non-locality of higher dimensional and multiparty systems, which were not detected by the current available methods.

(ii) Horodecki et. al. have re-expressed the Bell-CHSH inequality for two-qubits in terms of another easily accessible inequality that involve the quantity $M(\rho)$, where the two-qubit state is described by the density operator ρ. The quantity $M(\rho)$ is equal

to the sum of the two largest eigenvalues of the correlation matrix. Till today, there does not exist any similar-looking inequality that involves the quantity $M(\sigma)$, where σ describes the higher dimensional or multipartite system.
(iii) If problem (ii) is solved then the next problem would be to connect the quantity $M(\sigma)$ with the teleportation fidelity of the multiparty teleportation protocol in which the multiparty system described by the density operator σ act as a resource state shared between different parties.
(iv) In this thesis, we have obtained the state-dependent lower and upper bound of the expectation value of the Svetlichny operator but one may also explore the state-dependent lower and upper bound of Mermin's inequality to detect the non-locality of biseparable state.
(v) One may also deal with the problem of controlled quantum teleportation using the biseparable state and accordingly define the power of the controller in such a way that it may be linked with the non-locality of the entangled qubit lying in the biseparable state.

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List of Publications

1. **Anuma Garg** and Satyabrata Adhikari; *Teleportation Criteria Based on Maximum Eigenvalue of the Shared $d \otimes d$ Dimensional Mixed State: Beyond Singlet Fraction*, International Journal of Theoretical Physics **60**, 1038 (2021), **SCI**, Impact Factor (1.4).

2. **Anuma Garg** and Satyabrata Adhikari; *Strength of the nonlocality of two-qubit entangled state and its applications*, Physica Scripta **98**, 055101 (2023), **SCI**, Impact Factor (2.9).

3. **Anuma Garg** and Satyabrata Adhikari; *Detection of the genuine non-locality of any three-qubit state*, Annals of Physics **455**, 169400 (2023), **SCI**, Impact Factor (3).

4. **Anuma Garg** and Satyabrata Adhikari; *Estimation of Power in the Controlled Quantum Teleportation through the Witness Operator*, Accepted in The European Physical Journal D (EPJ D) (2024), **SCI**, Impact Factor (1.8).

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Conferences attended and Paper presented

1. **Attended** one day seminar on “Life and works of Prof. M.N. Saha and Prof. S.N. Bose” organized by JIIT, Noida on September 15, 2018.

2. **Attended** online symposium on quantum information and computation (Quantum Talks) conducted during 29th June to 3rd July 2020 in IIIT Hyderabad, India.

3. **Attended** the Young Quantum 2020 organized by Harish-Chandra Research Institute Chhatnag Road, Jhusi, Uttar Pradesh during 12-16 October, 2020.

4. **Attended** in the symposium NQSTS 2021 held online during 26th July 2021 to 3rd August 2021 in IIIT Hyderabad, India.

5. **Received best presenter award** in research scholar category for **presenting** a paper entitled *"Usefulness of Shared Entangled State in Quantum Teleportation by Eigenvalue Criteria"* in First International Conference on Agriculture, Science, Engineering & Management (ICASEM- 2021) organized by Sanskriti University, Mathura, UP, India & RSP Research Hub, Coimbatore, Tamilnadu, India(23-24 October 2021).

6. **Presented** a research paper titled as *"Eigenvalue Criterion for Quantum Teleportation Protocol"* in 5th International Conference on Recent Advances in Mathematical Sciences and its Applications (RAMSA-2021) organized by the Department of Mathematics at Jaypee Institute of Information Technology, Noida, Uttar Pradesh, (December 02-04, 2021).

7. **Presented** a research paper titled as *"Maximum eigenvalue Criteria in Quantum Teleportation for the shared $d \otimes d$ NPT entangled state"* in International Conference in Quantum Computing and Communications [QCC-2O23]at Baba Farid College, Bhatinda, Punjab during February 08-11, 2023.