

MATRIX GAMES WITH LINGUISTIC INFORMATION

A thesis submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

by

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(2K18/PHDAM/501)

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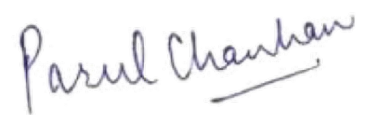
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I, Parul Chauhan (2K18/PHDAM/501), a PhD student in the Department of Applied Mathematics, hereby declare that the thesis entitled “**Matrix Games with Linguistic Information**” which is submitted by me to the Department of Applied Mathematics, Delhi Technological University, Delhi in partial fulfillment of the requirement for the award of the degree of “Doctor of Philosophy in Mathematics”, is original and not copied from any source without proper citation. This work has not previously formed the basis for the award of any Degree, Diploma Associateship, Fellowship or other similar title or recognition.



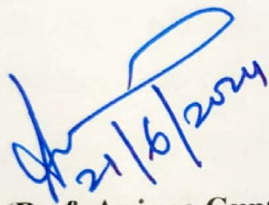
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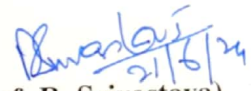
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CERTIFICATE

Certified that **Ms. Parul Chauhan** (2K18/PHDAM/501), has carried out her research work presented in this thesis entitled “**Matrix Games with Linguistic Information**” for the award of **Doctor of Philosophy in Mathematics** from Delhi Technological University, New Delhi, under my supervision. The thesis embodies results of original work and studies are carried out by the student herself and the contents of the thesis do not form the basis for the award of any other degree to the candidate or to anybody else from this or any other University/Institution.



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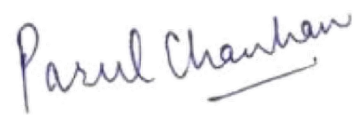
I am grateful to my grandparents, parents, and brother, who kept me safe under the shades and are the eternal source of my strength. Their unconditional love and perpetual support are the sole reason my thesis sees the light of day. I also want to thank my in-laws for their patience and heartiest cooperation during the Ph.D. duration. I am also highly indebted to my husband, whose care, understanding, and motivation kept me going ahead in life. Further, I express my profound gratitude to my daughter, whose lovely smiles kept me fresh and motivated during my Ph.D. work.

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*To my husband, Gaurav,
who encouraged me to follow my aspirations through his constant support,
&
my daughter, Naira,
who inspired me to remain strong in challenging times.*

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Abstract

Decision-making is a non-deterministic process. The antagonistic situations arise in many parts of real life, giving rise to uncertainty and imprecision. Furthermore, it is very common, when real situations are concerned, to deal with data vaguely. Game theory gives a mathematical background for such conflicting events. The imprecision in games can be of several types, such as randomness and fuzziness. The fuzziness in games is handled by fuzzy set theory. From this point of view, fuzzy games investigate and address those real-world issues that can be represented by a game with some degree of vagueness. Moreover, human activities and decisions are too ill-defined to be represented by conventional or fuzzy quantitative means. A possible way to solve such a situation is using a fuzzy linguistic approach, which creates the need for computing with words (CW). Beginning with the 2-tuple linguistic model, several computation models are introduced in the literature to deal with CW.

Further, game theory is the study of strategic information communication through language in a rigorous and stylized way. However, stylization and rigor cannot eradicate vagueness: there is always a remainder of linguistic vagueness at the heart of the conceptual tools used in game theory. This creates a foundational dilemma for conventional game theory. Therefore in our study, we have addressed the matrix games with linguistic information and have worked towards developing such games to enhance their applicability. This development supports the process of decision analysis in an uncertain qualitative environment. The thesis entitled “**Matrix Games with Linguistic Information**” comprises six chapters followed by the summary and future scope. The work in this thesis is motivated by the existing computational models that are implemented to resolve the uncertainty pertaining to matrix game problems. A good amount of analysis is done on the existing literature. The primary purpose of the work done in this thesis is to handle the vagueness and impreciseness prevailing in matrix games in the form of fuzzy

linguistic variables. The bibliography and the list of publications are provided at the end of the thesis.

The introductory **Chapter 1** presents an overview of fuzzy games, computing with words, linguistic models, and their elementary applications anticipated in distinct decision-making models, followed by their implementation in the game problems. Further, this chapter discusses the notion of randomness and fuzziness involved in any qualitative concept. Some basic concepts used throughout the thesis have been defined along with the motivation of the research work. Thus, the current chapter creates a background for this thesis's work and motivates the work carried out in this thesis.

The **Chapter 2** entitled, "A methodology for solving bimatrix games under 2-tuple linguistic environment" establishes the basis for a theory of bimatrix games that have payoffs as linguistic 2-tuples. In literature, the matrix games with 2-tuple linguistic information are considered and solved. However, the theory of bimatrix games having 2-tuple linguistic payoffs is pristine and yet to be explored. The use of linguistic 2-tuple variables to represent practical game situations has shown to be a powerful approach. The chapter is based on a research paper entitled, "A methodology for solving bimatrix games under 2-tuple linguistic environment", published in **International Journal of Systems Science: Operations & Logistics**, Taylor & Francis (SCIE, Impact Factor: 4.2).

The chapter mentioned above constitutes a game problem with payoffs belonging to a linguistic term set where all plausible linguistic descriptors provided by experts have a symmetric and uniform distribution. It might not be suitable in practical life decision problems since the experts may prefer linguistic labels distributed non-uniformly and non-symmetrically. Several computational models are established in literature to deal with unbalanced linguistic data. Therefore, in **Chapter 3** entitled, "Matrix games with unbalanced linguistic information," we propose a newly constructed methodology to handle matrix games with proportional linguistic payoffs. The proportional linguistic payoffs consist of two linguistic terms. However, a generalization of the proportional linguistic terms set is provided as a distribution assessment linguistic set. Hence, this chapter also discusses and solves matrix games with payoffs as linguistic distribution assessment. The chapter is based on the research papers titled, "Matrix games with proportional linguistic payoffs", published in **Soft Computing**, Springer (SCIE, Impact Factor: 4.1)

and “Matrix Games with Linguistic Distribution Assessment Payoffs”, published in: **Lecture Notes on Data Engineering and Communications Technologies**. Springer.

Chapter 4 entitled, “A novel cloud model based on multiplicative unbalanced linguistic term set” proposes a novel concept of the cloud model based on a multiplicative unbalanced linguistic term set. Further, based on the defined concept, a generalized multiplicative unbalanced linguistic scale function is introduced to map linguistic concepts to numerical intervals. The proposed approach can be perceived as a convenient technique for multiple criteria decision-making (MCDM) problems. Numerical illustrations are presented to discuss the significance of the proposed methodology. The chapter is based on a research paper titled, “A novel cloud model based on multiplicative unbalanced linguistic term set,” published in **The Journal of Supercomputing**, Springer (SCIE, Impact Factor: 3.3).

In **Chapter 5** entitled, “Solving Probabilistic Multiplicative Unbalanced Linguistic Games using Linguistic Cloud model”, the concept of probabilistic multiplicative unbalanced linguistic game is proposed in which the players record their expression in terms of unbalanced linguistic terms that have probabilities attached to them. The chapter is based on a research paper titled, “Solving Probabilistic Multiplicative Unbalanced Linguistic Games using Linguistic Cloud model”, published in **The Journal of Supercomputing**, Springer (SCIE, Impact Factor: 3.3).

Chapter 6 entitled, “The unbalanced 2-tuple intuitionistic fuzzy linguistic numbers and aggregation operators” presents unbalanced 2-tuple intuitionistic fuzzy sets in which the intuitionistic fuzzy linguistic numbers are composed of linguistic 2-tuples belonging to an unbalanced multiplicative set. The methodology presented in this chapter manipulates imprecise and uncertain information in decision-making in intuitionistic fuzzy set theory. The chapter is based on the research paper titled, “The unbalanced 2-tuple intuitionistic fuzzy linguistic numbers and aggregation operators” (communicated in “International Journal of Fuzzy Systems”).

Chapter 6 is followed by the summary of the research work carried out in this thesis. In addition, the future scope of the thesis has been discussed briefly.

Finally, the thesis ends with the bibliography and list of publications.

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Chapter 1

Introduction

Real-world decision-making problems incorporate uncertainty and erroneous information within their framework of definition such that it is inappropriate to model such problems via numerical values. In such situations, using a linguistic approach toward decision-making has provided effectual results. Matrix game decision-making problems possessing imprecision in their information are the major focus of this thesis. The present introductory chapter initially provides a brief understanding of uncertainty in games via a deep comprehension of fuzzy games and intuitionistic fuzzy games. Further, the chapter discusses the utility of the linguistic approach in games. For that purpose, several linguistic computational models are explained along with their implementation in matrix games. Furthermore, as it is impractical for a linguistic concept to only have a single kind of uncertainty; the cloud model explained in this chapter is capable of capturing the uncertainties like randomness along with fuzziness of a qualitative concept. Henceforth, the fundamentals of this chapter are proficient in providing a platform to motivate the research work carried out in this thesis.

Decision-making is an important real-life phenomenon arising in every competitive situation. The general problem of how to make decisions in a competitive environment is a common and important one. These competitive situations arise in the real world in business and other facets of human activity such as finance, politics, economics, etc. The conventional decision-making problems are those of a single decision-maker (DM). However, the game theory involves decision-making situations in which two or more DMs or players can be involved. A game formally describes the situation in strategic form. It depends on the basic assumption that some well-defined exogenous objectives be pursued by the DMs having a preliminary knowledge about the expectations/behavior of other DMs. A game is modeled when two or more individuals conflict over some issues, and the action of one individual depends upon the action taken by the opponent. The individuals involved in the game are called *players*. The possible actions taken by the players are called *strategies*. When each player selects a strategy, it will provide an outcome to the game in terms of *payoffs* to all players. The objective of each player is to maximize his/her payoff. At the heart of every game-theoretic problem, there lies a decision problem. Hence, game theory is a mathematical way to obtain the strategic interactions among DMs who select several strategies from the set of admissible strategies. The modern game theory is believed to have evolved from the pioneering work of Neumann & Morgenstern [171]. Since then, a great deal of research has been devoted to implementing game models that are widely used to analyze the behaviors in the field of management of technology [53, 54], marketing research [157], international cooperation [85, 187], supply chain management [47], electric power [195], finance [75], and many more. Game theory can be classified into two areas: cooperative and non-cooperative. This thesis is based on the non-cooperative theory of games that include two-person zero-sum as well as two-person non-zero sum games. In a zero-sum game, one player wins, and the other loses. A two-person zero-sum game is considered with a situation where the number of players or DMs is two, and one player's gain is quantitatively the same as the other's loss, so the net value is zero. This is far too restricted for many games, especially games in economics or politics, where both players can win something, or both can lose something. In a non-zero sum game, each player possesses his/her own payoff matrix and aims at maximizing his/her own payoff. Two-person non-zero sum or bimatrix game is no longer assumed to

be zero-sum or even constant sum. In such game, each player possesses her or his own payoff matrix.

The traditional game theory is based on a fully rational behavior assumption. In the past, several researchers formulated and solved matrix games considering precise payoffs. However, in general, in real games, the players usually do not act fully rationally. Besides, the data of traditional game theory is crisp, and such formulation is far from realistic. This is because often, there exist some very complex conditions in some conflict or cooperation games, like, asymmetric information among players, uncertain factors in decision-making situations, etc., such that some payoffs are uncertain and are not able to be evaluated exactly by the players due to the lack of complete information. In such situations, the information is uncertain and vague. Further, the behavior of players in a game depends on the structure of the game being played. This involves the decisions they face and the information they have when making decisions, how their decisions determine the outcome, as well as the preferences they have over the outcomes. The structure also incorporates the possibility of repetition, the implementation of any correlating devices, and alternative forms of communication. Any imprecision regarding the structure of the game has consequences for the outcome. Nevertheless, in the real world, decision-making often takes place in an environment in which the objectives, the constraints, and the outcomes faced by the players are not known in a precise manner. Ambiguities can exist if the components of the game are specified with some vagueness or when the players have their own subjective perception of the game. As game players are involved in uncertainties, several attempts have been made to generalize the game to a fuzzy game. By assigning a degree to a set in which the element possesses the defined characteristics, fuzzy sets introduced by L.A. Zadeh [255], provide a powerful tool to handle the imprecision in the evaluation of real-game situations. The fuzzy set theory can accommodate the uncertainty of fuzziness in game problems. Fuzzy sets have emerged as an uncertain system that addresses practical world issues. As a result, numerous decision-making problems based on fuzzy set theory have since been thoroughly examined in the literature.

1.1 Fuzzy Games

In a game, the uncertainty is due entirely to the unknown decisions of the other players and is inherent in the model. The degree of uncertainty is reduced through the assumption that each player knows the desires of the other player and the assumption that they will take whatever actions will aid them in attaining their goals. The uncertainty in games was first handled by Harsanyi [89], who introduced impreciseness in games with a probabilistic method and developed the theory of Bayesian games. However, real-game situations may possess other types of uncertainties, such as fuzziness. Amid multiple theoretical and practical developments, the theory of fuzzy sets outshines as a research area of study of decision-making possessing fuzziness as uncertainty. While the rapid development of fuzzy set theory to decision-making problems where only incomplete or uncertain information is available is the subject of substantial research, it is believed that it brings about another problem of how to potentially present a decision that could be easily understandable by humans. The advancement in the field of fuzzy set theory was so expeditious that it is recognized worldwide as a prominent research area and is studied by various researchers and DMs, especially in the theory of games. The matrix games having payoffs with fuzzy value are termed fuzzy games. A fuzzy game is an intersection of objectives and demarcations presented inside a multistage mechanism, where human intelligence can manipulate fuzzy concepts and fuzzy answer instructions. The concept of fuzzy games was first introduced by Aubin [10] in cooperative games and by Butnariu [35,36] in non-cooperative games. The work of Butnariu was further extended by Billot [28]. Dubois & Prade [69] described the fuzzy game with crisp sets of strategies and fuzzy payoffs due to some imprecision in the knowledge of the associated payoffs. Garagic & Cruz [83] used an approach based on the principle of the fuzzy controller to study games with fuzzy preferences. Li [113] studied several types of fuzzy multiobjective matrix games. The solution of fuzzy non-cooperative games was first given by Campos [41,42]. Maeda [149] defined three kinds of mini-max equilibrium strategies based on the fuzzy max order and proposed a bimatrix game method for solving a particular type of fuzzy matrix game that has payoffs of the type triangular fuzzy numbers (TFNs). On the other hand, Li [117] provided a faster approach to computing fuzzy values of matrix games with payoffs

of TFNs. Apart from these, an intense research has been done on zero-sum fuzzy matrix games [6, 33, 43, 105, 112, 143, 213] and non-zero sum fuzzy matrix games [2, 78, 79, 172, 211]. Table 1.1 summarizes the literature primarily centered on the fuzzy approach toward zero-sum and non-zero sum game problems. In this table, the chronological order of the publications, along with their key advantages and limitations, is given.

Present-day science & technology are featured with complex processes and phenomena for which complete information is not always available. For such cases, mathematical models are developed to handle various systems containing uncertain elements. A large part of these models are based on fuzzy set theory. In classical set theory, the notion of element membership to a set is a bivalent concept governed by the values 0 (no membership) and 1 (membership). However, fuzzy set theory brought about the concept of partial membership of an element, which is indicated by a number between 0 and 1, having for 0 non-membership, for 1 full membership, and for the numbers between 0 & 1 different degrees of membership (called partial membership). This means that the membership of an element to a fuzzy set is a single value between 0 and 1. However, the degree of non-membership of an element may not always be equal to 1 minus the membership degree because there may be some hesitation degree. Therefore, a generalization of the concept of fuzzy sets was introduced by Atanassov [8, 9] as Intuitionistic Fuzzy (IF) sets (IFSs) that can be viewed as an alternative approach to define fuzzy set in cases where available information is not sufficient for the definition of an imprecise concept employing a fuzzy set. Consequently, human decision-making processes involving human expertise and knowledge, which are inherently imprecise, may be emulated by the IFSs. In such sets, membership and non-membership are expressed using two values whose sum must be less than or equal to 1. The difference between this sum and 1 is called hesitancy. The next section discusses the implementation of IFS theory in matrix games theory.

1.2 Intuitionistic Fuzzy Games

In some game theoretic environments, there are situations where due to insufficiency in the information, the evaluation of membership and non-membership value is impossible as per our requirements. Consequently, there remains an in-

deterministic part on which hesitation survives. Certainly, fuzzy set theory is not appropriate to deal with such problems. In such situations, IFS theory serves better to model games as games with IF payoffs. As an extension of Zadeh's fuzzy set, IFS is also applied to decision-making problems, which can describe much more information than the fuzzy set. The IF concept is important in depicting real-world situations through decision-making and game problems. IFS can capture compound features and convey multi-faceted information, so it plays a significant role in the research of fuzzy games. The IFS allows us to model the hesitation and uncertainty of the player in estimating payoffs, which should be taken into account in a real problem through an additional degree. The IFS is useful in matrix game modeling because, in some real situations, players may describe their negative feelings, i.e., their degrees of dissatisfaction with the outcomes of the games. On the other hand, players could only approximately estimate their payoffs with some hesitation degrees. However, it is possible that players are not so sure about them. Thus, the IFS may provide players with a natural tool for modeling such uncertain situations.

Nayak & Pal [169] introduced bimatrix games with IF payoffs and conceptualized the meaning of Nash equilibrium for such games. According to Li & Nan [115], it is possible to solve two nonlinear programming models that are developed from two auxiliary nonlinear bi-objective programming models in order to find the solutions to matrix games with payoffs of Atanassov's IFSs. Further, to realize the day-to-day problems, researchers use fuzzy and vague sets to discuss the game problems. For example, in the paper [27], Bhaumik et al. consider a matrix game whose payoffs are triangular intuitionistic fuzzy numbers and apply a robust ranking technique to rank fuzzy numbers to solve the matrix game. Different types of score function and accuracy function approaches are used in the matrix games, but the robust ranking technique to solve matrix games is rare in use. A mathematical programming method for matrix games with payoffs represented by interval-valued IFSs is suggested by Li [116]. Xia [237] developed some interval-valued IF matrix games based on the t-norm and t-conorm. Further, simple non-cooperative games with IF information have also been defined in the literature by Yang et al. [250]. Also, the multi-objective bimatrix games under the IF environment have been defined by Yang & Wang [248]. Li & Liu [122] developed a new methodology, a parameterized nonlinear programming method for

solving matrix games with IF payoffs. Apart from these, numerous researchers have used the IFS theory in dealing with uncertainty prevailing in matrix game problems [15, 120, 166, 167, 189, 190, 208, 209].

Based on the literature, it can be summarized that the idea of IFS is useful in matrix games in the sense that it can indicate the players' preference information in terms of support, opposition and neutralization. The intuitionistic fuzzy optimisation (IFO) method has two advantages: This approach provides the most comprehensive toolkit for solving optimisation problems as it takes acceptance and rejection levels into account. Furthermore, compared to the similar crisp problems, the solutions to IFO problems are able to fully achieve the aim or objectives. Table 1.2 shows a summary of existing literature based on IF games in the chronological order of publication, along with a synopsis of the key contributions followed by their advantages and limitations.

The theories and literature discussed till now are different problems, and depending on the aspects presented by each problem, we can deal with different types of precise numerical values. But in other cases, the problems present complex qualitative aspects to assess utilizing precise values. In the latter case, using the fuzzy linguistic approach [256, 257] has provided very good results. Most decision analysis problems involve multiple attributes that exhibit both quantitative and qualitative nature. We can deal with quantitative attributes with different types of precise numerical values. However, it is not proper to represent qualitative aspects that often involve vague, imprecise, and uncertain information similarly. In such cases, experts often use linguistic descriptors to express their assessments regarding their uncertain knowledge of the problem. The next section discusses the significance of computing with words (CW) approaches and the limitations of the existing literature.

1.3 Computing with Words

In retreating from precision in the face of overpowering complexity, it is natural to explore the use of what might be called linguistic variables that is, variables whose values are not numbers but words or sentences in a natural or artificial language. The motivation for using words or sentences rather than numbers is that linguistic characterizations are generally less specific than numerical ones.

For example, in speaking of age, when we say “Player is *young*,” we are less precise than when we say, “Player is 25.” In this sense, the label “*young*” may be regarded as a linguistic value of the variable “Age” with the understanding that it plays the same role as the numerical value 25 but is less precise and hence, less informative. The same is true of the linguistic values “*very young*, *not young*, *extremely young*” etc. in place of numerical values 21, 22, 23, etc. When a problem is solved using linguistic information, it implies the need for CW [258, 259]. Figure 1.1 explains the process of computing with words.

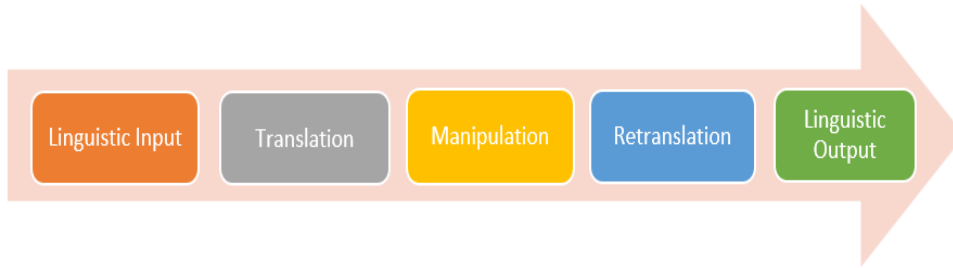


Figure 1.1: Process of computing with words

An extensive range of research is performed in decision-making using CW [57, 59, 71, 90, 91, 156, 205]. Herrera & Herrera-Viedma [93] explained the process of linguistic decision analysis by explaining the steps for solving decision problems under linguistic information. Tong & Bonissone [203] implemented the linguistic approach to decision-making with fuzzy sets. Figure 1.2 shows the scheme of a linguistic decision-making problem. To accomplish the objective of choosing an

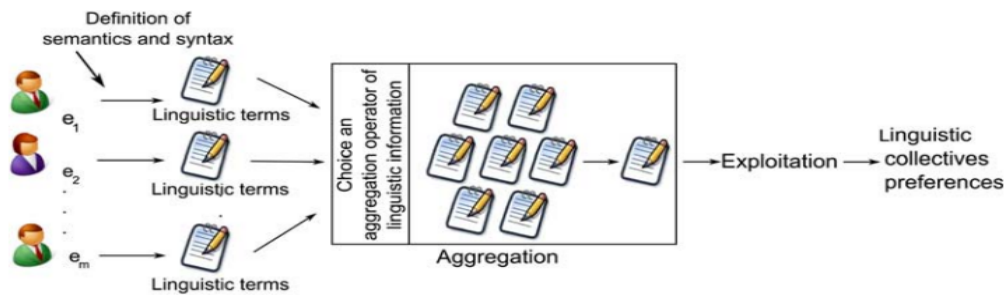


Figure 1.2: Scheme of a linguistic decision-making problem [180]

appropriate linguistic descriptor for a term set and its semantics, an important aspect to analyze is the granularity of information, i.e., the cardinality of the term set. For example, a set of seven terms for the linguistic term “*Speed*” could be given

as $L = \{\ell_0 : \text{Nil}, \ell_1 : \text{Very Low}, \ell_2 : \text{Low}, \ell_3 : \text{Average}, \ell_4 : \text{High}, \ell_5 : \text{Very High}, \ell_6 : \text{Very very high}\}$. These linguistic terms are assigned semantics of different membership functions, such as triangular, trapezoidal, etc. The triangular membership assigned to the linguistic terms may be viewed graphically as in Figure 1.3.

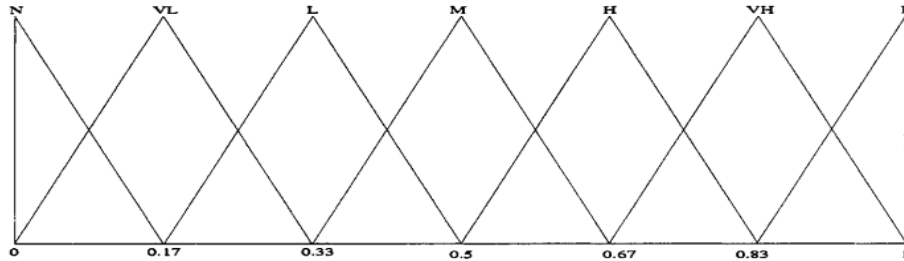


Figure 1.3: A set of seven terms with semantics

Although linguistic decision-making has proved to be a great approach, the limitation of this approach appears because the computational techniques used in the specialized literature, such as extension principle [31, 55] and symbolic method [58] present a common drawback, the “loss of information,” that implies a lack of precision in the final results. In both approaches, the results usually do not exactly match any of the initial linguistic terms, then an approximation process must be developed to express the result in the initial expression domain. This limitation of information loss is overcome by Herrera & Martínez [92, 95] by representing the linguistic information with a pair of values called 2-tuple, composed of a linguistic term and a number. For example, if we want to view the symbolic translation of the numerical value 3.25 on the set of labels, $L = \{\ell_0 : \text{Nothing}, \ell_1 : \text{Very Low}, \ell_2 : \text{Low}, \ell_3 : \text{Medium}, \ell_4 : \text{High}, \ell_5 : \text{Very High}, \ell_6 : \text{Perfect}\}$ then the 2-tuple that expresses the equivalent information is (Medium, .25) (see Fig 1.4). The main advantage of this representation is to be continuous in its domain; therefore, it can express any counting of information in the universe of the discourse. This model called the “2-tuple linguistic (2-TL) model” is a symbolic model that extends the use of indexes modifying the fuzzy linguistic approach representation by adding a parameter to the basic linguistic representation to improve the accuracy of the linguistic computations after the retranslation step keeping the CW scheme. Being the most beneficial model for CW processes due to its more straightforward interpretation and competence in expressing any linguistic data in a continuous domain, the 2-TL model has many applications. Over the sub-

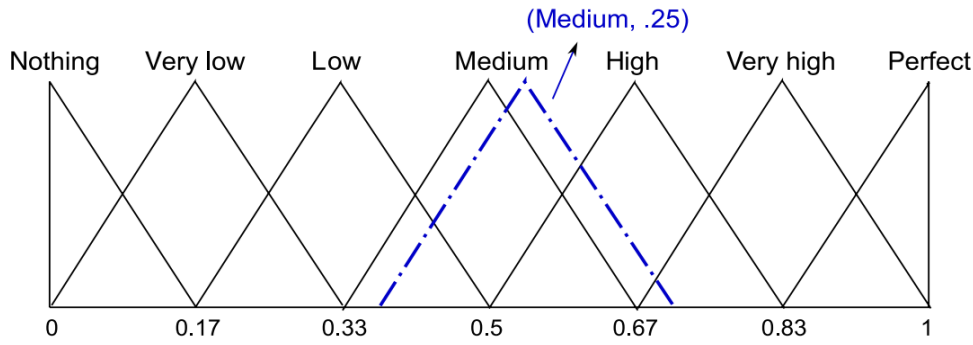


Figure 1.4: 2-tuple linguistic representation [92]

sequent years, many researchers have proposed variant versions of this model, stating its mathematical formulations, operations, and aggregation operators. The succeeding section highlights the extensions of the 2-TL model and its applications in linguistic decision-making.

1.3.1 Linguistic Computational Models (LCMs)

It is known that Herrera & Martínez's 2-TL model is alleged to address only the linguistic term (LT) sets having the symmetric distribution of linguistic labels. However, there exist problems that need to assess their variables with LT sets that are not uniformly and symmetrically distributed. Such types of sets are termed as *unbalanced linguistic term sets (ULTSs)*. There are two types of ULTS. The Type *I* consists of the ULTS in which the unbalanced linguistic information appears due to the nature of the linguistic variables that participate in the problem as it happens. In another type *II*, it seems in problems dealing with scales for assessing preferences where the experts need to assess several terms in a side of the reference domain higher than in the other one (refer Figure 1.5, Figure 1.6).

To address different types of ULTSs, several LCMs are defined in the litera-

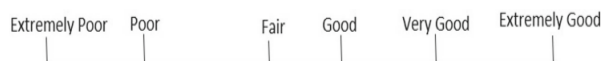


Figure 1.5: ULTS Type *I*

ture to deal with unbalanced linguistic information. A summary of a few of these unbalanced LCMs is stated next.



Figure 1.6: ULTS Type *II*

Linguistic Hierarchy Model [97]

In this paper, a methodology is developed to represent, manage, and accomplish processes of CW with ULTS without loss of information. For that purpose, firstly, an unbalanced linguistic representation model is defined that assigns semantics to the linguistic terms. Therefore, a process is outlined to assign semantics to the linguistic terms belonging to a ULTS. Then, these ideas are formalized by employing a semantic representation algorithm that represents each term by means of a parametric membership function that is assigned using a linguistic hierarchy structure. Secondly, a computational model is presented for ULTS based on the 2-TL model to accomplish the processes of CW without loss of information. Herrera et al. [97] accustomed the conception of linguistic hierarchy to acquire linguistic representations of unbalanced linguistic values in the format of 2-tuple. A linguistic hierarchy is determined as a set of levels.

Although this methodology is very useful for modeling different real-world problems dealing with linguistic terms assessed in ULTS, such as evaluation processes, decision making, and information retrieval, but it proved to be a very computationally expensive and complex approach to decision-making.

Symbolic Proportional Model [221]

This model provides a new version of the 2-TL model. In this model, linguistic information is represented by proportional 2-tuples, such as $(0.2A, 0.8B)$ for the case when someone's grades in the answer scripts of a whole course are distributed as 20%A and 80%B. This approach is based on the concept of "symbolic proportion". Under linguistic contexts and based on canonical characteristic values (CCVs) of linguistic labels, many aggregation operators are defined to handle CW of sophisticated linguistic description. These aggregation operators not only can operate in a computational stage for CW without any loss of information but also deal with linguistic labels, which do not have to be symmetrically distributed around a medium label and without the traditional requirement of having an "equal distance" between them. Moreover, since the concept of CCV, which is defined in

an ordered LT set, can be extended to the linguistic proportional 2-tuple set, that is generated by these aggregation operators still can work when members of a proportional 2-tuple set represent the initial linguistic information. Thus this new model not only provides a suitable space in a computational stage for CW but also provides an opportunity such that experts in various decision models can evaluate the performances of alternatives by not just one label but with (more flexible) proportional 2-tuples.

Although, this model makes the manipulation of aggregating linguistic 2-tuple information easy and efficient. However, under a deeper observation, one will find that to generate “accurate” information from this model, the semantic of linguistic terms in the ordered LT set used in the model can only be defined by fuzzy numbers which vary in “one dimensional” characteristic, in which all fuzzy numbers have different center but almost all with the same “width.” Further, this model is limited to proportional 2-tuples in which symbolic proportions are assigned to two successive ordinal terms in a LT set. Thus, a generalization is required in this manner.

Linguistic Distribution Assessment Model [262]

This model is a generalization of the symbolic proportional model in which the symbolic proportions are assigned to all the linguistic terms present in the LT set. For example, if $S = \{\ell_{-2} = \text{very poor}, \ell_{-1} = \text{poor}, \ell_0 = \text{average}, \ell_1 = \text{good}, \ell_2 = \text{very good}\}$ is an LT set, and a cricket captain uses the members of the set S to assess a player’s performance. The player takes part in 10 cricket games, in which 7 times was evaluated as ℓ_2 , 1 time as ℓ_1 , and 2 times as ℓ_{-1} . Then, the captain’s evaluations may be interpreted in the form of distribution assessments of S as $\{(\ell_{-2}, 0), (\ell_{-1}, 0.2), (\ell_0, 0), (\ell_1, 0.1), (\ell_2, 0.7)\}$. Further, this model also presents some aggregation operators for linguistic distribution assessments. This model aims to propose concepts of distribution assessments of LT sets and study the consistency and consensus measures for linguistic preference relations based on distribution assessment. However, it proves to be a more complicated approach in the field of decision-making.

Considering both Wang and Hao’s model [221] and Zhang et al.’s model [262], it is envisioned that the symbolic proportions involved in them sum up to 1. Henceforth, both these models are inadequate to handle incomplete information. So,

to overcome this limitation, Guo et al. [86] proposed a new proportional linguistic distribution model for modeling incomplete linguistic assessments addressed in MADM problems. However, given varying subjective conditions, it is challenging for DMs to provide exact symbolic proportions in linguistic distribution assessments, as it is not always easy for DMs to provide symbolic proportions through precise values in an uncertain decision environment. In that case, a better approach is to use interval symbolic proportions to address this issue instead of exact ones. For this, Dong et al. [68] defined linguistic distribution assessments with interval symbolic proportions and developed the transformation functions among the multi-granular unbalanced linguistic distribution assessments with interval symbolic proportions.

Numerical Scale Model [67]

In the symbolic proportional and linguistic distribution assessment models, the semantics of linguistic terms in the LT set can only be defined for symmetrical trapezoidal fuzzy numbers. The Numerical Scale model presents the concept of the numerical scale for the LT set and argues that the key to the computational techniques based on the 2-TL model is to set a suitable numerical scale with the purpose of making transformations between linguistic 2-tuples and numerical values. The main aim of this model is to further complete the 2-TL model for CW by providing an optimization model to compute the numerical scale of the LT set. This model proposes the concept of a transitive calibration matrix and defines its symmetry, transitivity, and consistent index. The optimization model makes the transitive calibration information of the DM more consistent. In particular, this proposal can set suitable numerical scales for different DMs. To make the computations easier, an algorithm is presented to automatically construct the transitive calibration matrix for decision problems using linguistic preference relations.

This model presents an advanced theory for CW. However, such methodologies already exist in the literature. Thus a comparative study is required in this theory.

A Method for Unbalanced Linguistic Term Sets and its Application in Group Decision Making [38]

In this method, a new linguistic computational model based on symbolic models is proposed. For that, a representation model of ULTS is defined; and the values expressed by unbalanced linguistic terms are converted into a special linguistic

domain, a basic equal-distance LT set. The model tries to solve two problems: one is defining an unbalanced linguistic representation model, and the other is defining computational techniques based on the representation model to solve group decision-making (GDM) problems. The method is easy to implement, and the result is also consistent. This methodology is useful for modeling different real-world problems in ULTS, such as evaluation processes, decision-making, and information retrieval. However, for computation purposes, the ULTS is first converted into a special linguistic domain called a basic LT set. This means that the original meaning of ULTS is missing.

Due to the relevance of linguistic decision-making in real problems and the necessity of linguistic computational methods in CW, many researchers have tried to improve the processes of CW by developing different models [1, 18, 37, 39, 76, 96, 103, 176, 252, 265, 266, 269]. However, concerning the above LCMs, it is envisioned that the representation of the ULTS is made by using a set of linguistic terms distributed uniformly, i.e., the ULTS is first converted to a set of uniformly distributed terms, and then the computations are made for a symmetrically distributed LT set. As a consequence, the literal meaning of a ULTS is missing. Therefore, it became essential to define an LT set in which the usual meaning of “unbalanced” remains preserved. For this purpose, the following ULTS are introduced to preserve the unbalanced virtue of an LT set.

A new 2-tuple linguistic approach for unbalanced linguistic term sets [153]

In this approach, a methodology is defined to manage unbalanced linguistic information. For that, initially, a novel approach is designed to represent the semantics of the linguistic terms that belong to a ULTS. The ULTS is defined as $\mathbb{LS} = \{t_{a^{-n}}, t_{a^{-(n-1)}}, \dots, t_{a^0}, \dots, t_{a^m}\}$ where $a > 1$, $n, m \in \mathbb{Z}^+$. This set has an additional property that it is a multiplicative ULTS. The term t_{a^e} is a plausible value for a linguistic variable. Also, a translation function is defined that transforms a precise number to its corresponding unbalanced 2-tuple. Further aggregation operators of such 2-tuples are also proposed.

This method is advantageous for two reasons; Firstly, a multiplicative linguistic scale set is utilized to represent unbalanced linguistic information where the terms are unevenly distributed. Henceforth, the authors in this paper can subjugate the limitation by employing a multiplicative linguistic scale set to substitute the

given unbalanced information. In this way, throughout the procedure, a formal definition of ULTS is preserved. Secondly, the proposed method has a more straightforward design. As a consequence, it is computationally less expensive and less complicated in comparison to the other LCMs. In this model, the case where $n \neq m$ is handled by adding a dummy variable in the ULTS such that n and m are equal. It is envisioned that the addition of a dummy variable into the set does not provide us with the optimal solution in several cases, suchlike cases of multi-criteria decision problems, fuzzy linguistic-based game-theoretic models, financial issues, and so forth, which may lead to a limitation.

Apart from these, many similar techniques consider distinguishing the positions of possible linguistic terms. All these techniques are capable of dealing with the uncertainty of the fuzziness of the decision-making processes. Analogous to these LT sets, the probabilistic linguistic term sets (PLTSs) allow experts to express their preferences regarding one linguistic term over another. Moreover, such sets are competent to handle the uncertainty prevailing regarding randomness. Furthermore, the PLTSs introduced by Pang et al. [174] can reflect the true weights of all evaluations over an object, so that comprehensive and accurate evaluation information can be obtained from the DMs. In the GDM environment, an opinion may take the form of a collection of possible linguistic terms associated with their probabilities of being the real value. PLTSs are proposed and developed for this scenario. The theory of PLTSs has been studied and analyzed by many researchers [11, 104, 230, 239, 252, 254]. Yu et al. [253] defined operators and comparisons of PLTSs. Wang et al. [219] also provided some order relations and operations on the set of PLTS. Moreover, Wu et al. [235] extended the concept of PLTSs to PLTS with interval uncertainty. Further, the interval-valued PLTSs have also been applied in multi-criteria GDM by Bai et al. [12]. The next section presents a new type of PLTS and its properties and applications.

Probabilistic multiplicative unbalanced linguistic term set and its application in matrix games [154]

PLTSs are suitable for enunciating evaluators' complex linguistic perceptions more accurately within the intricate qualitative setting. Usually, the PLTS is based on a balanced concept and does not serve as a good information representation for an unbalanced qualitative concept. Therefore, to reflect experts' distinct preferences

and uncertainties, the authors propose a new PLTS called the probabilistic multiplicative unbalanced linguistic term set (PM-ULTS), where both probabilities and non-uniform spacing of the linguistic labels are considered simultaneously. This model considers the hesitancy of DMs' preferences and reflects their opinions more accurately by effectively distinguishing the spacing difference for each linguistic label. Moreover, this PM-ULTS is capable of handling both types of ULTS. The novel operational laws developed for PM-ULTS are closed, i.e., the outcome obtained after applying the operation does not exceed the bounds of LT sets, thereby overcoming the unreasonableness in logic proficiency. In addition, these laws can efficiently deal with the persisting decision information by expressing it using the PM-ULTSs and can avoid losing the given priority information validly. The information measure for the probabilistic multiplicative unbalanced linguistic term is calculated to transform the probabilistic linguistic information. The information measure is primarily based on the bijective functions such that it avoids the loss of information in the re-translation process.

Apart from the fuzzy linguistic computations, the linguistic concepts also arise in the IF environment. Although IF theory has been successfully applied in some areas, the IFS is also characterized by the membership degree and the non-membership degree, whose sum is less than or equal to 1. However, all the earlier defined approaches in Section 1.2 are unsuitable to describe the membership degree and the non-membership degree of an element to a set by linguistic variables based on the given LT sets, which can reflect the DMs' confidence level when they are making an evaluation. To overcome this limit, Zhang [264] proposed the concept of an IF linguistic set consisting of linguistic membership and non-membership degrees. Chen et al. [50] developed an approach to multiple attribute GDM based on linguistic IF numbers. Garg & Kumar [84] introduced some aggregation operators for linguistic IFS and provided its application to the GDM process.

All the models defined above have provided remarkable results in decision-making. These models have been developed and applied as a computational basis to CW. Motivated by the above analysis and the ability of LCMs, as mentioned above, to advert vague and imprecise information (balanced and unbalanced), the next section discusses the applications of these LCMs to improve the accuracy and understandability of the CW processes in the theory of matrix

games.

1.3.2 Matrix Games with Linguistic Information

Game theory is an important research area of operation research, which mainly concerns competitive and skillful interaction between rational, intelligent DMs. So, how to make decisions in competitive situations is an interesting and relevant research problem. Numerous studies have been reported in the literature associated with matrix games with crisp/precise payoffs. However, in many real-world situations, the payoffs of the matrix game are not known precisely due to the presence of uncertainty or insufficient information about the problem data. Moreover, if the game players record their information in terms of linguistic data rather than real or fuzzy numbers, it appears as linguistic uncertainty in the game problem. To deal with this linguistic uncertainty, Delgado et al. [56] first introduced matrix games defined by linguistic labels by providing an approach to solve such “ill-defined games”. However, in a more general scenario, the linguistic games are said to be introduced by Arfi [4, 5], who defined a linguistic fuzzy game as a game with linguistic fuzzy strategies, linguistic fuzzy preferences, and the rules of reasoning and inferences of the game operate according to linguistic fuzzy logic, not Boolean logic. Arfi concluded that there is always an optimum strong Nash equilibrium which is Pareto optimal, thereby lifting many of the dilemmas that emerge in crisp game theory in two-player and social games. Since then, numerous researchers have devoted their time and effort to solving matrix games with linguistic payoffs. Sahni et al. [181] developed a generalized approach to solve matrix games with payoffs represented by 2-TL values. To do so, the work uses the generalized probabilistic 2-tuple average operator to calculate the expected value of the game and formulates a pair of nonlinear programming models with a flexible parameter to solve this class of games. Sharifian et al. [191] showed how fuzzy linguistic preference relation can be used in game theory by constructing a fuzzy IF-THEN rule set to derive preferences according to the difference between payoffs. Table 1.3 presents a summary of matrix games having linguistic information with their description as well as advantages and limitations.

1.4 Randomness and Fuzziness of Concepts

As the primary vehicles of human thinking, natural languages inevitably become one of the most essential and important research perspectives. But the uncertainties of natural languages, as mentioned in the earlier sections, especially the uncertainties of linguistic concepts, also lead to challenging problems. For example, how to formally describe qualitative concepts in natural languages? How to perform the natural transformation between qualitative concepts and quantitative values, and how to mimic the soft-reasoning ability of human thinking in terms of qualitative concepts? Among the uncertainties involved in natural language, randomness, and fuzziness are the two most important uncertainties which have attracted much attention from the researchers. Although randomness and fuzziness are quite different in nature, many linguistic concepts contain simultaneous randomness and fuzziness. For example, the linguistic terms “about”, “maybe”, “probably”, “sometimes”, and “usually” are not only random concepts but also embody fuzziness due to the vague boundary of the extension. However, people do not care about that when adopting the terms for emotional and mental expression. About randomness and fuzziness of linguistic concepts, the most representative theories and tools are the probability theory and fuzzy sets theory, based on which people have proposed many uncertain representation models to deal with the fuzziness and randomness of concepts, respectively.

In the fuzzy set theory, for a given set in the universe, the larger the membership degree of an element, the more the possibility of the element belonging to the related fuzzy concept will be. Thus, the membership functions of fuzzy concepts can usually be determined by using statistical methods. Therefore, formally describing the inherent relation between randomness and fuzziness and effectively integrating both uncertainties becomes a key challenge in research on the uncertainties of linguistic concepts. To describe the uncertainties of linguistic concepts, especially the randomness, and fuzziness, and implement the uncertain transformation between linguistic concepts and quantitative values, the cloud model defined by Li et al. [111] has provided remarkable results. Lou et al. [147] developed an integrated decision-making method for product design scheme evaluation based on cloud model and EEG data. Wang et al. [214] proposed a multi-dimension cloud model-based approach for water quality assess-

ment. Further, Wang et al. [223] developed a new algorithm to convert linguistic variables into clouds, then presented several cloud arithmetic aggregation operators and constructed a GDM process based on the proposed operators. Moreover, recent studies highlight the advantage of the cloud model in solving various problems [102, 127, 132, 160, 161, 196, 233]. Further, Gao [81] introduced a new optimization algorithm inspired by cloud model theory. The innovations of the algorithm are the estimation of good solution regions and new solution production according to the cloud model theory. Deng et al. [60] developed a novel hybrid water quality time series prediction method based on cloud models and fuzzy forecasting.

The cloud model, which mainly reflects the uncertainty in real life and the concepts in human knowledge, has many practical problems occurring in uncertain environments, especially in situations where the information about all the criteria weights, criteria values, and expert weights are uncertain linguistic variables. Although the cloud model is a description of the qualitative concept, it is depicted by three numerical characteristics: the expected value EX, the entropy EN and the hyper entropy HE. In the paper [223], the authors define a linguistic assessment scale that can be used to perform the transformation between the clouds and the linguistic variables of any LT set of odd labels. This scale can be used to transform linguistic assessment information into numerical assessment values. Further, in the paper [224], Wang et al. propose a decision-making method based on the cloud model to solve multi-criteria GDM in situations where the criteria weights are known completely, but the DMs' weights are unknown and the criteria values are interval linguistic variables. Based on the trapezium clouds [100, 263], a multi-criteria decision-making (MCDM) method with interval-valued intuitionistic linguistic numbers is proposed by Wang et al. in [225]. Various other approaches have been developed concerning the representation of linguistic information using cloud model [177, 226, 227, 268]. These cloud models are incapable of providing interval concepts and cannot distinguish the difference in spacing of each linguistic term. To solve this issue, Wang et al. [231] proposed an asymmetric trapezoidal cloud model to manage the irregularly distributed linguistic terms effectively.

To sum up, the cloud model is a good tool for dealing with linguistic information. By combining the probability theory and fuzzy set theory, the normal cloud

models can express the fuzziness and randomness of linguistic concepts, which are the two most important aspects among the uncertainties involved in linguistic concepts. The next section discusses some basic concepts and reviews some significant notions that are useful throughout the thesis.

1.5 Preliminaries

This section presents the preliminaries containing a few basic concepts that create the foundations of the work performed in this thesis.

1.5.1 Two-person Zero sum Matrix Game

In a general sense, game theory is the mathematics of strategy comprising at least two individuals. We call them the players of the game. They choose their respective strategies to win the game by making some decisions. As described in [178], game theory studies how players should rationally play the game. There may be conflict because different players will generally value outcomes differently. A two-player zero-sum game is the most elementary one, comprising two players only. This type of game is assessed by assuming that both the players apply strategies that could be pure, mixed, or continuous [175]. We recall a zero-sum game, according to Barron [17].

Definition 1.5.1. [17] A crisp two-player zero sum matrix game is represented as a triplet (S^r, S^s, A) , where

- (i) S^r is the set of ‘ r ’ possible strategies of player I .
- (ii) S^s is the set of ‘ s ’ possible strategies of player II .
- (iii) A is the payoff matrix of order $r \times s$ of player I corresponding to player II , and $neg(A)$ refers to the matrix consisting of payoffs of player II .

Definition 1.5.2. [17] A matrix game having $A = (a_{ij})_{r \times s}$ as the matrix of payoffs has the lower value

$$v^- = \max_{i=1, \dots, r} \min_{j=1, \dots, s} a_{ij}$$

and the upper value

$$v^+ = \min_{j=1, \dots, s} \max_{i=1, \dots, r} a_{ij}$$

The lower value v^- is that minimum amount that player I is secured to get, and the upper value v^+ is that assured maximum amount that player II might lose. We say that the game has a value if the upper value equals the lower value, and we express it as $v^+ = v^- = v$, and the row and column (i^*, j^*) giving payoff $a_{i^*j^*} = v^- = v^+$ are called optimal, or more precisely a saddle point in case the pure strategies are applied. Further, $a_{ij^*} \leq a_{i^*j^*} \leq a_{i^*j}$, $\forall i = 1, 2, \dots, r$ and $j = 1, 2, \dots, s$.

Definition 1.5.3. [17] A mixed strategy is a vector $\Phi = (\phi_1, \phi_2, \dots, \phi_r)$ for player I and $\Psi = (\psi_1, \psi_2, \dots, \psi_s)$ for player II , where

$$\phi_i \geq 0, \sum_{i=1}^r \phi_i = 1 \quad \& \quad \psi_j \geq 0, \sum_{j=1}^s \psi_j = 1.$$

ϕ_i represents the probability of using row ' i ' by player I , and ψ_j , the probability of using column ' j ' by player II . We represent the set of mixed strategies having ' k ' components as

$$S^k = \{(z_1, z_2, \dots, z_k) | z_p \geq 0, p = 1, 2, \dots, k, \sum_{p=1}^k z_p = 1\}.$$

In this terminology, elements $\Phi \in S^r$ and $\Psi \in S^s$ represent the mixed strategies for player I and player II , respectively.

Now, in case of mixed strategies, the expected payoff of players is defined as follows:

Definition 1.5.4. [17] Given a choice of mixed strategy $\Phi = (\phi_1, \phi_2, \dots, \phi_r)$ for player I and $\Psi = (\psi_1, \psi_2, \dots, \psi_s)$ for player II , selected autonomously, the expected payoff of player I is

$$\begin{aligned} E(\Phi, \Psi) &= \sum_{i=1}^r \sum_{j=1}^s a_{ij} \text{Prob}(I \text{ uses } i \text{ and } II \text{ uses } j) = \sum_{i=1}^r \sum_{j=1}^s a_{ij} \text{Prob}(I \text{ uses } i) \text{Prob}(II \text{ uses } j) \\ &= \sum_{i=1}^r \sum_{j=1}^s \phi_i a_{ij} \psi_j \\ &= \Phi A \Psi^T \end{aligned}$$

1.5.2 Two-person Non-zero Sum Game

The zero-sum games consider that whatever amount one player wins is the same amount the other loses. However, this is way too limiting for various games,

like politics or economics, where both players may win or lose something. There are situations where the interests of both players may not be exactly the opposite. These situations bring about the concept of non-zero sum games, also called bimatrix games. In two-person non-zero sum games, both players have their payoff matrix. Both players aim to maximize their own individual payoff.

Definition 1.5.5. [17] Suppose the payoff matrices of player I and player II be $A = [a_{ij}]_{r \times s}$ and $B = [b_{ij}]_{r \times s}$, respectively. When player I plays i^{th} row and player II plays j^{th} column, the payoff is represented by (a_{ij}, b_{ij}) in which a_{ij} represents the payoff to player I and b_{ij} to player II .

The respective rows and columns are termed as pure strategies. Moreover, the mixed strategy for Player I is $\Phi \in S^r = \{\Phi = (\phi_1, \phi_2, \dots, \phi_r) | \phi_i \geq 0, i = 1, 2, \dots, r, \sum_{i=1}^r \phi_i = 1\}$, and for player II is $\Psi \in S^s = \{\Psi = (\psi_1, \psi_2, \dots, \psi_s) | \psi_j \geq 0, j = 1, 2, \dots, s, \sum_{j=1}^s \psi_j = 1\}$. Each player will have their own expected payoffs as

$$E_I(\Phi, \Psi) = \Phi A \Psi^T \text{ for player } I,$$

$$E_{II}(\Phi, \Psi) = \Phi B \Psi^T \text{ for player } II.$$

Further, Bector & Chandra [22] defined a bimatrix game as $BG = (S^r, S^s, A, B)$.

Next we present the Nash equilibrium defined in the case of mixed as well as pure strategies.

Definition 1.5.6. [17] $\Phi^* \in S^r$ and $\Psi^* \in S^s$ form a Nash equilibrium if

$$E_I(\Phi, \Psi^*) \leq E_I(\Phi^*, \Psi^*) \text{ for every mixed } \Phi \in S^r$$

and,

$$E_{II}(\Phi^*, \Psi) \leq E_{II}(\Phi^*, \Psi^*) \text{ for every mixed } \Psi \in S^s.$$

If (Φ^*, Ψ^*) is a Nash equilibrium, denote $v_A = E_I(\Phi^*, \Psi^*)$ and $v_B = E_{II}(\Phi^*, \Psi^*)$ as the optimal payoff to each player. In matrix form, (Φ^*, Ψ^*) is a Nash equilibrium if

$$E_I(\Phi^*, \Psi^*) = \Phi^* A \Psi^{*T} \geq \Phi A \Psi^{*T} = E_I(\Phi, \Psi^*), \text{ for every } \Phi \in S^r$$

$$E_{II}(\Phi^*, \Psi^*) = \Phi^* B \Psi^{*T} \geq \Phi^* B \Psi^T = E_{II}(\Phi^*, \Psi), \text{ for every } \Psi \in S^s$$

Moreover, a Nash equilibrium in case of pure strategies are i^* and j^* such that $a_{ij^*} \leq$

$$a_{i^*j^*} \text{ and } b_{i^*j} \leq b_{i^*j^*}.$$

Next, we provide the methodology required to solve any arbitrary two-person non-zero sum game.

Non-linear Programming Method for Non-zero Sum Two-person Games [17] :

This method provides a way for computing all the Nash equilibrium for an arbitrary two-person non-zero sum game. This method of computing the solution of a nonlinear program lays on the foundation of the method provided by [108]. The method proceeds as follows:

Consider the two-person game with matrices (A, B) . Then $\Phi^* \in S^r, \Psi^* \in S^s$ is said to be a Nash equilibrium iff they satisfy, along with scalars p^* & q^* the nonlinear programming problem

$$\begin{aligned} & \max_{\Phi, \Psi, p, q} \Phi A \Psi^T + \Phi B \Psi^T - p - q \\ & \text{s.t.} \\ & A \Psi^T \leq p J_r^T \\ & B^T \Phi^T \leq q J_s^T \quad (\text{equivalently } \Phi B \leq q J_s) \\ & \phi_i \geq 0, \psi_j \geq 0, \Phi J_r = 1 = \Psi J_s^T \end{aligned} \tag{1.5.1}$$

where, J_k is the unit vector consisting of all 1s. Additionally, $p^* = E_I(\Phi^*, \Psi^*)$, $q^* = E_{II}(\Phi^*, \Psi^*)$.

1.5.3 The basics of the 2-tuple Linguistic model

The 2-TL model introduced by [92] is a strong and accurate model representing the linguistic information using the pair $(\ell_\varepsilon, \alpha)$, where ℓ_ε denotes a linguistic term belonging to a linguistic term set and α is a numerical value belonging to $[-0.5, 0.5)$ representing the value of the symbolic translation.

Definition 1.5.7. [92] Let $L = \{\ell_0, \ell_1, \dots, \ell_g\}$ be defined as the finite and totally ordered LTS having cardinality $g + 1$ and $\varsigma \in L$ be a numerical value resulting from a symbolic aggregation operation. Then the function which transforms ς into a 2-TL variable is given by:

$$\Delta: [0, g] \rightarrow L \times [-0.5, 0.5) \equiv \bar{L}$$

$$\Delta(\varsigma) = (\ell_\varepsilon, \alpha)$$

with $\varepsilon = \text{round}(\varsigma)$ & $\alpha = \varsigma - \varepsilon \in [-0.5, 0.5)$ where $\text{round}(\cdot)$ assigns the closest index label to ς .

Definition 1.5.8. [92] The distribution of distinct linguistic labels in 2-TL model is defined on a finite ordered scale set exhibiting the following properties:

- (i) L is ordered: $\ell_\vartheta \geq \ell_\varepsilon$, if $\vartheta \geq \varepsilon$.
- (ii) The negation operator is given as $\text{neg}(\ell_\vartheta) = \ell_\varepsilon$ such that $\varepsilon = g - \vartheta$.
- (iii) Maximization operator: $\max(\ell_\vartheta, \ell_\varepsilon) = \ell_\vartheta$ if $\vartheta \geq \varepsilon$.
- (iv) Minimization operator: $\min(\ell_\vartheta, \ell_\varepsilon) = \ell_\vartheta$ if $\vartheta \leq \varepsilon$.

Definition 1.5.9. [92] Let $L = \{\ell_0, \ell_1, \dots, \ell_g\}$ be a linguistic set and $(\ell_\varepsilon, \alpha)$ be a 2-TL variable. Then there always exists Δ^{-1} ,

$$\Delta^{-1}: \bar{L} \rightarrow [0, g]$$

$$\Delta^{-1}(\ell_\varepsilon, \alpha) = \varepsilon + \alpha = \varsigma.$$

so that it yields an equivalent number $\varsigma \in [0, g] \subset \mathbb{R}$.

Remark 1.5.1. We may convert a linguistic term into a 2-TL variable by attaching 0 as symbolic translation, i.e. $\ell_\varepsilon \in L \implies (\ell_\varepsilon, 0) \in \bar{L}$.

Further, Herrera & Martínez [92] presented the comparison of 2-TL information by using conventional lexicographic order.

Definition 1.5.10. Let $(\ell_\vartheta, \alpha_\vartheta)$ and $(\ell_\varepsilon, \alpha_\varepsilon)$ be two 2-TL variables.

- if $\vartheta < \varepsilon \implies (\ell_\vartheta, \alpha_\vartheta) < (\ell_\varepsilon, \alpha_\varepsilon)$.
- if $\vartheta = \varepsilon$ then,
 1. if $\alpha_\vartheta = \alpha_\varepsilon \implies (\ell_\vartheta, \alpha_\vartheta) = (\ell_\varepsilon, \alpha_\varepsilon)$;
 2. if $\alpha_\vartheta < \alpha_\varepsilon \implies (\ell_\vartheta, \alpha_\vartheta) < (\ell_\varepsilon, \alpha_\varepsilon)$;

The operation of negation is found to be an elementary notion of linguistic variables. The following definition of negation of 2-TL variable is broadened in light of conventional negation operator for linguistic variables.

Definition 1.5.11. [92] Let $L = \{\ell_\varepsilon \mid \varepsilon = 0, \dots, g\}$ be the finite LTS with cardinality $g + 1$, then the 2-TL negation operator is given as

$$\text{neg}((\ell_\varepsilon, \alpha)) = \Delta(g - (\Delta^{-1}(\ell_\varepsilon, \alpha))).$$

It is worth noting that if $\alpha = 0$, the negation operator is given as $\text{neg}(\ell_\varepsilon) = \ell_{g-\varepsilon}$, which is similar to a standard negation of the linguistic terms.

1.5.4 2-tuple unbalanced LCM

The unbalanced 2-TLCM as discussed earlier, is introduced by Malhotra & Gupta [153] to cope with unbalanced linguistic information. To do so, an efficient method is designed to characterize the interpretation of the linguistic terms belonging to the ULTS.

Definition 1.5.12. [153] Let $\mathbb{L}\$ = \{t_{a^{-n}}, t_{a^{-(n-1)}}, \dots, t_{a^0}, \dots, t_{a^m}\}$, $a > 1$, $n, m \in \mathbb{Z}^+$ be a multiplicative ULTS having cardinality $n + m + 1$ such that n and m may not be equal. Moreover, the set $\mathbb{L}\$$ is ordered if the given properties hold:

- $\mathbb{L}\$$ is an ordered set: $t_{a^\varepsilon} \geq t_{a^\vartheta}$ for $\varepsilon \geq \vartheta$;
- $\max(t_{a^\varepsilon}, t_{a^\vartheta}) = t_{a^\varepsilon}$ for $\varepsilon \geq \vartheta$;
- $\min(t_{a^\varepsilon}, t_{a^\vartheta}) = t_{a^\varepsilon}$ for $\varepsilon \leq \vartheta$.

Definition 1.5.13. [153] The negation operator for such 2-tuples is defined as:

$$\text{Neg}(t_{a^\varepsilon}, \gamma) = (\text{Neg}(t_{a^\varepsilon}), -\gamma)$$

where $\text{Neg}(t_{a^\varepsilon})$ is defined as follows:

- If $n = m$, then $\text{Neg}(t_{a^\varepsilon}) = t_{a^{-\varepsilon}}$
- If $n < m$, then $\text{Neg}(t_{a^\varepsilon}) = t_{a^{-\varepsilon}}$, for $\varepsilon \leq n$ and $\text{Neg}(t_{a^\varepsilon}) = t_{a^{m-\varepsilon}}$ otherwise.
- If $n > m$, then $\text{Neg}(t_{a^\varepsilon}) = t_{a^\varepsilon}$, if $\varepsilon \leq m$ and $\text{Neg}(t_{a^\varepsilon}) = t_{a^{\varepsilon-n}}$ otherwise.

Definition 1.5.14. [153] Let $\mathbb{L}\$$ be the multiplicative ULTS as defined above and $\mathbb{R}$ be the set of real numbers. Then the numerical scale function

$$NS: \mathbb{L}\$ \rightarrow \mathbb{R}$$

such that

$$NS(t_{a^\varepsilon}) = a^\varepsilon$$

Definition 1.5.15. [153] Let $\beta \in [a^{-n}, a^m]$ be a numeric value supporting the symbolic aggregation result. Then the translation function that translates β to its analogous 2-tuple unbalanced linguistic value is

$$\Omega: [a^{-n}, a^m] \rightarrow \mathbb{L}\mathbb{S} \times [-0.5, 0.5]$$

$$\Omega(\beta) = (t_\lambda, \gamma)$$

$$\begin{cases} \lambda = a^{\varepsilon_L} \text{ or } a^{\varepsilon_R}, & \text{for } \varepsilon_L, \varepsilon_R \in \{-n, \dots, m\} \\ \gamma = \frac{\beta - \lambda}{d_L + |d_R|}, & \text{for } \gamma \in [-0.5, 0.5] \end{cases}$$

where, d_L and d_R represent nearest left and right distance of λ from β respectively.

Definition 1.5.16. [153] Let $\mathbb{L}\mathbb{S} = \{t_{a^{-n}}, t_{a^{-(n-1)}}, \dots, t_{a^0}, \dots, t_{a^m}\}$ be a ULTS and let (t_λ, γ) be a 2-tuple corresponding to numerical value $\beta \in [a^{-n}, a^m] \subset \mathbb{R}$. An inverse translation function Ω^{-1} returning equivalent numerical information from a corresponding 2-tuple linguistic information is defined as:

$$\Omega^{-1}: \mathbb{L}\mathbb{S} \times [-0.5, 0.5] \rightarrow [a^{-n}, a^m]$$

$$\Omega^{-1}(t_\lambda, \gamma) = \gamma \times (d_L + |d_R|) + \lambda = \beta.$$

1.5.5 Normal Cloud model

The normal cloud model, as described by Li et al. [111], is a cognitive model that implements the uncertain translation between linguistic concepts and quantitative values and describes the uncertainties of linguistic concepts, particularly the randomness and fuzziness.

Definition 1.5.17. [111] Let U be the universe of discourse, and $\tilde{T}$ be a qualitative concept defined over U . If $x \in U$ is a random instantiation of the concept $\tilde{T}$ that satisfies $x \sim N(EX, EN'^2)$, $EN' \sim N(EN, HE^2)$, and $y \in [0, 1]$ defined as

$$y = e^{-\frac{(x-EX)^2}{2(EN')^2}}$$

be the certainty degree of x , then the distribution of x in the universe U is called as a normal cloud and (x, y) is called a normal cloud drop. Here,

- EX is the mathematical expectation of the cloud drops.
- Entropy EN represents the uncertainty measurement of the qualitative concept.
- Hyper entropy HE is the uncertain degree of entropy EN .

Given the numerical attributes (EX, EN, HE) of a cloud, the forward generator of this cloud is given in the following algorithm:

Forward normal cloud generator

Input: three parameters EX, EN, HE and the number of cloud drops c .

Output: c cloud drops and their certainty degree.

Steps:

1. Generate a normally distributed random number EN'_i with expectation EN and variance HE^2 .
 2. Generate a normally distributed random number x_i with expectation EX and variance EN'_i .
 3. Calculate $y_i = e^{-\frac{(x_i - EX)^2}{2(EN'_i)^2}}$.
 4. x_i is a cloud drop in the universe, and y_i is the degree of certainty of x_i belonging to the concept $\tilde{T}$.
 5. Repeat steps 1 – 4 until c cloud drops have been generated.
-

1.6 Research Gaps

Although the above-discussed literature is capable enough to capture the uncertainty underlying matrix games with uncertain information, the following research gaps have originated in the existing studies.

1. The two-person non-zero sum games considered till now are either based on numerical information or fuzzy data. Such type of data causes information loss while applying fusion processes while aggregating the final data. Moreover, the defuzzification methods used to solve fuzzy bimatrix games are restricted to a particular solution procedure. Hence, a more generalized methodology must be developed to draw the value of a bimatrix game directly.
2. Concerning zero-sum games, it is envisaged that the LT set utilized in the prevailing literature for addressing the information is balanced because its terms are symmetrically and uniformly distributed. However, from a more logical viewpoint, some problems are not well suited to deal with a balanced

LT set; instead, incorporating linguistic information that is non-uniformly distributed is required for real-time decision problems. Hence, it is necessary to address matrix games with payoffs belonging to a ULTS.

3. Speaking of the representation of linguistic information in terms of clouds, either the existing cloud models are based on symmetric linguistic data or the unbalanced linguistic distribution assessments utilized in the existing models are primarily based on the hierarchical unbalanced linguistic model, which is a very complex and computationally expensive approach. Apart from this, the ULTS used in the existing literature is first converted to a balanced set first, and then it is worked upon. In this way, the unbalanced property of an LT set loses its literal meaning. Hence, a more appropriate and straightforward model is needed to be employed for the application of linguistic cloud models.
4. In order to deal with unbalanced linguistic information, several PLTSs have been introduced in the literature. However, an additive scale set is still used to represent the data; therefore, the information does not retain its literal meaning. The linguistic information in probabilistic game problems is still a type of uncertainty that needs to be taken care of. To address multiplicative unbalanced linguistic information, various authors have developed accurate and innovative cloud decision support models. Such models are required to be implemented in game problems to handle the various type of uncertainties, like fuzziness and randomness.
5. Considering IFS theory, analogous to IFS, an intuitionistic fuzzy linguistic set (IFLS) is outlined by a linguistic membership degree and a linguistic non-membership degree, respectively. Several authors have significantly solved many decision-making problems in the realm of IFLSs. However, it has been noticed that membership and non-membership of an IF variable are taken to be in the same evaluation scale set. If we try to understand the scenario of membership and non-membership in the real sense, there is no clarity on how to differentiate between membership and non-membership because both belong to the same domain. Additionally, the existing theories are unsuitable for defining such operational laws and aggregation operators of IFLSs, which are closed under their defined operations. Hence it is crucial

to investigate such IFLSs that have a clear picture of the origin of the degree of membership and non-membership of an element.

1.7 Motivation

Human beings are always making decisions beneath the linguistic environment to tackle ill-structured problems successfully. As our physical world is always changing, it seems to be a common attribute of human intelligence to make decisions with knowledge and solve the pertinent problems that we encounter within our real-life conflicting situations. Decision-making is guided by the cognitive process, which depends on an individual's perception, i.e., in the form of natural language. The process attracts the researchers to model and present the linguistic data using new tools and methodologies. Researchers in game theory and its allied areas have allowed us to deal with the cognitive process of human behavior concerning imprecise linguistic premises. Several methods have been developed in the literature for managing such information. Generally, the problems present quantitative aspects easily accessible through precise numeric values; nevertheless, in some cases, they present qualitative aspects that are complicated to handle by precise numeric values. The introduction of various LCMs has improved the accuracy of game problems and facilitated the processes of CW by managing linguistic information. The 2-TL model, as well as various other unbalanced LCMs, have delivered remarkable advancement in the uncertainty domain pertaining to the field of matrix game theory. Moreover, only a few LCMs have been implemented in matrix games, that too in a very straightforward way so that information loss occurs in a huge manner. Further, the unbalanced information prevailing in the game problems has not yet been taken care of in the existing theories. Apart from this, the representation of information in terms of fuzzy linguistic variables may only eradicate the fuzziness of the problem. Randomness is still an intact uncertainty that needs to be addressed in game problems.

This thesis is motivated by the existing literature and the research gaps prevailing in the existing literature. The actual motivation behind this thesis is that no such methodologies are defined to address linguistic uncertainty prevalent in game problems without making any information loss. Therefore, in this thesis, we have analyzed the prosecution of balanced as well as unbalanced information

emerging in the games due to unavoidable real-life circumstances. We proposed several methodologies for meticulously modeling the uncertainty in intricate game problems with broad applications and further validated these methodologies with appropriate data. All these methodologies are capable enough to deal with game problems with payoffs in the form of “words” rather than precise or fuzzy numbers. Henceforth, the primary purpose of this current study is to develop accurate and innovative models to address linguistic concepts in matrix games by emphasizing the aforementioned momentous research gaps.

1.8 Organization of the Thesis

This thesis is organized into six chapters. Chapter 1 is introductory. The outline of the proposed research study in Chapter 2 onwards is explained as follows:

Chapter 2 presents a methodology to solve two-person non-zero sum games having a 2-TL environment. The LT set utilized in the current theory is a balanced LT set having all linguistic terms distributed symmetrically. This study provides the concept of linguistic bimatrix games, which is considered to be a new study in the literature addressing such types of games. Even though several methodologies are available in the literature to solve constant-sum linguistic matrix games, the approach proposed by us is an improvement to offer acceptable results faster.

It is noteworthy that the DMs cannot always express the qualitative information that is distributed uniformly and symmetrically in real-world decision systems. In most situations, experts have to use LT sets having the non-uniform distribution of linguistic labels. Several pioneering studies have been put forward to support the context of ULTS. But no methodology has been yet applied to game problems. Henceforth, in **Chapter 3**, we propose a newly constructed method to handle matrix games with unbalanced linguistic information. For that purpose, first, we consider matrix games that have proportional linguistic payoffs. The proportional linguistic payoffs consist of two linguistic terms. However, a generalization of the proportional linguistic terms set is also provided as a distribution assessment linguistic set. Hence, in this chapter, matrix games with payoffs as linguistic distribution assessment are also discussed and solved.

Addressing decision-making issues with linguistic information to apprehend uncertainty has received much interest. However, with rapid societal development,

linguistic decision-making problems have become increasingly complex and challenging. In such a scenario, it is necessary to introduce models that are lucrative for defining linguistic labels and can synthetically explain the notion of randomness and fuzziness of the decision information simultaneously. Numerous cloud models are defined to describe an ambiguous mapping between qualitative notions and their quantitative manifestations. However, the existing research on cloud models is based on a balanced linguistic term set and cannot address information in an unbalanced linguistic form. Henceforth, to thoroughly capture the intrinsic fuzziness and vagueness of decision information, the work proposed in **Chapter 4** aims to introduce a cloud model based on multiplicative unbalanced linguistic term sets.

Further in **Chapter 5**, the work done in Chapter 4 is extended to the application of such multiplicative ULTSs, that have linguistic terms with probabilities associated with them towards matrix games. In such types of games, the players record their expressions in terms of probabilistic multiplicative unbalanced linguistic terms. This type of game is a useful technique for multiple criteria analysis. Further, to thoroughly capture the uncertainty involved in the payoffs of such games, they are converted to probabilistic unbalanced linguistic clouds that describe the uncertainties of qualitative payoffs by implementing the uncertain transformation between payoffs and quantitative values. This model overcomes the inherent limitation of existing linguistic games where only the fuzziness of concepts is evaluated.

Chapter 6 introduces the concept of unbalanced 2-tuple intuitionistic fuzzy linguistic numbers. The IFLS is outlined by a linguistic membership degree and a linguistic non-membership degree. By using IFLS, the DMs can consider a linguistic hesitancy degree, where they cannot simply convey their perception using one linguistic term. Adding to the contribution of the present research, in this chapter, we try to develop an intuitionistic model in terms of a multiplicative unbalanced LT set.

A summary followed by the future scope of the research work is evinced to conclude the thesis after Chapter 6.

Table 1.1: Summary of literature based on fuzzy games

Title	Year	Paper	Description	Advantages/Limitations
Fuzzy Games : A description of the concept.	1978	[35]	The notion of fuzzy sets is introduced. A heuristical discussion of fuzzy games and their relationship to classical games is presented.	- The intuitive meaning of fuzzy games is defined. - Usual properties of fuzzy sets are provided. - Limited to two players.
Fuzzy Linear Programming models to solve Fuzzy Matrix Games	1989	[41]	A fuzzy linear programming problem is proposed for each player in a two-person zero sum matrix game with imprecise values in the matrix. The approach is demonstrated as a generalisation of the standard method for solving a classical game. The use of certain approaches for ranking fuzzy numbers has led to the development of auxiliary models that are suggested as solutions to these issues.	- The proposed method is a generalization of the classical method, hence easy to compute. - The methods used are more operative, and simple and preserve the linearity and dimensionality of the constraints. - The game problems are only limited to payoffs of the type TFNs.
Max-min solutions for fuzzy multiobjective matrix games	1994	[186]	Fuzzy multiple payoff matrices for two-person zero sum games are examined. A fuzzy goal's degree of achievement is determined, and the max-min strategy with regard to that degree of completion is looked at. To account for the uncertainty of human judgement, each player has an ill-defined objective for the payout. Two techniques have been devised for computing the solutions where membership functions of fuzzy objectives and a form function of fuzzy number entries in a fuzzy payout matrix can be represented as linear functions.	- Multiple payoffs are considered in games because a decision-making problem under conflict involves multiple objectives. - Only the linear membership functions of fuzzy goals and shape functions of L-R fuzzy numbers for fuzzy payoffs are considered.
Equilibrium Solutions for Multiobjective Bimatrix Games incorporating Fuzzy Goals	1995	[172]	Equilibrium solutions in terms of the degree of attainment of a fuzzy goal for games in fuzzy and multiobjective environments are examined. A fuzzy goal for a payoff and the equilibrium solution for the degree of attainment of a fuzzy goal is defined.	- Basic methods are employed to aggregate multiple fuzzy goals. - The equilibrium solutions are shown as equivalent to the solution of a mathematical programming problem. - The proposed solutions belong to the set of Pareto-optimal equilibrium solutions only after fulfilling some special conditions.
A Fuzzy Multi-Objective Programming Approach to Solve Fuzzy Matrix Games	1999	[65]	The value of the matrix game with fuzzy numbers has been considered as a fuzzy number in this paper. Based on both the linear programming models of a classical game and operations of fuzzy numbers, fuzzy multiobjective linear programming models are established and a corresponding effective solution method which allows us to use directly the Simplex Method is proposed. These models are shown as generalizations of those used to solve a classical game. The models obtained along this paper have expressions for TFNs.	- The linear programming used to solve game problems is defined just as a generalization of classical games, making it easier to compute. - Subjective criteria are used to rank fuzzy numbers. - The expressions are limited to TFNs.
Characterization of the Equilibrium Strategy of the Bimatrix Game with Fuzzy Payoff	2000	[149]	Bimatrix games with fuzzy payoff matrices are considered. Two types of concepts of equilibrium strategy are defined. Their relationships are investigated. It is proved that the Nash equilibrium strategies exist in any fuzzy bimatrix game.	- The proof that Nash equilibrium strategies exist for any fuzzy bimatrix game is done. - No application of the work done is provided.
Multiobjective fuzzy two-person non-zero-sum games	2001	[173]	Two-person non-zero sum games with single and multiple fuzzy payoffs are considered. Equilibrium solutions in terms of the degree of attainment of a fuzzy goal for such games are examined.	- The equilibrium solutions are equal to optimal solutions to certain mathematical programming problems that are easier to solve. - Restricted to linear fuzzy payoffs.
An alternative approach to the Bimatrix Non-cooperative Game with Fuzzy Multiple Objectives	2002	[48]	The feasibility of using fuzzy multi-objective quadratic programming to describe and resolve the bimatrix multi-objective non-cooperative game is explored. The model is solved by simultaneously maximizing the achievement level of each goal. A revised genetic approach is used to find stratifying results.	- The method is quite straightforward and clear to describe and resolve the traditional bimatrix multi-objective non-cooperative game. - Studying results are also acceptable and easily understandable. - A more efficient evolving algorithm is required to solve the multi-objective game model using fuzzy coefficients in payoff matrices.
On characterization of equilibrium strategy of two-person zero-sum games with fuzzy payoffs	2003	[150]	Based on fuzzy max order, three kinds of concepts of minimax equilibrium strategies are defined and their properties are investigated.	- It is shown that the set of minimax equilibrium strategies coincides with sets of Nash equilibrium strategies of a family of parametric bimatrix games with precise payoffs. - The approach is a natural extension of classical approaches in game theory to games with fuzzy payoffs. - Limited to bimatrix games.
Matrix Games with fuzzy goals and fuzzy linear programming duality	2004	[20]	A two-person zero sum matrix game with fuzzy goals is conceptualized in a more generalized way. The equivalence of such games to a primal-dual pair of fuzzy linear programming problems is shown.	- It is proved that a matrix game with fuzzy goals is equivalent to solving a fuzzy linear programming problem. - The game theoretic results are not exactly similar to the crisp ones, although they might hold under some special conditions.

Table 1.1: (Continued)

Title	Year	Paper	Description	Advantages/Limitations
Duality in linear programming with fuzzy parameters and matrix games with fuzzy payoff	2004	[21]	A dual for linear programming problems with fuzzy parameters is introduced and it is shown that a two-person zero sum matrix game with fuzzy payoffs is equivalent to a primal-dual pair of such fuzzy linear programming problems.	- The duality in linear programming with fuzzy parameters is introduced. - The introduced duality theory plays a key role in the development of a modified procedure and its justification for solving such a game. - A generalized theory is missing.
2-person zero sum game approach for fuzzy multiple attribute decision making problems	2006	[49]	Weights of an MCDM problem with a fuzzy decision matrix by formulating it as a two-person zero sum game with an uncertain payoff matrix is obtained. The equilibrium solution and the resolution method for the MCDM game have also been developed.	- The defined innovative decision problem has dependent fuzzy triangular entries as a crisp two-person zero sum game with an uncertain payoff matrix. - The solution approach is reduced to solving a linear programming problem. - Handling the dependency among attributes to obtain accurate weights is still a major task.
An approach for solving fuzzy games	2006	[148]	A Nash equilibrium in a fuzzy environment is computed and is represented by a fuzzy approximate Nash equilibrium in a space of discrete mixed strategies. For discrete mixed strategies, the relationship between the discrete degree and the approximate degree is discussed. Based on the fuzzy regret degree, a genetic algorithm for computing a fuzzy Nash equilibrium is given.	- Computation complexity is reduced in discrete probability space. - The algorithm has good convergence and high efficiency.
Solution of Fuzzy Matrix Games: An application of the extension principle	2007	[112]	For the two-person zero sum game, where the pay-offs can be represented as fuzzy numbers and are only roughly understood, a solution approach is created. Two two-level mathematical programmes are developed based on the extension principle. They are then converted into two regular one-level linear programmes so they may be managed by using a dual formulation and variable substitution technique.	- The value of the game is expressed by the membership function rather than by a point value that avoids information loss. - More information is provided for making better decisions. - Restricted to zero sum games.
Fuzzy two-person zero-sum games	2008	[34]	Fuzzy Monte Carlo methods are used to get approximate optimal fuzzy mixed strategies for fuzzy two-person zero sum games. The games are fuzzified and the optimal fuzzy values for the players and optimal fuzzy mixed strategies are defined.	- The fuzzy values for both players are defined. - It is shown that the value of Player <i>I</i> is always less than or equal to the value of Player <i>II</i>. - It remains an open question whether the value of Player <i>I</i> is equal to the value of Player <i>II</i>, ever.
Studying interval valued matrix games with fuzzy logic	2008	[51]	The interval-valued game matrices are considered. The results of classical strictly determined matrix games are extended to fuzzily determined interval matrix games.	- The model parallels the classical game context in a closely analogous way. - The theory of optimal mixed strategies for interval valued games is introduced, once again mimicking the classical model of converting the game into a linear programming problem. - The assumptions made to solve game problem as linear programming problem seem unrealistic.
Lexicographic method for matrix games with payoffs of triangular fuzzy numbers	2008	[114]	Fuzzy optimization problems are established for two players. Then, based on the order relation of TFNs, the fuzzy optimization problems are decomposed into three linear programming models. Finally, maximin and minimax strategies for players and the fuzzy value of the game can be obtained by solving two corresponding auxiliary linear programming problems.	- The auxiliary linear programming problems constructed from the initial matrix game are easily computed using Simplex method. - The defined models extend the classical models. - No conclusion is made about the initial fuzzy programming problems.
Two-Person Non Zero-Sum Bimatrix Game with Fuzzy Payoffs and its Equilibrium Strategy	2009	[88]	Fuzzy bimatrix games with fuzzy payoffs are considered. The concept of minimax equilibrium strategies is defined based on fuzzy max order.	- The concept of equilibrium strategies for fuzzy bimatrix games is introduced. - Limited to some particular concepts.
Solving bimatrix games with fuzzy payoffs by introducing Nature as a third player	2009	[106]	The payoffs are assumed to be fuzzy intervals with bounded support. A new equilibrium concept is defined for this game by introducing Nature as a third player and using α -cuts. Sufficient conditions for its existence and a procedure for its computation are provided.	- Nature has no objective function. However, its strategies are chosen in the α-cuts of the fuzzy entries of the payoff matrices of both players. - The concept of α-maxmin solution reflects simultaneously the strategic aspect of the game and the decision-making under uncertainty element involved in the game as a result of the presence of uncertainty in the payoffs of players expressed in terms of fuzzy sets. - The approach is only restricted to bimatrix games.

Table 1.1: (Continued)

Title	Year	Paper	Description	Advantages/Limitations
Fuzzy Nash Equilibrium in Fuzzy Games Using Ranking Fuzzy Numbers	2010	[44]	A new method for finding the pure strategy Nash equilibriums, to realistically analyze the games with fuzzy payoffs are investigated. Using ranking fuzzy numbers, a fuzzy preference relation is constructed over payoffs. The priorities of payoffs are considered as the degree of being Nash equilibriums.	- The priorities of payoffs were interpreted as the degree of being Nash equilibrium. - Each cell has the possibility of being Nash equilibrium.
Nash equilibrium strategy for fuzzy non-cooperative games	2011	[52]	The fuzzy environment is used to analyse two-person zero sum games. Equilibrium strategies are suggested to exist in the asymmetric fuzzy environment. Nonetheless, there might not always be Nash equilibrium tactics. Consequently, two unique scenarios are showcased wherein solutions for Nash equilibrium are present. Crisp bimatrix games with parameters are introduced as a notion to explore the presence of (weak) Pareto-Nash equilibrium solutions for fuzzy matrix games. The (weak) Pareto Nash equilibrium strategies for the fuzzy matrix games are introduced by solving the parametric bimatrix games.	- All types of equilibrium strategies based on general asymmetrical TFNs are defined, while no such strategies are defined in literature before this. - A correlation is established between fuzzy games and crisp games.
Solving constrained matrix games with payoffs of triangular fuzzy numbers	2012	[119]	The goal of this study is to create a methodical approach to solving limited matrix games with TFN payoffs. For constrained matrix games with payoffs of TFNs and the values, the notions of alpha-constrained matrix games are introduced. Any Alpha-constrained matrix game has an interval-type value since it is demonstrated by the linear programming duality theorem that players' gain-floor and loss-ceiling always share a similar interval-type value. Furthermore, it is demonstrated that every limited matrix game with TFN payoffs always has a TFN-type fuzzy value by applying the fuzzy set representation theorem.	- No study before this paper has yet been attempted for constrained matrix games with payoffs of TFNs since there is no effective way to simultaneously incorporate the payoffs' fuzziness and strategies' constraints into classical and/or fuzzy matrix game methods. - The mean and the lower and upper limits of the TFN type fuzzy value of the constrained matrix game with payoffs of TFNs can be directly obtained by solving the derived three linear programming models with all data taken from only 1-cut and 0-cut of payoffs. Hence the fuzzy value of the constrained matrix game with payoffs of TFNs can be easily and explicitly constructed.
Fuzzy Nash Equilibriums in Crisp and Fuzzy Games	2013	[45]	A fuzzy Nash equilibrium is introduced to determine a graded representation of Nash equilibriums in crisp and fuzzy games. This interpretation shows the distribution of equilibriums in the matrix form of a game and handles uncertainties in payoffs. In addition, a new method to rank fuzzy values with the user's viewpoint is investigated. By this means, the definition of satisfaction function, which provides the result of comparison in the form of real value, is developed when users have preferences regarding the payoffs.	- The most significant advantages of using the proposed method are the range of game-theoretic problems that can be analyzed and the information about equilibriums that can be obtainable to players. - The proposed method incorporates player viewpoints in the domain of fuzzy value and transforms it into another fuzzy value. - Comparing the results with the fuzzy Nash equilibrium, the results obtained through this strategy were more sensitive to the payoffs.
A New Method for Solving Bimatrix Game using LCP under Fuzzy Environment	2013	[82]	A new method is proposed for solving the Fully Fuzzy Bimatrix Game using the Linear Complementarity approach. In such a game, the player's payoff matrix is represented by trapezoidal fuzzy numbers. The given Fully Fuzzy Bimatrix Game is transformed into the Fully Fuzzy Linear Complementarity Problem. The obtained solution is the solution of the given fuzzy bimatrix game.	- A fully fuzzy bimatrix game is solved without introducing any artificial variable. - There is no objective function to be optimized. Hence, an optimum strategy is achieved at a minimum number of iterations. - A negative value of probability represents the interval number spread of the given problem's solution output.
An effective methodology for solving matrix games with fuzzy payoffs	2013	[66]	An effective methodology for solving matrix games with payoffs expressed by trapezoidal fuzzy numbers is discussed. The concept of Alpha-matrix games is introduced and it is proved that the players' fuzzy values are always identical, hereby, any matrix game with payoffs expressed by trapezoidal fuzzy numbers has a fuzzy value, which is also a trapezoidal fuzzy number.	- A new concept of Alpha-matrix games is introduced to address fuzzy games. - The players' defuzzification values closely depend on not only the choice of the ranking function but also the parameters and the adequacies, which are difficult to appropriately determine a priori.
Effect of defuzzification methods in solving fuzzy matrix games	2015	[182]	This paper deals with two-person matrix games whose elements of the payoff matrix are fuzzy numbers. Then the corresponding matrix game has been converted into a crisp game using different defuzzification techniques. The value of the matrix game for each player is obtained by solving corresponding crisp game problems using the existing method.	- The fuzzy game is converted to a crisp game, and is then solved like a traditional game. - Several defuzzification techniques are used to convert a fuzzy game into a crisp game, and are then compared. - Defuzzification causes information loss. - The solution of a fuzzy game is evaluated without converting it to a classical game.
A new approach for the solution of fuzzy games	2018	[105]	A new approach is proposed to solve the games with imprecise entries in its payoff matrix. All these imprecise entries are assumed to be trapezoidal fuzzy numbers. Also the proposed approach provides fuzzy optimal solution of the fuzzy valued game.	- The proposed method is capable of giving fuzzy solutions for the fuzzy matrix game. - Limited to payoffs of the form trapezoidal fuzzy numbers.

Table 1.2: Summary of literature based on intuitionistic fuzzy games

Title	Year	Paper	Description	Advantages/Limitations
Intuitionistic Fuzzy Bi-Matrix Games	2007	[169]	An attempt is made to conceptualize the meaning of the Nash equilibrium solution for non-cooperative bi-matrix games. A methodology is defined to take advantage of the relationship between IFS and matrix game theories.	- The defined approach helps to automate the solution by the players of the fuzzy strategies chosen and their corresponding fuzzy preferences needed to play the game. - This approach is especially appropriate for the DM whose preference for attributes and alternatives are both unknown.
The Linear Programming Approach to Matrix Games with Payoffs of Intuitionistic Fuzzy Sets	2009	[165]	The definition of matrix games with payoffs of IFSs is given and according to the concept of the Pareto solutions, the concept of solutions for the game is defined. It is proven that the solutions of matrix games with payoffs of IFSs can be obtained through solving a pair of linear programming models.	- Although the method proposed in this paper is illustrated with the market share problem, it may also be applied to game problems in IFSs with a wide spectrum of possibilities. - The solution of the IF matrix game for Players is obtained through solving a pair of linear programming models derived from two auxiliary nonlinear bi-objective programming models based on operations of the IFSs and the concept of solutions of the IF matrix game. - More effective methods may be investigated.
A nonlinear programming approach to matrix games with payoffs of Atanassov's intuitionistic fuzzy sets	2009	[115]	The concept and methodology of matrix games with payoffs of IFSs are developed. In this methodology, the concepts of the solutions for matrix games with payoffs of IFSs are defined based on the degree of membership and the degree of non-membership. It is proven that matrix games with payoffs of IFSs have solutions, which can be obtained through solving a pair of nonlinear programming models derived from two auxiliary nonlinear bi-objective programming models.	- Solving matrix games with payoffs of the IFSs is turned to solving a pair of nonlinear bi-objective programming problems. - A pair of nonlinear programming models is derived from the two auxiliary nonlinear bi-objective programming models using the weighted average method. - Other methods may be explored to solve the bi-objective problem.
Bi-matrix Games with Intuitionistic Fuzzy Goals	2010	[169]	An application of IF programming to a two-person bimatrix game (pair of payoff matrices) for the solution with mixed strategies using linear membership and non-membership functions is presented. The IF goal for a choice of a strategy in a payoff matrix is also introduced.	- Since the strategy spaces of player I and player II could be polyhedral sets, the constrained IF bimatrix game may also be conceptualized on the lines of crisp constrained bimatrix games. - It is shown that the solution of such a game is the optimal solution of a mathematical programming problem.
Mathematical-Programming Approach to Matrix Games With Payoffs Represented by Atanassov's Interval-Valued Intuitionistic Fuzzy Sets	2010	[116]	The concept of solutions of matrix games with payoffs represented by interval-valued IFSs (IVIFSs) is defined, and some important properties are studied using multiobjective-programming and duality-programming theory. It is proven that each matrix game with payoffs represented by IVIFSs has a solution, which can be obtained through solving a pair of auxiliary linear/nonlinear-programming models derived from a pair of nonlinear bi-objective interval programming models.	- The payoffs of matrix games seem to be suitably expressed with IVIFSs, which are characterized by a membership function and a non-membership function, whose values are intervals rather than real numbers. - The concept of solutions of the matrix game with payoffs represented by IVIFSs may be given in a similar way to that of the Pareto optimal solutions. - The linear interval programming model defined in the paper is not a standard method to solve. Hence it is necessary to transform the problem into a standard problem before solving it.
Two-Person Zero-Sum Matrix Game Based on Intuitionistic Fuzzy Set	2010	[267]	A concept of possibility measure for IFSs is introduced, and the credibility measure for IFSs are defined. Based on the credibility measure of IFSs, a type of concept of minimax equilibrium strategies σ -credible min-max equilibrium strategy is defined. An iterative algorithm based on IF simulation that improves on the fuzzy simulation is designed to find the equilibrium strategies.	- The IF simulation is provided. - An algorithm based on IF simulation is designed to seek the minimax equilibrium strategies. - Other techniques than the iterative technique may be used for simulation.
A Lexicographic Method for Matrix Games with Payoffs of Triangular Intuitionistic Fuzzy Numbers	2010	[166]	The concept of triangular IF numbers (TIFNs) and their arithmetic operations and cut sets are introduced as well as the ranking order relations. Secondly, the concept of solutions for matrix games with payoffs of TIFNs is defined. A lexicographic methodology is developed to determine the solutions of matrix games with payoffs of TIFNs for both Players through solving a pair of bi-objective linear programming models derived from two new auxiliary IF programming models.	- Using the lexicographic method, two simpler auxiliary linear programming models are constructed to generate the maximin and minimax strategies for players and the value of the IF matrix game. - It has been shown that the models of the IF matrix game proposed in this paper extend those of the fuzzy matrix game. - Some applications in game problems using IFSs with a wide spectrum of possibilities may be applied.

Table 1.2: (Continued)

Title	Year	Paper	Description	Advantages/Limitations
Intuitionistic Fuzzy Optimization Technique for Nash Equilibrium Solution of Multi-objective Bi-Matrix Games	2011	[170]	An application of the IFO model to a two-person multiobjective bimatrix game (pair of payoff matrices) for the Nash equilibrium solution(NES) with mixed strategies is presented. The IF goal for a choice of a strategy in a payoff matrix is introduced.	• A model for studying multi-objective bimatrix games with IF goals is presented. • The degree of acceptance and the rejection of objectives and constraints are introduced together, which both cannot be simply considered as a complement to each other, and the sum of their value is less than or equal to 1. • Other relevant applications of the proposed theory could be implemented in the paper.
Application of Triangular Intuitionistic Fuzzy Numbers in Bi-matrix Games	2012	[188]	The concept of TIFNs and their cut sets are introduced as well as the inequality relations between two TIFNs. Secondly, a bimatrix game with payoffs of TIFNs is considered and an attempt is made to conceptualize the meaning of a Nash equilibrium solution for such games.	• This process helps to automate the solution by the players of the pure strategy and their corresponding preferences needed to play an IF bi-matrix game. • This game approach is especially appropriate for the DM, whose preference for attributes and alternatives are both unknown. • The method may be applied to other competitive decision-making problems.
A Bi-objective Programming Approach to Solve Matrix Games with Payoffs of Atanassov's Triangular Intuitionistic Fuzzy Numbers	2012	[118]	In the methodology defined in this paper, the concepts and ranking order relations of TIFNs are defined. A pair of bi-objective linear programming models for matrix games with payoffs of TIFNs is derived from two auxiliary IF programming models based on the ranking order relations of TIFNs defined in this paper.	• The results show that IFS/IFN may express information more abundant and flexible than the fuzzy set when it is used to deal with uncertainty in game theory. • The constructed linear programming models of matrix games with payoffs of TIFNs in this paper extend those of fuzzy matrix games with payoffs of TFNs. • The proposed models and method may be extended to matrix games with payoffs of trapezoidal IF numbers.
Multi-objective bimatrix game under intuitionistic fuzzy environment	2012	[248]	Since determining payoffs is frequently dependent on the judgment and intuition of the players in the game, the payoff of each strategy is described as IFSs. Then, an IF weighted average operator is used to obtain the integral payoff matrices. Based on the score functions of IFSs, we propose a non-linear programming model to find the mixed Nash equilibrium solution.	• This paper considers bimatrix games with multi-objectives. • The payoffs of strategies are expressed with IFSs due to the fact that in some situations determining them is usually dependent on the intuition of the players in the game. • The investigation of methodology in single-objective games and zero sum games is required.
Solution Of Matrix Game In Intuitionistic Fuzzy Environment	2013	[14]	An IF matrix game has been considered and its solution method has been proposed using the defuzzification method. Score functions have been defined to construct the problem and a solution method is provided.	• The IF number is considered as a membership and a non-membership function which represents the acceptance and rejection degree of a DM. • The example given, establishes the theory on strong ground. It has a strong impact on modern socio-economic structures where conflicting interests exist. • There is a scope to apply other score functions for such defuzzification.
Linear programming approach to matrix games with intuitionistic fuzzy goals	2013	[164]	In the methodology defined in this paper, the concepts of IFS goals and the solutions of matrix games with IFS goals are proposed. It is proven that solutions of matrix games with IFS goals can be obtained through solving the developed auxiliary linear programming models, which are the generalizations of matrix games with fuzzy goals.	• The matrix games discussed in this paper are ones in which uncertainty only appears in the goals of players, i.e., the goals of players are expressed with the IFSs whereas payoffs of players are real numbers rather than the IFSs. • The proposed models in this paper provide not only the least degree of satisfaction with the attainment of the IFS goals but also the largest degree of dissatisfaction and the degree of hesitation in the attainment of the IFS goals. • The games in which both goals and payoffs are IFSs must be investigated.
A Fast Approach to Bimatrix Games with Intuitionistic Fuzzy Payoffs	2014	[72]	A bimatrix game model with intuitionistic fuzzy payoffs (IFPBG) is put forward. For that purpose, two kinds of nonlinear programming algorithms are discussed with the Nash equilibrium of IFPBG. Further, the Nash equilibrium of the algorithm is proved by the fixed point theory and the algorithm is simplified by linear programming methods.	• Under the complicated decision environment, the model and method of this paper are simple and practical than other general equilibrium game models because of the "linear exchange". • Further study could be focused on more people's bidding online IF matrix game model.

Table 1.2: (Continued)

Title	Year	Paper	Description	Advantages/Limitations
An interval-valued programming approach to matrix games with payoffs of triangular intuitionistic fuzzy numbers	2014	[121]	In the defined methodology, the concept of solutions for matrix games with payoffs of TIFNs is introduced. A pair of auxiliary IF programming models for players are established to determine optimal strategies and the value of the matrix game with payoffs of TIFNs. Based on the cut sets and ranking order relations between TIFNs, the IF programming models are transformed into linear programming models, which are solved using the existing simplex method.	- The defined methodology is quite effective and practical for solving matrix games with payoffs of TIFNs. - It is easily seen that matrix games with payoffs of TIFNs are a generalization of matrix games with payoffs of TFNs. - The computation amount for determining player I's gain-floor and player II's loss-ceiling is larger.
A Note on "A Lexicographic Method for Matrix Games with Payoffs of Triangular Intuitionistic Fuzzy Numbers"	2015	[210]	In this paper, it is pointed out that Nan et al. [166] have used some mathematical incorrect assumptions in their proposed method, and the existing method is also modified.	- The error in the existing method is pointed out and the modified method is proposed to find the exact solution of such matrix games in which payoffs are represented by TIFNs. - No application of the improved proposed method is provided.
Interval-valued intuitionistic fuzzy matrix games based on Archimedean t-conorm and t-norm	2017	[237]	The IF aggregation operators based on Archimedean t-conorm and t-norm are employed to aggregate the payoffs of players. To derive the solution of the matrix game with interval-valued IFNs, several mathematical programming models are developed based on Archimedean t-conorm and t-norm. The proposed models can be transformed into a pair of primal-dual linear programming models, based on which, the solution of the matrix game with interval-valued IFNs is obtained.	- The proposed method is an extension of the existing ones and can provide more choices for players. - It is proved that the theorems being valid in the existing matrix game with interval-valued IFNs are still true when the general aggregation operator is used in the proposed matrix game with interval-valued IFNs.
A Linear Programming Approach to Solve Constrained Bi-matrix Games with Intuitionistic Fuzzy Payoffs	2018	[3]	Based on dual programming theory, two linear programming models are developed for solving any constrained bimatrix game. Then, a classic example of a bimatrix game called the Rock-scissors-cloth game which considers players' preference for strategies is used to show the validity of the proposed models and method. Further, constrained bimatrix games with payoffs represented by IFNs are taken into account.	- The models and method proposed in this paper are more effective in terms of breadth of applications and simplicity of calculation. - This paper is the first one to investigate the constrained bimatrix games with IF information. - Constrained bimatrix games with linear constraints only are considered.
Fuzzy Matrix Games with Intuitionistic Fuzzy Goals and Intuitionistic Fuzzy Linear Programming Duality	2020	[204]	In this paper, a two-person zero sum matrix game with fuzzy goals or constraints that can be introduced by IFNs is defined. Such a game is equal to two crisp linear programming problems for two players which establish a primal-dual problem in the sense of linear programming duality in an IF situation.	- The primal-dual problems and IF inequalities depend upon the pessimistic and optimistic approach of the DM. - The matrix games are assumed to have a pessimistic view points, the result can be recognized for optimistic cases as well.
Bi-Matrix Games with General Intuitionistic Fuzzy Payoffs and Application in Corporate Environmental Behavior	2022	[128]	A new asymmetric general intuitionistic fuzzy number (GIFN) and its cut sets are first defined. Then, the order relationship of GIFNs and the definitions of α and β -bimatrix games are proposed. Afterward, the constructed general IF quadratic program is converted into an interval bi-objective program based on the order relationship of GIFNs. Furthermore, the interval bi-objective program is converted into a multi-objective quadratic program based on the combination of interval order relationship and the player's acceptance degree.	- During the processes of model conversion and solution, this paper not only skillfully combines interval order relation with interval goal program but also considers the player's acceptance degree. These processes are rigorous, meaningful, and reasonable. - This paper provides a novel way to address GIF information that can retain as much information as possible during the game process. The proposed method can offer more detailed information regarding the Nash equilibrium value, including the tendency of a player's Nash equilibrium value at different $\langle \alpha, \beta \rangle$ levels. - This paper only considers the quadratic function form of the membership and non-membership function of GIFN.
A new approach to solve intuitionistic fuzzy bi-matrix games involving multiple opinions	2023	[194]	This paper presents an approach to solving bimatrix game problems with multiple experts in an IF environment. An aggregation operator-based technique is formulated to fuse the opinion of a group of experts to obtain the collective payoff matrices corresponding to both the players in the game. A solution procedure is designed to solve the fuzzy bimatrix game problem with the collective payoff matrices.	- This paper has made a detailed study of fuzzy bimatrix games with IF payoffs and pointed out some limitations of the existing solution methods. - This methodology considers the payoff values of more than one expert. - More competitive decision problems need to be explored.

Table 1.3: Summary of literature based on linguistic matrix games

Title	Year	Paper	Description	Advantages/Limitations
Vague linguistic matrix game approach for multi-criteria decision making with uncertain weights	2013	[249]	For handling the uncertainty in setting the weights of criteria, a fuzzy matrix game related with the initial decision matrix is established. This fuzzy game model is then transformed to a linear programming. By solving this programming, the weights of criteria are obtained. The weighted arithmetic average operator of vague linguistic numbers yields the aggregated ranking values of alternatives.	• Since the weight vector is one of the most important parts of MCDM problems, this model introduces an approach based on maximizing the deviation between the alternatives for comparing with the game approach. • The proposed approach has limitation that the linearization of the game programming contains additional calculations.
Matrix Games with Interval-Valued 2-Tuple Linguistic Information	2018	[193]	In this paper, a two-player constant-sum interval-valued 2-TL matrix game is construed. The value of a linguistic matrix game is proven as a non-decreasing function of the linguistic values in the payoffs, and hence, a pair of auxiliary linguistic linear programming (LLP) problems is formulated to obtain the linguistic lower bound and the linguistic upper bound of the interval-valued linguistic value of such class of games. The duality theorem of LLP is also adopted to establish the equality of values of the interval linguistic matrix game for players.	• The methodology shows promising results in the game problems where the payoffs to the given strategies of two players are not numerical or linguistic variables but interval-valued linguistic information. • The interval linear programming can be adopted as the alternative approach to solve such class of game that results in the interval-valued linguistic value of the game as well as the mixed strategies also appearing as an interval.
Solving matrix games with linguistic payoffs	2019	[185]	In order to solve the matrix game with linguistic payoffs a solution methodology has been proposed. This proposed methodology is based on fuzzy representation of linguistic payoffs and defuzzification of fuzzy payoffs. In this method, the linguistic variable has been represented by fuzzy number. Here, widely known Yager's Ranking Index method has been utilized for defuzzification of fuzzy number.	• The elements of payoff matrix are considered as descriptive words that characterized linguistic variables. • This paper is capable of solving real-life decision making problems. • This paper has a limitation that the fuzzy matrix game problem has been first converted into a matrix game problem whose payoffs are precise valued and then it is solved using the linear programming method.
Methodology for Interval-Valued Matrix Games with 2-Tuple Fuzzy Linguistic Information	2019	[151]	In this paper, a two-player zero sum interval-valued 2-tuple fuzzy linguistic (IVTFL) game is introduced and a methodology is developed to evaluate its saddle point and optimal interval-valued linguistic value of the game. In this direction, an auxiliary pair of interval-valued LLP problem is constructed, that is further transformed into conventional interval linear programming problem to obtain optimal strategy sets of both players as the region that is not only completely feasible but also totally optimal.	• To validate the suggested solution scheme, the transformed ILP problems are solved using best-worst case approach, enhanced-interval linear programming method and LLP technique of solving interval linguistic matrix game problems. • The duality principle is also taken into consideration.
Matrix games with 2-tuple linguistic information	2020	[192]	In this contribution, a two-player constant-sum 2-TL matrix game is described, and an LLP approach is proposed to solve this class of games. Moreover, an innovative game theory approach is proposed for solving linguistic MCDM problems. The expediency of the proposed method is that it naturally generates the optimal weights for the attributes in an intermediate step, in contrast to other existing methods where the weights are presumed to be known a priori.	• The proposed approach presents a unified framework because it can be adopted to solve linguistic game problems, LLP problems and linguistic MCDM problems. • The proposed approach adds a feature in the situation when there are multiple DMs involved in a group MCDM problem. Each DM, based on his expertise, has a different payoff matrix for the same set of alternatives-attributes. • The theory of non-zero sum linguistic games is missing.
On matrix games with 2-tuple intuitionistic fuzzy linguistic payoffs	2021	[206]	This paper presents an extensive study of matrix games with qualitative payoffs. The work uses 2-tuple intuitionistic fuzzy linguistic values (2-TIFLVs) to represent the payoffs of the matrix game. The mathematical formulation and concepts of the solutions for matrix games with payoffs represented by 2-TIFLVs are developed. Paper also shows that matrix games with payoffs of 2-TIFLVs have solutions that can be obtained by transforming the matrix game in a pair of linear/nonlinear programming problems.	• The authors have defined maximin and minimax strategies for Players <i>I</i> and <i>II</i> under 2-TIFL environment. • Paper has transformed the given matrix game problem in a pair of linear/nonlinear bi-objective programming models and solved them using existing methods. • No real-life application in decision-making is depicted.

Table 1.3: (Continued)

Title	Year	Paper	Description	Advantages/Limitations
Matrix games with linguistic intuitionistic fuzzy Payoffs: Basic results and solution method	2021	[207]	This work presents a valuable study on matrix games with payoff represented by linguistic intuitionistic fuzzy numbers (LIFNs). First, the paper defines some new operational laws for LIFNs based on linguistic scale function (LSF) and studies their properties in detail. Next, a new aggregation operator called “generalized linguistic intuitionistic fuzzy weighted average (GLIFWA)” operator is defined for aggregating LIFNs. Several properties and special cases of GLIFWA operator are also discussed.	- The solution methods are developed using a pair of auxiliary linear/nonlinear-programming models derived from a pair of nonlinear bi-objective programming. - The LSF based aggregation operator has been used to calculate the expected value of the game to consider different semantic situations under different environment.
The two-person and zero-sum matrix game with probabilistic linguistic information	2021	[162]	This study takes the probabilistic linguistic information as the input of a two-person zero sum matrix game, and addresses the vague information by triangular membership functions. Such a game with probabilistic linguistic information is a useful technique for multiple criteria analysis. In addition, it outputs the same form of information as the inputs to increase the interpretability compared with other uncertain matrix games.	- The defined methodology increases the feasibility, flexibility and application scope of the game theory and reduces the burden of players evolved in the game. - The retranslation process from membership degrees to the corresponding probabilistic linguistic information is proposed to enable the interpretability of the output results. - The study is limited to triangular probabilistic membership function only.
Probabilistic Linguistic Matrix Game Based on Fuzzy Envelope and Prospect Theory with Its Application	2022	[129]	This paper uses PLTSs to express players’ payoff values, and aims to develop an integrated method based on fuzzy envelope and prospect theory under a probabilistic linguistic environment for solving matrix games. In this method, an improved probabilistic ordered weighted averaging operator is defined. Then, a novel trapezoidal fuzzy envelope for PLTSs is proposed and some related theorems are analyzed. Next, based on the defined cosine distance measure for PLTSs, the players’ psychological behavior in the game is considered by establishing the prospect value function.	- The study of matrix games under a probabilistic linguistic environment expands the scope of application of game theory. - The definition of trapezoidal fuzzy envelope for probabilistic linguistic information enriches the defuzzification technology for linguistic information. Moreover, the fusion of prospect theory commendably captures the DMs’ psychological behavior regarding gain and loss, which makes the proposed method more suitable for solving practical decision-making problems. - The developed method fails to investigate the multi-objective problems under a probabilistic linguistic environment.
Bi-matrix Games with 2-tuple Linguistic Information	2022	[179]	A new type of bimatrix game called the bi-matrix game with 2-TL information is proposed and the LLP model is used to obtain the mixed set. The problem of linguistic information is transformed into a mathematical problem that can be solved quickly, which reduces the uncertainty and complexity caused by linguistic information.	- The method proposed in this paper is desirable because it can be used to solve linguistic processing problems and linguistic game problems. - Various major properties and theorems that are used throughout the text, are left unproved.
Solutions of matrix games involving linguistic interval-valued intuitionistic fuzzy sets	2023	[168]	This paper explores the theory of matrix games under a qualitative information environment. We use linguistic interval-valued intuitionistic fuzzy numbers (LIVIFNs) to describe the payoff values as suggested by experts. The LIVIFNs are more efficient tools that provide experts with a flexible information modeling capability to describe their ambiguous and uncertain perceptions in the form of linguistic terms. The solution of this class of matrix games is attained by resolving a duo of linear or nonlinear programming problems originating through nonlinear bi-objective programming problems.	- The benefit of adopting LIVIFN payoff values is that it allows DMs to describe ambiguous and imprecise facts in a more detailed manner. - Based on the order relation of LIVIFNs, the nonlinear bi-objective programming problems for each player are reduced to a corresponding pair of linear/nonlinear programming problems to determine the maximin and minimax strategies corresponding to the players. - A more generalized methodology is required.
Probabilistic multiplicative unbalanced linguistic term set and its application in matrix games	2023	[154]	This paper proposes a new PLTS called the probabilistic multiplicative unbalanced linguistic term set (PM-ULTS), where both probabilities and non-uniform spacing of the linguistic labels are considered simultaneously. Specific operational laws are defined to preserve the resultant linguistic labels and corresponding probability information. This study initiates the design of a unified two-person linguistic matrix game model with the new PLTS as a parameter. Additionally, the proposed game model involves a re-translation process to convert the output back into the original probabilistic unbalanced linguistic domain without information loss.	- The defined methodology addresses the imprecise information by the information measure function. Such a two-player probabilistic unbalanced linguistic matrix game is considered a convenient technique for multiple decision scenarios. - This methodology can generate the optimal weights for the given alternatives in an intermediate step, unlike the other existing methods where the weights are presupposed to be well-known a priori. - It is necessary to capture probabilistic information of the linguistic labels. The elicitation of the probabilistic information is not known.

Chapter 2

A methodology for solving bimatrix games under 2-tuple linguistic environment

In a game problem, when the situation is formulated as the gain of one equals the loss of another, the game is called a zero-sum game. However, it is not always true that one player's win is the other's loss. Such a game is referred to as a non-zero-sum game or a bimatrix game. The work provided in this chapter¹ is primarily elaborated on linguistic bimatrix games under 2-tuple linguistic environment. This class of bimatrix games consists of payoffs having entries as linguistic 2-tuples that are based on a finite subscript-symmetric additive linguistic set defined in the domain $[-g, g]$, where g is a positive integer. In this direction, a novel linguistic equilibrium concept is defined to solve such games, and a methodology based on a nonlinear linguistic program is proposed to compute mixed Nash equilibrium. The proposed methodology is sufficient to evaluate the Nash equilibrium and the maximum expected payoffs corresponding to each player simultaneously in a single program. The necessary and sufficient conditions for the existence of Nash equilibrium are provided along with the computation procedure. Finally, some hypothetical working examples are given to show the practical applicabil-

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ity of the defined methodology. The proposed method is also applied to solve two-player constant-sum linguistic games.

2.1 Introduction

Beginning with the seminal work of Neumann & Morgenstern [171], the tremendous growth and advancement in games have attracted researchers to work in the field of game theory [17, 19, 26, 46, 54, 163], in which one is the winning player while the other is a loser. This signifies that one player always maximizes his profit, and the other minimizes his loss. Nevertheless, as a matter of fact, both players want to maximize their profit. In that case, the theory of the bimatrix game is relevant to control the contradictory situation. Recently, much attention has been focused on bimatrix games [25, 74, 99], where there are two players, and each player has a finite number of available strategies. In a bimatrix game, the players' interests are not required to be precisely opposed. Both players have a finite number of available strategies, and each player attempts to maximize their payoff.

The traditional game theory assumes that players precisely determine every probable situation to select the payoffs. However, in real-life scenarios, such restrictive assumptions limit game theory scope as it is difficult to assess the information precisely because of the impreciseness of the game problems. This impreciseness in games was handled with the help of probabilistic methods and the theory of Bayesian games. But this theory was only confined to one kind of imprecision. To the solution, several impreciseness types are handled well in fuzzy set theory as introduced by [255]. Fuzziness in game problems has had much more influence from an extensive range of research [22, 66, 83, 186]. Larbani [107] presented a survey of non-cooperative fuzzy games in normal form. Various solution methods are implemented to solve fuzzy matrix games like defuzzification methods [184], interval parametric techniques [183], signed distance methods [182], etc. Since the 1960s, mathematicians, and computer scientists have been studying computational aspects of finding Nash equilibria in bimatrix games [13, 32, 80, 108] as well as in fuzzy bimatrix games [48, 82, 88, 110, 149]. However, it was examined that for the study of systems that are too complex or ill-defined for implementing traditional quantitative methods, the representation of the result in numerical form becomes improper. In dealing with accuracy in the face of overpowering intricacy, it is essential to learn the importance of linguistic variables as introduced by Zadeh in [257]. Since it is challenging to clarify the pre-

cise relationship between payoffs and different strategic profiles, especially when each player's strategy space is numerous, linguistic variables have provided incredible matrix game results.

Further, to overcome the loss of information incurred in the process of CW, Herrera & Martínez introduced the 2-TL model [92] for computing linguistic approaches, and it is implemented to solve two-person constant-sum matrix games having payoffs as 2-TL variables. However, no concepts of “linguistic bimatrix games” are yet to be accepted broadly. Hence, it is a crucial problem to define the concepts of equilibrium solutions in bimatrix games with linguistic payoffs and investigate the characteristics of solutions of such games. This study aims to introduce a novel concept of “Linguistic Bimatrix Games having 2-tuple linguistic payoffs”. This is done by emphasizing the following research gaps:

- The bimatrix games considered till now are either based on numerical information or fuzzy data. Such type of data causes information loss while applying fusion processes while aggregating the final data.
- The defuzzification methods used to solve fuzzy bimatrix games are restricted to a particular solution procedure. Hence, a more generalized methodology requires to be developed to directly draw the value of a bimatrix game.
- In the existing methodologies, researchers construct two different auxiliary equations for both players separately and then solve them individually to find their expected payoff. This enhances the time complexity of the computation methods.

For filling the aforementioned research gaps, first, the linguistic bimatrix games having 2-TL payoffs are introduced, and then a methodology is defined for solving such games. The motivation for doing this is to incorporate ambiguity of human judgments in bimatrix game problems having two independent players that always try to maximize their payoffs, irrespective of what other player tries to undertake. The primary reason for this is to handle the linguistic uncertainty in non-zero sum games. For example, two competing companies launch a new product with almost the same features in the market. Each company aims to sell maximum quantities by selecting its market strategies like advertisements, free coupons, etc. Now to approximate the demand for this product, we have to count on the

subjective opinions of experts with great managing experiences. The demand may be expressed in terms of language like “around 50 units”. Thus to deal with such linguistic ambiguity in bimatrix games, we introduce this novel concept.

This chapter contributes to the theoretical development by proposing a new concept of solutions for linguistic bimatrix games and by proving the existence of this new concept. The payoffs in such games belong to a special type of set called the “Subscript-symmetric LT set (SSLTS)”. Further, the linguistic expected payoffs and the Nash equilibrium concerning linguistic bimatrices are defined. Further, a nonlinear linguistic program has been defined to compute Nash equilibrium in a linguistic bimatrix game. The proposed program is further simplified and solved in terms of linguistics. The existence and computation of mixed Nash equilibrium are provided that guarantee the necessary and sufficient conditions required for the existence and computation of mixed Nash equilibrium in linguistic bimatrix games. Additionally, the proposed methodology is used to solve two-person constant-sum linguistic matrix games, too. Thus, it can be said that the developed approach may be used to solve any arbitrary two-person game, be it a constant-sum or a non-zero sum game. Technically, this theory provides Nash equilibrium and the value of the game using Maple software to provide better and more accurate results. Unlike traditional games, the values of both the players in this chapter are evaluated in a single Maple program reducing the time and computation complexity.

This entire chapter is laid out as follows. Section 2.2 introduces the linguistic bimatrix games with payoffs as 2-TL variables belonging to the interval $[-g, g]$. Further in this section, we define pure and mixed Nash equilibrium, and the methodology for finding the mixed Nash equilibrium used to maximize the players’ payoffs has also been defined. Two numerical examples have been presented in Section 2.3. Section 2.4 discusses the relevance of the proposed methodology in solving two-person games. Section 2.5 offers some conclusions of the chapter.

2.2 Linguistic Bimatrix Games having payoffs as 2-TL Variables

In this section, we’ll define a methodology to solve the linguistic bimatrix games having 2-TL payoffs where the payoffs belong to predefined SSLTS. The present

section is divided into two subsections. The details of the basic definitions and terminologies beneficial for the development of the methodology to solve two-person non-zero sum linguistic bimatrix games are stated in Subsection 2.2.1. The existence and evaluation of mixed Nash equilibria in such games are then provided in Subsection 2.2.2 followed by the computation procedure used to compute Nash equilibrium in Subsection 2.2.3.

2.2.1 Linguistic Bimatrix Games

In the existent game problems, the decision framework is usually identified by a number of feasible strategies, intricate connections between strategic alternatives, and their complex impact on payoffs. Thus, estimating the precise evaluation of the payoff bimatrices becomes almost impossible. In this type of situation, we may identify the uncertain payoffs in the form of fuzzy linguistic variables.

Consider a game having player I and player II as two players and let $R = \{1, 2, \dots, r\}$ and $S = \{1, 2, \dots, s\}$ denote the sets of all pure strategies available for player I and II , respectively. Suppose the payoff matrix of player I is $\tilde{A} = [\tilde{a}_{ij}]_{r \times s}$ and of player II is $\tilde{B} = [\tilde{b}_{ij}]_{r \times s}$. The payoffs $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ of player I and II respectively, come up when player I selects a pure strategy $i \in R$ and player II selects a pure strategy $j \in S$. The payoffs of both players are represented together as $(\tilde{a}_{ij}, \tilde{b}_{ij})$. These payoffs $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ where $i \in R, j \in S$ are the linguistic variables belonging to a predefined SSLTS, or are 2-TL variables for any linguistic variable ℓ_ϵ . Now, the notion of matrix games can be extended to linguistic bimatrix games in the following manner.

Definition 2.2.1. A 2-person non-zero sum linguistic bimatrix game may be defined as $\tilde{G} = (S^r, S^s, LT, \tilde{A}, \tilde{B})$ where S^r and S^s are the strategy sets of player I and II respectively, and the mixed strategies of player I and player II are represented by

$$\Phi = (\phi_1, \phi_2, \dots, \phi_r)^T \in S^r = \{(\phi_1, \phi_2, \dots, \phi_r) : \phi_i \geq 0, i = 1, 2, \dots, r, \sum_{i=1}^r \phi_i = 1\}, \text{ and}$$

$$\Psi = (\psi_1, \psi_2, \dots, \psi_s)^T \in S^s = \{(\psi_1, \psi_2, \dots, \psi_s) : \psi_j \geq 0, j = 1, 2, \dots, s, \sum_{j=1}^s \psi_j = 1\}$$

respectively, where T denotes the transposition. $LT = \{\ell_{-g}, \dots, \ell_0, \dots, \ell_g\}$ is the SSLTS for players with cardinality $2g + 1$. $\tilde{A}$ and $\tilde{B}$ are the linguistic payoff matrices of player I

and player II respectively, having payoffs as 2-TL variables. We write this game in matrix form as $(\tilde{A}, \tilde{B})$.

The SSLTS used in this chapter is taken from the paper [240]. Further, some group operations on 2-TL variables have also been implemented based on the paper [152]. The set, its properties, and the group operations implemented on its linguistic terms are provided as follows.

Definition 2.2.2. Subscript-symmetric Linguistic Term set [240] Let $LT = \{\ell_{-g}, \dots, \ell_0, \dots, \ell_g\}$ be a finite and totally ordered predefined linguistic term set following that

- LT is ordered set i.e. $\ell_{\vartheta} > \ell_{\varepsilon} \iff \vartheta > \varepsilon$, and
- For any $\ell_{\varepsilon} \in LT$, $Neg(\ell_{\varepsilon}) = \ell_{-\varepsilon}$.

The traditional 2-TL model introduced by [92] defined earlier in the introductory section is stipulated over the interval $[0, g]$. Nevertheless, this model may also be stated simply for the domain $[-g, g]$ by using SSLTS named LT . The idea of Δ and Δ^{-1} operators, theoretically validated by [92] for the set LT , are given below. For a comprehensive and complete elucidation of the theory for the existing 2-TL model, one can refer to the introductory chapter [1].

Definition 2.2.3. [151] Let $LT = \{\ell_{-g}, \dots, \ell_0, \dots, \ell_g\}$ be a set of finite linguistic terms with cardinality $2g + 1$ and suppose $x \in [-g, g]$. Then, the function Δ that converts x into its corresponding 2-TL variable is identified as

$$\Delta : [-g, g] \rightarrow LT \times [-0.5, 0.5] \equiv \overline{LT}$$

$$\Delta(x) = (\ell_{\varepsilon}, \alpha)$$

where $\varepsilon = round(x) \in LT$ and $\alpha = x - \varepsilon \in [-0.5, 0.5]$. Here, $round(.)$ is the standard round operation, ℓ_{ε} is a linguistic term nearest to β , and α is the symbolic translation.

It is clear that Δ is bijective; thus, its inverse may be specified as

$$\Delta^{-1} : \overline{LT} \rightarrow [-g, g] \text{ s.t. } \Delta^{-1}(\ell_{\varepsilon}, \alpha) = \varepsilon + \alpha$$

Moreover, similar to the comparison of conventional 2-TL variables stipulated on lexicographic ordering, the comparison of the 2-TL variables based on SSLTS is defined as follows.

Definition 2.2.4. [151] Let $(\ell_\vartheta, \alpha_\vartheta)$ and $(\ell_\varepsilon, \alpha_\varepsilon)$ be two 2-TL variables based on LT . Then

- If $\vartheta < \varepsilon$, then $(\ell_\vartheta, \alpha_\vartheta) < (\ell_\varepsilon, \alpha_\varepsilon)$.
- If $\vartheta = \varepsilon$, then
 - * If $\alpha_\vartheta = \alpha_\varepsilon$, then $(\ell_\vartheta, \alpha_\vartheta)$ and $(\ell_\varepsilon, \alpha_\varepsilon)$ represent the same information.
 - * If $\alpha_\vartheta < \alpha_\varepsilon$, then $(\ell_\vartheta, \alpha_\vartheta) < (\ell_\varepsilon, \alpha_\varepsilon)$.
 - * If $\alpha_\vartheta > \alpha_\varepsilon$, then $(\ell_\vartheta, \alpha_\vartheta) > (\ell_\varepsilon, \alpha_\varepsilon)$.

Further, Malhotra & Gupta [152] have defined the 2-TL additive operator based on the above-mentioned SSLTS. However, we'll be reviewing the following theorem in this chapter as it would be useful throughout the work.

Theorem 2.2.1. [152] Let $LT = \{\ell_{-g}, \dots, \ell_0, \dots, \ell_g\}$ be the SSLTS with odd cardinality $2g + 1$ and let $\alpha_\varepsilon \in [-0.5, 0.5]$ be the numerical value depicting the value of symbolic translation corresponding to ℓ_ε . Then $\overline{LT}$ forms a direct group under the binary operation 'o' defined as : $(\ell_\vartheta, \alpha_\vartheta) \circ (\ell_\varepsilon, \alpha_\varepsilon) = (\ell_\vartheta \circ \ell_\varepsilon, \alpha_\vartheta \circ \alpha_\varepsilon)$, where

$$\begin{aligned} \ell_\vartheta \circ \ell_\varepsilon &= \ell_\vartheta * \ell_\varepsilon \\ &= \ell_{\vartheta * \varepsilon} \\ &= \begin{cases} \ell_{\vartheta + \varepsilon + g}, & \text{if } \vartheta + \varepsilon < -g \\ \ell_{\vartheta + \varepsilon}, & \text{if } -g \leq \vartheta + \varepsilon \leq g \\ \ell_{\vartheta + \varepsilon - g}, & \text{if } \vartheta + \varepsilon > g \end{cases} \end{aligned}$$

and,

$$\begin{aligned} \alpha_\vartheta \circ \alpha_\varepsilon &= \alpha_\vartheta \oplus \alpha_\varepsilon \\ &= \begin{cases} \alpha_\vartheta + \alpha_\varepsilon, & \text{if } \alpha_\vartheta + \alpha_\varepsilon \in (-0.5, 0.5) \\ \alpha_\vartheta + \alpha_\varepsilon - \text{round}(\alpha_\vartheta + \alpha_\varepsilon), & \text{else} \end{cases} \end{aligned}$$

where $\ell_\vartheta, \ell_\varepsilon \in LT$ and $\alpha_\vartheta, \alpha_\varepsilon \in [-0.5, 0.5]$.

For detailed proof of the theorem, one may refer to the paper [152].

Proceeding further, the evaluation of the expected payoffs of both players is given as follows:

Definition 2.2.5. Suppose player I and player II select the mixed strategy $\Phi \in S^r$ and $\Psi \in S^s$ respectively, then the expected payoffs of player I and player II are:

$$E_I(\Phi, \Psi) = \Phi \tilde{A} \Psi^T$$

and,

$$E_{II}(\Phi, \Psi) = \Phi \tilde{B} \Psi^T$$

respectively. Moreover, if $(\tilde{A}, \tilde{B})$ is a bimatrix game, then the safety value of player I is the value($\tilde{A}$) and of player II is the value($\tilde{B}^T$).

Now before heading towards the main theorem, we first define the scalar multiplication of 2-TL variables as follows.

Definition 2.2.6. For any $\kappa \in [0, 1]$ and $(\ell_\varepsilon, \alpha) \in LT \times [-0.5, 0.5]$, we write

$$\kappa \cdot (\ell_\varepsilon, \alpha) = (\ell_{\hat{\varepsilon}}, \kappa \cdot \alpha), \text{ where } \hat{\varepsilon} = \text{round}(\kappa \cdot \varepsilon)$$

We next proceed toward the theorem outlining the Nash equilibrium.

Theorem 2.2.2. A pair of mixed strategies $(\Phi^* \in S^r, \Psi^* \in S^s)$ is a Nash equilibrium iff

$$E_I(\Phi^*, \Psi^*) = \Phi^* \tilde{A} \Psi^{*T} \geq \Phi \tilde{A} \Psi^{*T} = E_I(\Phi, \Psi^*), \text{ for every } \Phi \in S^r, \quad (2.2.1)$$

$$E_{II}(\Phi^*, \Psi^*) = \Phi^* \tilde{B} \Psi^{*T} \geq \Phi^* \tilde{B} \Psi^T = E_{II}(\Phi^*, \Psi), \text{ for every } \Psi \in S^s. \quad (2.2.2)$$

Proof. First suppose that the pair (Φ^*, Ψ^*) is a Nash equilibrium. Then

$$\begin{aligned} E_I(\Phi^*, \Psi^*) &= \Phi^* \tilde{A} \Psi^{*T} \\ &= [\phi_1^* \ \phi_2^* \ \dots \ \phi_r^*]_{1 \times r} [\tilde{a}_{ij} = (\ell_{ij}, \alpha_{ij})]_{r \times s} \begin{bmatrix} \psi_1^* \\ \psi_2^* \\ \vdots \\ \psi_s^* \end{bmatrix}_{s \times 1} \\ &= \circ_{i=1}^r \circ_{j=1}^s \left(\ell_{\hat{ij}}, \Phi^* \alpha_{ij} \Psi^* \right), \text{ where } \hat{ij} = \text{round}(\Phi^* i j \Psi^*) \\ &\geq \circ_{i=1}^r \circ_{j=1}^s (\ell_{\bar{ij}}, \Phi \alpha_{ij} \Psi^*), \text{ where } \bar{ij} = \text{round}(\Phi i j \Psi^*) \text{ } (\Phi^* \text{ and } \Psi^* \text{ are Nash equilibrium}) \\ &= \Phi \tilde{A} \Psi^{*T}, \text{ for every } \Phi \in S^r \end{aligned}$$

Similarly, $E_{II}(\Phi^*, \Psi^*) \geq E_{II}(\Phi^*, \Psi)$, for every $\Psi \in S^s$.

Here, $\circ$ denotes the 2-TL binary operator as defined in Theorem 2.2.1. The converse follows trivially. $\square$

It is to be noted that the pure strategy Nash equilibrium would also be the row i^* and column j^* such that $\tilde{a}_{ij^*} \leq \tilde{a}_{i^*j^*}$ and $\tilde{b}_{ij^*} \leq \tilde{b}_{i^*j^*}$ where $i \in R$, $j \in S$. The above-mentioned ordering is valid because of the lexicographic ordering of 2-TL variables. Hence, for bimatrix games, a Nash equilibrium in pure strategies is a pair that is, the largest first component in a column and the largest second component in a row simultaneously. We illustrate this with the help of an example.

Example 2.2.1. A company owner observed during a health emergency that the absence of employees in his company was affecting the company's profits, especially in the department of sales and the department of operations, very badly. He introduced some schemes to make profits back on track. The schemes include Work from Home (WFH), Work from Office (WFO) (this scheme involves working of all employees from the office; however, this may affect the health of employees resulting in a bad environment to work), and roster-based attendance (i.e., only 30% of employees of both the departments attend office each day maintaining required safety measures). He met employees of both departments, offered them these schemes, and tried to observe profit improvement with each scheme for a month. The linguistic bimatrix game that models this situation is as follows:

Sales/Operations	WFH	WFO	Roster
WFH	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_0)	(ℓ_1, ℓ_2)
WFO	(ℓ_0, ℓ_1)	(ℓ_0, ℓ_0)	(ℓ_0, ℓ_2)
Roster	(ℓ_2, ℓ_1)	(ℓ_2, ℓ_0)	(ℓ_2, ℓ_2)

The linguistic payoffs belong to the LT set

$$LT = \{\ell_{-2} : \text{Very Low}, \ell_{-1} : \text{Low}, \ell_0 : \text{Average}, \ell_1 : \text{High}, \ell_2 : \text{Very High}\}$$

The payoff $\tilde{a}_{12}$ depicts that if the sales team works from home and the operations team works from the office, then the sales get high profits, and the operations team gets average profits.

Now to establish pure Nash equilibrium points, we put a bar above the first component that is the largest in each column and the second component that is the largest in each row. In this way, the pair of linguistic terms in which both components have a bar is a Nash

equilibrium in pure strategies. So, in this problem, we have

Sales/Operations	WFH	WFO	Roster
WFH	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_0)	$(\ell_1, \bar{\ell}_2)$
WFO	(ℓ_0, ℓ_1)	(ℓ_0, ℓ_0)	$(\ell_0, \bar{\ell}_2)$
Roster	$(\bar{\ell}_2, \ell_1)$	$(\bar{\ell}_2, \ell_0)$	$(\bar{\ell}_2, \bar{\ell}_2)$

This means that there is a unique pure Nash equilibrium at $(3, 3)$, which means that if both teams come to the office on a roster basis, they can stabilize their profits without compromising their health.

Now it might be possible that a pure Nash equilibrium does not always exist; for example, consider the 2×2 game whose bimatrix is given by

$$\begin{bmatrix} (\ell_0, \ell_1) & (\ell_1, \ell_2) \\ (\ell_{-1}, \ell_2) & (\ell_2, \ell_1) \end{bmatrix}$$

In this game, there is no pair (i^*, j^*) which holds the earlier mentioned criteria for the existence of pure Nash equilibrium. Therefore, this game admits no Nash equilibria in pure strategies. The problem of finding Nash equilibrium can be further removed by the introduction of mixed strategies and thus by calculating the expected payoffs of the players.

2.2.2 Existence and Computation of Mixed Nash equilibrium

This section offers the introduction of the methodology to find mixed Nash equilibria in bimatrix games having entries as 2-TL variables. Based on the construction mentioned in previous sections, we contemplate the following theorem that guarantees the necessary and sufficient conditions required for the existence and computation of mixed Nash equilibrium in two-person non-zero sum linguistic bimatrix games.

Theorem 2.2.3. Consider a linguistic two-person non-zero sum game having matrices $(\tilde{A}, \tilde{B})$ for player I and player II . $\Phi^* \in S^r$, $\Psi^* \in S^s$ is a Nash equilibrium iff they satisfy,

along with linguistic variables $\tilde{p}^*$ and $\tilde{q}^*$ the nonlinear linguistic program

$$\Delta \left[\max_{\Phi, \Psi, \tilde{p}, \tilde{q}} \Delta^{-1} \left\{ \Phi \tilde{A} \Psi^T \circ \Phi \tilde{B} \Psi^T - \tilde{p} - \tilde{q} \right\} \right]$$

such that

$$\tilde{A} \Psi^T \leq \tilde{p} J_r^T \quad (2.2.3)$$

$$\tilde{B}^T \Phi^T \leq \tilde{q} J_s^T$$

$$\phi_i \geq 0, \psi_j \geq 0, \Phi J_r = 1 = \Psi J_s^T$$

where J_k is the unit vector consisting of all 1s. Further, $\tilde{p}^* = E_I(\Phi^*, \Psi^*)$, and $\tilde{q}^* = E_{II}(\Phi^*, \Psi^*)$, and $\circ$ denotes the 2-TL additive operator mentioned in Theorem 2.2.1.

Proof. Consider the players, namely player I & player II having mixed strategy vectors $\Phi = (\phi_1, \dots, \phi_r)$, $\Psi = (\psi_1, \dots, \psi_s)$, respectively, $\tilde{A}$ and $\tilde{B}$ be payoff matrices of both the players having payoffs as 2-TL variables. Player I maximizes over the rows of $\tilde{A}$ while player II maximizes over the columns of $\tilde{B}$.

According to the objective of player I

$$\begin{aligned} & \max_{\Phi} \Phi \tilde{A} \Psi^T \\ & \text{s.t.} \\ & J_r^T \Phi - 1 = 0 \\ & \Phi \geq 0 \end{aligned} \quad (2.2.4)$$

and according to the objective of player II

$$\begin{aligned} & \max_{\Psi} \Phi \tilde{B} \Psi^T \\ & \text{s.t.} \\ & J_s^T \Psi - 1 = 0 \\ & \Psi \geq 0 \end{aligned} \quad (2.2.5)$$

where T denotes the transpose of the matrix.

We know that a Nash equilibrium point (Φ^*, Ψ^*) consists of a pair of strategies where the objectives (2.2.4) and (2.2.5) are satisfied at the same time. More precisely, if (Φ^*, Ψ^*) is a Nash equilibrium, then

$$E_I(\Phi^*, \Psi^*) = \Phi^* \tilde{A} \Psi^{*T} = \max_{\Phi} \{ \Phi \tilde{A} \Psi^{*T} \mid J_r^T \Phi - 1 = 0, \Phi \geq 0 \}$$

$$E_{II}(\Phi^*, \Psi^*) = \Phi^* \tilde{B} \Psi^{*T} = \max_{\Psi} \{ \Phi^* \tilde{B} \Psi^T \mid J_s^T \Psi - 1 = 0, \Psi \geq 0 \}$$

We need to show that (Φ^*, Ψ^*) is a Nash equilibrium iff they satisfy along with the scalars $\tilde{p}^*, \tilde{q}^*$, the nonlinear program given in (2.2.3) which can be simplified as

$$\Delta \left[\max_{\Phi, \Psi, \tilde{p}, \tilde{q}} \Phi \Delta^{-1} (\tilde{A} \circ \tilde{B}) \Psi^T - \Delta^{-1}(\tilde{p}) - \Delta^{-1}(\tilde{q}) \right]$$

such that

$$\Delta^{-1}(\tilde{A}) \Psi^T \leq \Delta^{-1}(\tilde{p}) J_r^T \quad (2.2.6)$$

$$\Delta^{-1}(\tilde{B}^T) \Phi^T \leq \Delta^{-1}(\tilde{q}) J_s$$

$$\phi_i \geq 0, \psi_j \geq 0$$

Now, according to Theorem 2.2.2, $E_I(\Phi^*, \Psi^*) = \Phi^* \tilde{A} \Psi^{*T} \geq \Phi \tilde{A} \Psi^{*T}$ and $E_{II}(\Phi^*, \Psi^*) = \Phi^* \tilde{B} \Psi^{*T} \geq \Phi^* \tilde{B} \Psi^T$, it can clearly be observed that $E_I(\Phi^*, \Psi^*)$ and $E_{II}(\Phi^*, \Psi^*)$ are 2-TL numbers. In Equation (2.2.1), we successively select $\Phi = (0, 0, \dots, 1, \dots, 0)$ having 1 at the r^{th} spots, and in Equation (2.2.2), we select $\Psi = (0, 0, \dots, 1, \dots, 0)$ having 1 at the s^{th} spots. Doing so, we obtain that $E_I(\Phi^*, \Psi^*) \geq E_I(i, \Psi^*) = i \tilde{A} \Psi^*$ and $E_{II}(\Phi^*, \Psi^*) \geq E_{II}(\Phi^*, j) = \Phi^* \tilde{B} j$. In matrix form, it can be written as

$$E_I(\Phi^*, \Psi^*) J_r^T = \Phi^* \tilde{A} \Psi^{*T} J_r^T \geq \tilde{A} \Psi^{*T} \quad (2.2.7)$$

$$E_{II}(\Phi^*, \Psi^*) J_s = \Phi^* \tilde{B} \Psi^{*T} J_s \geq \Phi^* \tilde{B} \quad (2.2.8)$$

Since equations (2.2.7) and (2.2.8) are true for the pair (Φ^*, Ψ^*) , then these strategies must be a Nash point. This means that (2.2.1) and (2.2.2) hold true (due to definition). So pre-multiplying $\Phi \in S^r$ and post-multiplying $\Psi \in S^s$ in equations (2.2.7) and (2.2.8), respectively.

$$\left. \begin{aligned} E_I(\Phi^*, \Psi^*) \Phi J_r^T &= \Phi^* \tilde{A} \Psi^{*T} \Phi J_r^T \geq \Phi \tilde{A} \Psi^{*T} \\ E_{II}(\Phi^*, \Psi^*) J_s \Psi^T &= (\Phi^* \tilde{B} \Psi^{*T}) J_s \Psi^T \geq \Phi^* \tilde{B} \Psi^T \end{aligned} \right\} \left[\because \Phi J_r^T = J_s \Psi^T = 1 \right]$$

Next, we select the scalars $\tilde{p}^*$ and $\tilde{q}^*$ as

$$\tilde{p}^* = E_I(\Phi^*, \Psi^*) = \Phi^* \tilde{A} \Psi^{*T}$$

$$\tilde{q}^* = E_{II}(\Phi^*, \Psi^*) = \Phi^* \tilde{B} \Psi^{*T}$$

Then, by using the equivalent characterization of a Nash point, we get by equations (2.2.7) and (2.2.8) that

$$\Phi^* \tilde{A} \Psi^{*T} J_r^T = \tilde{p}^* J_r^T \geq \tilde{A} \Psi^{*T}$$

Also,

$$(\Phi^* \tilde{B} \Psi^{*T}) J_s = \tilde{q}^* J_s \geq \Phi^* \tilde{B}$$

Other constraints are fulfilled directly because $\Phi^* \in S^r$ and $\Psi^* \in S^s$. Hence, $(\Phi^*, \Psi^*, \tilde{p}^*, \tilde{q}^*)$ is a required feasible point. Next, we show that $(\Phi^*, \Psi^*, \tilde{p}^*, \tilde{q}^*)$ maximizes the function

$$\tilde{f}(\Phi, \Psi, \tilde{p}, \tilde{q}) = \Phi \tilde{A} \Psi^T \circ \Phi \tilde{B} \Psi^T - \tilde{p} - \tilde{q}$$

over the set of feasible points.

Note that, here, we use the Δ^{-1} function for further computations as ‘ $\circ$ ’ is not a pre-defined/valid operator in the 2-TL domain due to its non-closure property. As Δ is an isomorphic function, suppose $p = \Delta^{-1}(\tilde{p})$, $q = \Delta^{-1}(\tilde{q})$.

Since every feasible solution satisfies the constraints

$$\Delta^{-1}(\tilde{A}) \Psi^T \leq p J_r^T, \quad \Phi \Delta^{-1}(\tilde{B}) \leq q J_s$$

By multiplying the first inequality by Φ on the left and the second by Ψ^T on the right, we obtain

$$\Delta^{-1}(\Phi \tilde{A} \Psi^T) \leq p \Phi J_r^T, \quad \Delta^{-1}(\Phi \tilde{B} \Psi^T) \leq q J_s \Psi^T$$

$$\implies \Delta^{-1}(\Phi \tilde{A} \Psi^T) \leq p, \quad \Delta^{-1}(\Phi \tilde{B} \Psi^T) \leq q$$

$$\implies \Delta^{-1}(\Phi \tilde{A} \Psi^T \circ \Phi \tilde{B} \Psi^T) - p - q \leq 0 \quad (\text{By adding and rearranging})$$

So, $f(\Phi, \Psi, p, q) \leq 0$ for any feasible point. However, with $p^* = \Delta^{-1}(\Phi^* \tilde{A} \Psi^{*T})$ and $q^* = \Delta^{-1}(\Phi^* \tilde{B} \Psi^{*T})$, it can be concluded that $(\Phi^*, \Psi^*, p^*, q^*)$ is a feasible solution of (2.2.6) and,

$$f(\Phi^*, \Psi^*, p^*, q^*) = \Delta^{-1}(\Phi \tilde{A} \Psi^T \circ \Phi \tilde{B} \Psi^T) - \Delta^{-1}(\tilde{p}) - \Delta^{-1}(\tilde{q}) = 0$$

by the definition of p^* and q^* . The above step is justified as Δ is a bijective function, hence invertible, and further Δ satisfies the linearity property.

Hence, $(\Phi^*, \Psi^*, p^*, q^*)$ is feasible and offers the maximum value of the objective, and

so is a solution of the nonlinear program. Therefore we conclude that if there is a Nash equilibrium, it must solve the nonlinear programming problem.

Note 2.2.1. This must be noted that for the final linguistic results, we have to apply Δ to the maximized objective function. So we can say that $(\Phi^*, \Psi^*, p^*, q^*)$ satisfies the problem $\Delta \left[\max_{\Phi, \Psi, \tilde{p}, \tilde{q}} \Delta^{-1} \left\{ \Phi \tilde{A} \Psi^T \circ \Phi \tilde{B} \Psi^T - \tilde{p} - \tilde{q} \right\} \right]$ also.

Now for the converse, we need to show that any solution of the nonlinear program (2.2.3) must be a Nash point. Let $\Phi_1, \Psi_1, \tilde{p}_1, \tilde{q}_1$ be any arbitrary solution of (2.2.3). Let (Φ^*, Ψ^*) be a Nash point, and set $\tilde{p}^* = \Phi^* \tilde{A} \Psi^{*T}$, $\tilde{q}^* = \Phi^* \tilde{B} \Psi^{*T}$. We need to show that (Φ_1, Ψ_1) must be a Nash point of the game.

Since Φ_1, Ψ_1 are a solution, they must satisfy the constraints of the problem (2.2.3), i.e.,

$$\begin{aligned} \tilde{A} \Psi_1^T &\leq \tilde{p}_1 J_r^T \quad \text{and} \quad \Phi_1 \tilde{B} \leq \tilde{q}_1 J_s \\ \implies \tilde{A} \psi_1^T &\leq (\phi_1 \tilde{A} \psi_1^T) J_r^T \quad \text{and} \quad \phi_1 \tilde{B} \leq (\phi_1 \tilde{B} \psi_1^T) J_s \end{aligned}$$

This makes (Φ_1, Ψ_1) a Nash point. Hence, it is proved that any solution of the nonlinear program should give a Nash equilibrium, and the scalars should be the expected payoffs. $\square$

To solve the proposed nonlinear program, we follow the below-mentioned computation procedure.

2.2.3 Computation Procedure

The solution process to obtain the Nash equilibrium points and the maximized expected payoffs of both the players of a two-person non-zero sum game is introduced below.

Step 1 : Input two payoff matrices $(\tilde{A}, \tilde{B})$ corresponding to player *I* and player *II*, respectively.

Step 2 : Put $\tilde{A}$ and $\tilde{B}$ in the nonlinear program (2.2.3) given in Theorem 2.2.3.

Step 3 : Δ being a monotonic and linear bijection, apply Δ^{-1} to both sides of the

constraints of program (2.2.3), the problem becomes

$$\Delta \left[\max_{\Phi, \Psi, \tilde{p}, \tilde{q}} \Phi \Delta^{-1} (\tilde{A} \circ \tilde{B}) \Psi^T - \Delta^{-1} (\tilde{p}) - \Delta^{-1} (\tilde{q}) \right]$$

such that

$$\Delta^{-1} (\tilde{A}) \Psi^T \leq \Delta^{-1} (\tilde{p}) J_r^T$$

$$\Delta^{-1} (\tilde{B}^T) \Phi^T \leq \Delta^{-1} (\tilde{q}) J_s^T$$

$$\phi_i \geq 0, \psi_j \geq 0$$

Step 4 : Set $\Delta^{-1} (\tilde{A}) = A$, $\Delta^{-1} (\tilde{B}) = B$, $\Delta^{-1} (\tilde{p}) = p$ and $\Delta^{-1} (\tilde{q}) = q$, the nonlinear program becomes

$$\Delta \left[\max_{\Phi, \Psi, p, q} \Phi \Delta^{-1} (\tilde{A} \circ \tilde{B}) \Psi^T - p - q \right]$$

such that

$$A \Psi^T \leq p J_r^T \quad (2.2.9)$$

$$B^T \Phi^T \leq q J_s^T$$

$$\phi_i \geq 0, \psi_j \geq 0, \Phi J_r = 1 = \Psi J_s^T$$

Step 5 : Use the Maple commands mentioned in Figure 2.1 to solve the nonlinear program (2.2.9).

```

> with(LinearAlgebra) :
> A := Matrix([[a11, a12, ..., a1m], [a21, a22, ..., a2m], ..., [an1, an2, ..., anm]]);
> B := Matrix([[b11, b12, ..., b1m], [b21, b22, ..., b2m], ..., [bn1, bn2, ..., bnm]]);
> Phi := <[phi[1], phi[2], ..., phi[n]]>;
> Psi := <[psi[1], psi[2], ..., psi[m]]>;
> Cnst := {seq((A.Psi)[i] <= p, i = 1..n), seq((Transpose(Phi).B)[i] <= q, i = 1..m) add(phi[i], i = 1..n) = 1, add(psi[i], i = 1..m) = 1};
> with(Optimization);
> objective := expand(Transpose(Phi).A.Psi + Transpose(Phi).B.Psi - p - q);
> QPSolve(objective, Cnst, assume=nonnegative, maximize);
> QPSolve(objective, Cnst, assume=nonnegative, maximize, initialpoint=({q = 1, p = 2}));
> NLPsolve(objective, Cnst, assume=nonnegative, maximize);

```

Figure 2.1: Maple commands for solving the Nonlinear program

Step 6 : Step 5 gives the values of p, q, Φ, Ψ . Substituting Δ function back to these variables, the solution of the original linguistic nonlinear program may be obtained.

Note 2.2.2. On changing the initial point in code, we may achieve the pure Nash solution also, but it is, in fact, a waste as pure Nash points may be evaluated directly from the

game unless it is a very large game. There are two ways of solving this problem using Maple commands. Firstly, we may use the command QPSolve if the objective function is a quadratic function. Secondly, NLPsolve may be used for a general nonlinear objective function. If the expected payoffs of the game might be negative, then we have to drop the “assume=nonnegative” part and mention the nonnegativity of the strategies to constraints.

In the next section, we proceed towards illustrating the practicality of the proposed approach of maximizing the objective and finding Nash equilibrium.

2.3 Numerical Illustration

In the present section, we consider two examples to demonstrate the effectiveness of the proposed nonlinear linguistic program for solving 2-TL bimatrix game problems. Example 2.3.1 considers a general linguistic bimatrix game whose payoff entries belong to a predefined LT set belonging to the interval $[-g, g]$. Apart from this, Example 2.3.2 includes linguistic bimatrices of two players whose payoffs belong to an LT set in the interval $[0, g]$.

Example 2.3.1. Consider a general linguistic bimatrix game having payoffs belonging to a symmetrical linguistic terms set

$$LT = \{\ell_{-2} : \text{Very Bad (VB)}, \ell_{-1} : \text{Bad (B)}, \ell_0 : \text{Average (AVG)}, \ell_1 : \text{Good (G)}, \ell_2 : \text{Very Good (VG)}\}$$

and suppose the payoff matrices corresponding to player I and player II be

$$\tilde{A} = \begin{bmatrix} G & G & (AVG, 0.3) & (VG, -0.1) & VG \\ VG & (B, 0.4) & (G, 0.8) & G & AVG \\ VB & (AVG, 0.2) & G & AVG & VG \\ VG & B & AVG & (G, 0.2) & (AVG, 0.4) \end{bmatrix} = \begin{bmatrix} \ell_1 & \ell_1 & (\ell_0, 0.3) & (\ell_2, -0.1) & \ell_2 \\ \ell_2 & (\ell_{-1}, 0.4) & (\ell_1, 0.8) & \ell_1 & \ell_0 \\ \ell_{-2} & (\ell_0, 0.2) & \ell_1 & \ell_0 & \ell_2 \\ \ell_2 & \ell_{-1} & \ell_0 & (\ell_1, 0.2) & (\ell_0, 0.4) \end{bmatrix}$$

and,

$$\tilde{B} = \begin{bmatrix} AVG & VG & (AVG, 0.3) & G & VG \\ VG & (G, 0.1) & B & VG & (VB, 0.1) \\ B & (VB, 0.4) & VG & G & (AVG, 0.4) \\ G & VG & (B, 0.4) & (VB, 0.2) & AVG \end{bmatrix} = \begin{bmatrix} \ell_0 & \ell_2 & (\ell_0, 0.3) & \ell_1 & \ell_2 \\ \ell_2 & (\ell_1, 0.1) & \ell_{-1} & \ell_2 & (\ell_{-2}, 0.1) \\ \ell_{-1} & (\ell_{-2}, 0.4) & \ell_2 & \ell_1 & (\ell_0, 0.4) \\ \ell_1 & \ell_2 & (\ell_{-1}, 0.4) & (\ell_{-2}, 0.2) & \ell_0 \end{bmatrix}$$

respectively.

We first look for pure Nash points in this game and these points are given in Table 2.1.

Φ	Ψ	$E_I(\Phi, \Psi)$	$E_{II}(\Phi, \Psi)$
(1, 0, 0, 0)	(0, 1, 0, 0, 0)	ℓ_1	ℓ_2
(1, 0, 0, 0)	(0, 0, 0, 0, 1)	ℓ_2	ℓ_2
(0, 1, 0, 0)	(1, 0, 0, 0, 0)	ℓ_2	ℓ_2

Table 2.1: Table for pure Nash points

Now applying the operator ‘ $\circ$ ’ as in Theorem 2.2.1 to $\tilde{A}$ and $\tilde{B}$,

$$\tilde{A} \circ \tilde{B} = \begin{bmatrix} \ell_1 & \ell_1 & (\ell_0, -0.4) & (\ell_1, -0.1) & \ell_2 \\ \ell_2 & (\ell_0, -0.5) & (\ell_0, -0.2) & \ell_3 & (\ell_{-2}, 0.1) \\ \ell_{-1} & (\ell_{-2}, -0.4) & \ell_3 & \ell_1 & (\ell_2, 0.4) \\ \ell_1 & \ell_1 & (\ell_{-1}, 0.4) & (\ell_{-1}, 0.4) & (\ell_0, 0.4) \end{bmatrix}$$

and solving the nonlinear programming problem 2.2.3, the problem becomes,

$$\Delta \left[\max_{\Phi, \Psi, \tilde{p}, \tilde{q}} \Delta^{-1} \left\{ \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix} \begin{bmatrix} \ell_1 & \ell_1 & (\ell_0, -0.4) & (\ell_1, -0.1) & \ell_2 \\ \ell_2 & (\ell_0, -0.5) & (\ell_0, -0.2) & \ell_3 & (\ell_{-2}, 0.1) \\ \ell_{-1} & (\ell_{-2}, -0.4) & \ell_3 & \ell_1 & (\ell_2, 0.4) \\ \ell_1 & \ell_1 & (\ell_{-1}, 0.4) & (\ell_{-1}, 0.4) & (\ell_0, 0.4) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{bmatrix} - \tilde{p} - \tilde{q} \right\} \right]$$

such that

$$\tilde{A} \Psi^T \leq \tilde{p} J_r^T$$

$$\tilde{B}^T \Phi^T \leq \tilde{q} J_s^T$$

$$\phi_i \geq 0, \psi_j \geq 0, \Phi J_r = 1 = \Psi J_s^T$$

Then for the linguistic variables $\tilde{p}$ and $\tilde{q}$, let $\Delta^{-1}(\tilde{p}) = p$ and $\Delta^{-1}(\tilde{q}) = q$. So we write the Maple code given in Figure 2.1 for achieving the solution after applying Δ^{-1} to constraints. Solving this, we get the result as $p = 1.11$, $q = 1.05$, $\phi[1] = 0.47$, $\phi[2] = 0.05$, $\phi[3] = 0.48$, $\phi[4] = 0$ and $\psi[1] = 0$, $\psi[2] = 0$, $\psi[3] = 0.51$, $\psi[4] = 0.19$, $\psi[5] = 0.3$. This implies that the expected payoffs of players I and II are $\tilde{p} = E_I(\Phi, \Psi) = \Delta(p) = (\ell_1, 0.11)$ and $\tilde{q} = E_{II}(\Phi, \Psi) = \Delta(q) = (\ell_1, 0.05)$. The results indicate that with mixed Nash points $\Phi = (0.47, 0.05, 0.48, 0)$ and $\Psi = (0, 0, 0.51, 0.19, 0.3)$, player I gets an expected payoff which is “good” (with translational value 0.11) and player II also gets a “good” expected payoff (with translational value 0.05).

Next we solve the following example which is taken from the paper [192].

Example 2.3.2. Consider a constant-sum linguistic game with payoffs from the linguistic terms

$LT = \{\ell_0 : \text{Very Poor (VP)}, \ell_1 : \text{Poor(P)}, \ell_2 : \text{Moderately Poor(MP)}, \ell_3 : \text{Fair(F)}, \ell_4 : \text{Moderately Good(MG)},$

$\ell_5 : \text{Good (G)}, \ell_6 : \text{Very Good (VG)}\}$

Suppose

$$\tilde{A} = \begin{bmatrix} MG & MG & F & F & (MG, 0.3) \\ MG & (F, -0.4) & (VG, -0.1) & MG & VP \\ (F, 0.4) & VG & MG & MG & MG \end{bmatrix}$$

is the payoff matrix of player I , and since it is a constant-sum (more precisely, zero-sum) game, the payoff matrix for player II would be $\tilde{B} = \text{neg}(\tilde{A})$. This means that,

$$\tilde{B} = \text{neg}(\tilde{A}) = \begin{bmatrix} MP & MP & F & F & (MP, -0.3) \\ MP & (F, 0.4) & (VP, 0.1) & MP & VG \\ (F, -0.4) & VP & MP & MP & MP \end{bmatrix}$$

Applying Δ^{-1} to the matrices $\tilde{A}$ and $\tilde{B}$ and based on the commands given in Figure 2.1, we get the optimal solution of the above problem as presented in Table 2.2.

Φ	Ψ	$E_I(\Phi, \Psi)$ (p)	$E_{II}(\Phi, \Psi)$ (q)
(0.3343, 0.1086, 0.5571)	(0.5571, 0, 0, 0.3593, 0.0836)	3.6657	2.3343

Table 2.2: Optimal Solution of the problem given in Example 2.3.2

This implies that the expected payoffs of players I and II are $\tilde{p} = E_I(\Phi, \Psi) = \Delta(p) = (\ell_4, -0.3343)$ and $\tilde{q} = E_{II}(\Phi, \Psi) = \Delta(q) = (\ell_2, 0.3343)$. Thus, the lower value of the game would be $\tilde{p} = (\ell_4, -0.3343)$ and the upper value of the game would be $\text{Neg}(\tilde{q}) = (\ell_4, -0.3343)$.

Theorem 2.3.1. The computation time required to compute the Nash equilibrium of a linguistic two-person non-zero sum game having entries as linguistic 2-tuples using the proposed method has complexity of $O(s + r + rs + \min(s^2, r^2))$.

Proof. In this proof, we derive the computation time to solve a linguistic bimatrix game using computation procedure given in Section 2.2.3. Consider the payoff matrices of player I and II as $\tilde{A}_{ij}$ and $\tilde{B}_{ij}$, respectively, where i runs from 1 to r and j runs from

1 to s . Step 1 takes the input of two matrices making $2rs$ operations. Step 5 solves the nonlinear program with the help of Maple, and its complexity is as given in Table 2.3. In

Constraints	$O(rs) + O(rs) + O(s) + O(r) = O(rs) + O(s) + O(r)$
Objective	$O(rs) + O(rs) - O(1) - O(1) = O(rs)$
QPSolve/NLPSolve	$O(\min(s^2, r^2))$

Table 2.3: Time complexity of Maple program

total, complexity of the defined methodology is $O(2rs) + O(rs) + O(s) + O(r) + O(rs) + O(\min(s^2, r^2)) = O(s + r + rs + \min(s^2, r^2))$. $\square$

2.4 Comparison and Discussion Analysis

The present section aims to discuss the comparative analysis of the proposed model by discussing the simplicity of our methodology with the existing methods given in the literature.

2.4.1 Comparison of the proposed approach with the approach used to solve “matrix games with linguistic payoffs”

Singh & Gupta [192] discussed a two-player constant-sum 2-TL matrix game in their paper, in which such constant-sum matrix games are solved that have payoffs consisting of 2-TL variables. The authors constituted a linguistic linear programming (LLP) approach to solve these types of games. It is interesting to know that, since constant-sum games are a particular case of non-zero sum games, we can also use our proposed approach to solve the games introduced by them. According to the results obtained in the corresponding paper [192], the optimal values i.e. the expected payoffs of player I and player II of Example 2.3.2 are $(\ell_4, -0.3343)$ and $(\ell_2, 0.3343)$ respectively. Moreover, the value of the game is $(\ell_4, -0.3343)$. The obtained optimal solution is analogous to the results shown through the proposed methodology. This exhibits the consistency of the developed approach used for solving the non-zero sum as well as zero-sum matrix games having 2-TL payoffs in the interval $[0, g]$ as well.

Besides, our methodology proffers a generalization of the technique developed by the authors in the paper [192]. For solving complex decision-making problems, the methodology of our paper are more simpler and practical than the other gen-

eral game solution models. Despite the fact that there are several methodologies available in the literature to solve constant-sum linguistic matrix games, the approach proposed by us is an improvement to offer acceptable results at a faster rate. This could happen because one can find the results using a single nonlinear program only, for the task of forming two separate optimization problems for both the players have now been reduced and the safety values of both the players as well as the value of the game may now be computed using one nonlinear program, which we can solve easily using Maple code provided in the text. Moreover, the proposed approach is capable enough to solve comprehensive game problems. If such a nonlinear program was not constructed, in that case, we had to construct two nonlinear optimization problems, which might be somewhat tedious for solving large-scale matrix game problems.

2.4.2 Comparison of the proposed approach with the approach used to solve “fuzzy bimatrix games”

Larbani [106] introduced bimatrix games having fuzzy payoffs and provided a solution method to solve such games. For this, the game is first defuzzified by taking α -cuts of the payoffs. The resulting game is then treated as the game with uncertainty. This concept of defuzzification converts the fuzzy payoffs into numerical ones by using the process of α -cut. This approach is limited by its representation and the solution methods on linguistic terms. This limitation is the loss of information occurred because of expressing the results in the initial expression domain, and this loss implicates a lack of accuracy occurred in the results due to aggregation of linguistic information. However in our work, this limitation is overcome by expressing the linguistic information as 2-tuples that allow a continuous form of the linguistic data in its domain. Along with the 2-tuple representation of payoffs, we establish a solution method for computing with words in bimatrix games causing no loss of information. The loss of information caused by the need to express the results in the initial expression domain is discrete in the former paper. This loss is easily dealt with by the Δ function in our proposed methodology, which describes the conclusion in the original domain using Δ^{-1} function, without causing a single information loss in a continuous domain.

Hence to conclude, this study provides the concepts of linguistic bimatrix games,

which is considered to be a new study in the literature addressing such types of games. Considering the importance of the methodology proposed and its relevance in the theory of bimatrix games in the real-time scenario, it emphasizes us to take forward the present theory to solve games with incomplete information to handle the uncertainty and vagueness present in the real-world problems.

2.5 Conclusions

This chapter introduces a new explanation as well as the existence of Nash equilibrium in linguistic bimatrix games. Here, the elements of payoff matrices are in the form of descriptive words that characterize 2-TL variables. For this purpose, a methodology to solve linguistic bimatrix games is suggested to establish the solution. This approach is used to find the mixed Nash equilibrium as well as the maximized expected payoffs of both the players at the same time. Instead of forming two auxiliary equations for both the players separately, we have constructed a single nonlinear program which is supporting the computations of both the players simultaneously. Moreover, the computations become easier to perform because of the Maple programming code used for solving the nonlinear programming problem. The payoffs belong to a predefined LT set defined in the interval $[-g, g]$. The proposed methodology would be very much helpful in solving two-person constant-sum linguistic matrix games with payoffs in $[0, g]$ also and this is depicted with the help of an example in the paper as well. Moreover, our method is beneficial and supportive for solving large-scale two-person game problems.

The proposed approach is advantageous in solving two-person non-zero sum linguistic games, where the payoffs belong to the linguistic term set having balanced or symmetrical distribution data. So, in the near future, we aim to extend this methodology to linguistic games where the data belongs to an asymmetrical or unbalanced linguistic term set.

Chapter 3

Matrix Games with unbalanced linguistic information

This chapter¹² is implemented in two segments. The objective of the first segment is to present a novel approach for computing two-player constant-sum matrix games laid on the notion of a “symbolic proportional” linguistic term set. As it is not always possible to lay assessments based on uniformly and symmetrically distributed linguistic term sets; hence, the defined concept motivates the DMs to represent their opinions using 2-tuples composed of two proportional linguistic terms. The proportional 2-TL representation of payoffs concerns linguistic labels, which do not certainly have to be symmetrically distributed or do not have the conventional prerequisite of having uniform distance among them. This representation confers an opportunity to describe the linguistic payoffs of a matrix game using members of a continuous linguistic scale domain. This perspective of proportional 2-tuples provides DMs an approach to represent their opinions not by just using one label, rather by proportional linguistic labels of the form $(h\ell_\varepsilon, \zeta\ell_{\varepsilon+1})$, where ℓ_ε and $\ell_{\varepsilon+1}$ are two successive linguistic terms, with $0 \leq h, \zeta \leq 1$ and $h + \zeta = 1$. The next segment proposes a generalization of the concept of matrix games having proportional payoffs by representing the payoffs in the form of linguistic distribu-

¹The content of this chapter is based on the research papers “Matrix games with proportional linguistic payoffs”, *Soft Computing*, Springer, 25, 15067–15081 (2021). <https://doi.org/10.1007/s00500-021-06363-3> (SCIE, Impact Factor : 4.1)

²“Matrix Games with Linguistic Distribution Assessment Payoffs”, In: Saraswat, M., Sharma, H., Balachandran, K., Kim, J.H., Bansal, J.C. (eds) Congress on Intelligent Systems. *Lecture Notes on Data Engineering and Communications Technologies*, vol 111. Springer, Singapore. (2022). https://doi.org/10.1007/978-981-16-9113-3_56. (Conference Proceedings Citation Index (CPCI))

tion assessments. Such types of payoffs allow the players to express their opinion in terms of a whole linguistic set and, thus, are not limited to just one single or two linguistic terms. To solve the mentioned two types of games, the different linguistic linear programming (LLP) problems are framed and evaluated. The applicability of the proposed methods is shown with the help of some test examples.

3.1 Introduction

The overpowering complications of the socio-economic environment make it more uncertain and imprecise. Human beings usually make use of words in prevailing evaluation and decision-making processes without the use of a certain numeric value. CW models [92, 97, 98, 158] play a significant role in dealing with computational processes linguistically. A critical attribute of CW is that it operates linguistic inputs to achieve linguistic outputs quickly understandable by humans. Yager [247] indicates the significance of the translation and retranslation processes to realize the objectives of CW.

CW has played a significant role in dealing with game problems, mainly in economics, politics, and social sciences. Since the introduction of game theory, a lot of research has been committed to solving various types of game problems in exceptional situations, e.g., a group of players, the strategy sets available to the players, and a stipulation of payoffs for every association of strategies, etc. The ultimate goal of game theory is to find the optimal solution(s) for players. This optimization means that we look for such values which maximize or minimize the payoffs of respective players. Considering the importance of optimization in different research domains, several optimization methods have been presented so far. The optimization algorithms, which cover both types of maximization and minimization issues, have many applications in quantum differential evolution [40, 61, 62], decision-making, differential evolution algorithms [63, 64, 201] etc. Game theory and optimization have a conceptual overlap. In other words, game theory is a kind of approximating optimization. Traditionally matrix games have considered a precise or certain value of the payoffs. Nevertheless, there exist some complicated conditions in games, e.g., unbalanced information among the game's players, imprecise factors in the decision situations, and a degree of vagueness among the players' reluctance, such that some payoffs may be uncertain and not able to be evaluated exactly by the players. In this type of condition, it is helpful to design the problems as matrix games having fuzzy payoffs. Several researchers have utilized fuzzy sets [255] in matrix games [20–22, 35, 41]. Fuzzy games, however, are restricted to numbers and do not deal with the linguistic aspect of human interaction, making it difficult to express the information linguistically. Hence, to facilitate the players' straightforward selection of utilities, several authors [4, 5, 56, 185]

suggested matrix games based on a linguistic approach. In addition to these contributions, Singh et al. [192] amplify the dimension of matrix games under a 2-tuple linguistic structure. In that paper, the authors have implemented the 2-tuple linguistic model [92, 95] to solve two-player constant-sum linguistic matrix games.

The application of the 2-TL model only allows players to express information in terms of linguistic terms that belong to a balanced or symmetrical LT set. However, from a more realistic viewpoint, it is not always possible to express information in the form of balanced linguistic terms. So to address the uncertainty well, methodologies implementing ULTS are incorporated in various fields [1, 18, 38, 39, 96]. Moreover, Wang & Hao [221] established an advanced form of the linguistic representation model for unbalanced sets, namely the symbolic proportional model. This newly designed so-called symbolic proportional model is interestingly easy-going and is flexible in a computation stage for CW allowing DMs to evaluate their alternatives via proportional 2-tuples of the form $(\hbar\ell_i, \zeta\ell_{i+1})$, where ℓ_i, ℓ_{i+1} are represented as two successive linguistic terms, such that $\hbar, \zeta \in [0, 1]$, $\hbar + \zeta = 1$. However, this model has a limitation as the symbolic proportions are allocated only to 2 consecutive members of an LT set. To overcome this limitation, Zhang et al. [262] presented the notion of distribution assessments in an LT set. In this concept, symbolic proportions are assigned to all the linguistic terms, unlike the symbolic proportional approach.

In this work, the scope of 2-TL matrix games is extended within the aforementioned unbalanced linguistic framework introduced by Wang & Hao [221] and Zhang et al. [262] to manage the uncertainty to a large extent. For this purpose, we proceed one-by-one segmentwise. In the first segment, we consider matrix games as having payoffs in the form of symbolic proportions. An approach is developed to solve such constant-sum linguistic matrix games involving proportional linguistic information. The methodology introduced in this segment is an entirely new approach as it is concerned with the matrix games consisting of the payoffs of the type, say $(0.6M_1, 0.4M_2)$ where M_1 and M_2 denote month 1 and month 2 respectively. The notation represents the case when the production of a commodity is reported as 60% in month 1 and 40% in month 2. This representation of payoffs is based on the concept of "symbolic proportion." Under a proportional 2-tuple linguistic context of payoffs, not only the process of CW may be carried out without any loss of information in some sense, but the linguistic labels can

also be dealt with, without having to be symmetrically distributed around a central label and without having the necessity of being equidistant. Thus, a two-player constant-sum proportional matrix game is developed to model game problems having unbalanced proportional payoffs. To solve such game problems, a Proportional Linguistic Linear Programming (PLLP) methodology is proposed. Finally, this PLLP formulation is used to solve a multi-criteria decision-making (MCDM) problem based on actual-time linguistic information to strengthen the applicability of this new version of matrix game problems.

The next segment generalizes the concept of symbolic proportions in game problems. For that purpose, we consider matrix games having payoffs in the form of linguistic distribution assessments. This means that this study considers that the players may express their opinion in the form of distribution assessments. For example, if $LS = \{\ell_{-2} = \text{very poor}, \ell_{-1} = \text{poor}, \ell_0 = \text{average}, \ell_1 = \text{good}, \ell_2 = \text{very good}\}$ is an LT set, and a cricket captain uses the members of the set LS to assess a player's performance. The player takes part in 10 cricket games, in which 7 times was evaluated as ℓ_2 , 1 time as ℓ_1 , and 2 times as ℓ_{-1} . Then, the captain's evaluations may be interpreted in the form of a distribution assessments of LS as $\{(\ell_{-2}, 0), (\ell_{-1}, 0.2), (\ell_0, 0), (\ell_1, 0.1), (\ell_2, 0.7)\}$. The defined methodology allows the DMs to evaluate performances in which the players' opinions do not have to be restricted to one or two terms. To deal with such types of game problems, first, we define what matrix games having payoffs as linguistic distribution assessments mean. Next, we define the methodology in case the players apply pure strategies in the game. For the case of mixed strategies, we define the Linguistic Distribution Linear Programming (LDLP) approach to evaluate matrix games. This methodology gives information about what strategy would be beneficial for the player to win the game.

The organization of the current chapter is presented as follows: Section 3.2 presents a brief review of proportional linguistic variables and linguistic distribution assessment sets. Then we introduce a two-player constant-sum matrix game based on the proportional linguistic model in Section 3.3. Section 3.3.1 offers some basic terminologies and definitions of matrix games based on proportional linguistic information. In Section 3.3.2, a Proportional Linguistic Linear Programming is being formulated in relevance to the proportional matrix game. Section 3.3.3 illustrates the application of proportional matrix games in the MCDM prob-

lem, evaluating the perspective of doctors in the first-line treatment of constipation based on several alternatives chosen on different criteria. Section 3.4 generalizes the idea of matrix games having proportional linguistic payoffs to matrix games with payoffs as distributed assessment sets defined in Section 3.4.1. In Section 3.4.2, we introduce the LDLP approach to solve such matrix games. Section 3.5 analyses the results obtained in previous sections. Section 3.6 provides some conclusions of the chapter and recommends future works.

3.2 Preliminaries

For reviewing the basics of the two-person matrix games, one can refer to the Section 1.5 of the introductory chapter. Here, we proceed to briefly explain the proportional linguistic variables. Further we explain the generalisation of a proportional LT set as a linguistic distribution assessment set.

3.2.1 Proportional Linguistic Variables (PLVs)

The 2-TL model presented by Herrera & Martinez (2000) [92] comprises of linguistic terms belonging to an LT set $L = \{\ell_\varepsilon, \varepsilon = 0, 1, \dots, g\}$. Generally, we assume the index of granularity in information uncertainty to be an even positive integer. In literature, Wang & Hao (2006) [221] introduced an advanced version of the 2-TL model as given below. This model is called the “symbolic proportional model” and is proficient enough in handling linguistic information asymmetrically.

Definition 3.2.1. [221] Suppose $L = \{\ell_0, \ell_1, \dots, \ell_g\}$ is an ordered set, $I = [0, 1]$ and

$$IL \equiv I \times L = \{(\hbar, \ell_\varepsilon) : \hbar \in [0, 1] \ \& \ \varepsilon = 0, 1, \dots, g\}.$$

Consider the pair $(\ell_\varepsilon, \ell_{\varepsilon+1})$ of two succeeding members of the set L , then any two elements $(\hbar, \ell_\varepsilon), (\zeta, \ell_{\varepsilon+1}) \in IL$ are named as a symbolic proportion pair and $\hbar, \zeta$ are termed as pair of symbolic proportions of the pair $(\ell_\varepsilon, \ell_{\varepsilon+1})$ if $\hbar + \zeta = 1$. A symbolic proportion pair $(\hbar, \ell_\varepsilon), (1 - \hbar, \ell_{\varepsilon+1})$ is represented as $(\hbar \ell_\varepsilon, (1 - \hbar) \ell_{\varepsilon+1})$ and we denote the set of all symbolic proportions as $\bar{L}$, that is $\bar{L} = \{(\hbar \ell_\varepsilon, (1 - \hbar) \ell_{\varepsilon+1}), \hbar \in [0, 1], \varepsilon = 0, 1, \dots, g - 1\}$.

Definition 3.2.2. [221] **Negation of a Proportional Variable**

The negation of a PLV is described as: $Neg((\hbar \ell_\varepsilon, (1 - \hbar) \ell_{\varepsilon+1})) = ((1 - \hbar) \ell_{g-\varepsilon-1}, \hbar \ell_{g-\varepsilon})$

Definition 3.2.3. [221] **Position Index Function**

Let $L = \{\ell_0, \ell_1, \dots, \ell_g\}$ be an ordinal term set and $\bar{L}$ be the resulting ordinal proportional 2-tuple set. The position index function $\pi: \bar{L} \rightarrow [0, g]$ of an ordinal 2-tuple is defined as

$$\pi((\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1})) = \varepsilon + (1-\hbar), \varepsilon = 0, 1, \dots, g-1, \hbar \in [0, 1]$$

π is a bijection under identification convention and its inverse function $\pi^{-1}: [0, g] \rightarrow \bar{L}$ may be determined as

$$\pi^{-1}(z) = ((1-\zeta)\ell_\varepsilon, \zeta\ell_{\varepsilon+1})$$

where $\varepsilon = E(z)$, E is the integer part function, $\zeta = z - \varepsilon$ and $(1.\ell_g, 0.\ell_{g+1}) \equiv (0.\ell_{g-1}, 1.\ell_g)$.

Definition 3.2.4. [221] **Comparison of Proportional 2-tuples.**

For any $(\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1}), (\zeta\ell_\vartheta, (1-\zeta)\ell_{\vartheta+1}) \in \bar{L}$,

$$(\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1}) < (\zeta\ell_\vartheta, (1-\zeta)\ell_{\vartheta+1}) \iff \varepsilon + (1-\hbar) < \vartheta + (1-\zeta)$$

- When $\varepsilon < \vartheta$, then

(a) $(\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1}) = (\zeta\ell_\vartheta, (1-\zeta)\ell_{\vartheta+1})$ if $\vartheta = \varepsilon - 1$ and $\hbar = 0$ & $\zeta = 1$.

(b) Otherwise, $(\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1}) < (\zeta\ell_\vartheta, (1-\zeta)\ell_{\vartheta+1})$.

- When $\varepsilon = \vartheta$, then

(a) If $\hbar = \zeta$, then $(\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1}) = (\zeta\ell_\vartheta, (1-\zeta)\ell_{\vartheta+1})$.

(b) If $\hbar < \zeta$, then $(\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1}) > (\zeta\ell_\vartheta, (1-\zeta)\ell_{\vartheta+1})$.

(c) If $\hbar > \zeta$, then $(\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1}) < (\zeta\ell_\vartheta, (1-\zeta)\ell_{\vartheta+1})$.

It is worth noting that the above mentioned comparison techniques can further be characterized by using position indices. Thus, for any $(\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1}), (\zeta\ell_\vartheta, (1-\zeta)\ell_{\vartheta+1}) \in \bar{L}$,

$$(\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1}) < (\zeta\ell_\vartheta, (1-\zeta)\ell_{\vartheta+1}) \iff \pi(\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1}) < \pi(\zeta\ell_\vartheta, (1-\zeta)\ell_{\vartheta+1}),$$

$$Neg(\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1}) = \pi^{-1}(g - (\pi(\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1})))$$

Definition 3.2.5. [221] Let $L, \bar{L}$ be as defined in Definition 3.2.1 and suppose $\gamma: [0, g] \rightarrow L \times [-0.5, 0.5]$ be the function defined by Herrera & Martinez in [92]. Let $\bar{L}^T$ be the range

of the $\hbar$ function. Define $h: \bar{L} \rightarrow \bar{L}^T$ by

$$h((\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1})) = \begin{cases} (\ell_{\varepsilon+1}, -\hbar), & \text{if } 0 \leq \hbar \leq 1/2 \\ (\ell_\varepsilon, 1-\hbar), & \text{if } 1/2 < \hbar \leq 1 \end{cases}$$

The 2-tuples present on the right side of the above equation are traditional 2-tuples as defined by Herrera & Martinez [92]. h is a bijective function that can be used to convert traditional 2-tuples into proportional 2-tuples.

3.2.2 Distribution Assessment Sets [262]

Definition 3.2.6. Let $LS = \{\ell_{-t}, \dots, \ell_t\}$ be an LT set. Let $m = \{(\ell_u, \beta_u) | u = -t, \dots, t\}$, where $\ell_u \in LS$, $\beta_u \geq 0$, $\sum_{u=-t}^t \beta_u = 1$, $t \in \mathbb{Z}^+$ and β_u is the symbolic proportion of ℓ_u . Then m is said to be a distribution assessment of LS .

Definition 3.2.7. Let $m = \{(\ell_u, \beta_u) | u = -t, \dots, t\}$, $\ell_u \in LS$, $\beta_u \geq 0$, $\sum_{u=-t}^t \beta_u = 1$, where $t \in \mathbb{Z}^+$ be a distribution assessment of LS . The expectation of m is defined as follows:

$$E(m) = \sum_{u=-t}^t \beta_u \Delta^{-1}(\ell_u). \quad (3.2.1)$$

Definition 3.2.8. Comparison Operator of distribution assessments

If m_1 & m_2 are two distribution assessments of LS . Then

1. If $E(m_1) < E(m_2)$, then m_1 is less important than m_2 .
2. If $E(m_1) = E(m_2)$, then m_1 and m_2 are equivalent.

Definition 3.2.9. Negation Operator of distribution assessments

$$Neg(\{(\ell_u, \beta_u), u = -t, \dots, t\}) = \{(\ell_u, \beta_{-u}), u = -t, \dots, t\}$$

Definition 3.2.10. Let $\{m_1, \dots, m_n\}$, $n \in \mathbb{N}$ be a set of distribution assessments of LS , where $m_k = \{(\ell_u, \beta_u^k) | u = -t, \dots, t\}$, $k = 1, 2, \dots, n$. Let $\{w_1, w_2, \dots, w_n\}$ be the associated weights satisfying $w_i > 0$ and $\sum_{i=1}^n w_i = 1$. The weighted averaging operator of distribution assessments is:

$$DAWA_w(m_1, \dots, m_k) = \{(\ell_u, \beta_u), u = -t, \dots, t\}, \text{ where } \beta_u = \sum_{i=1}^k w_i \beta_u^i \quad (3.2.2)$$

In both classical and fuzzy environments, numerous studies have been conducted for solving matrix games. However, this setup has been extended further

to the linguistic games overcoming the limitation of providing precise membership values to the payoffs. The linguistic setup in games has provided a more flexible structure for the DMs to evaluate the game problems. This linguistic information may come from a balanced as well as an unbalanced LT set. The next section discusses the matrix games based on the information underlying an unbalanced (proportional approach) LT set.

3.3 Two-Player Constant-Sum Matrix Games based on Proportional Linguistic Information

In this section, we majorly focus on the non-conventional case of matrix games in which the players are capable of expressing knowledge in terms of unbalanced linguistic labels.

3.3.1 Terminologies and Definitions

Definition 3.3.1. Consider a two-player constant-sum proportional game $\tilde{G}$ as a quadruplet $(S^r, S^s, \bar{L}, \tilde{\mathcal{A}})$ where $S^r = \{\Phi = (\phi_1, \phi_2, \dots, \phi_r) \mid \phi_i \geq 0, i = 1, 2, \dots, r, \sum_{i=1}^r \phi_i = 1\}$, and $S^s = \{\Psi = (\psi_1, \psi_2, \dots, \psi_s) \mid \psi_j \geq 0, j = 1, 2, \dots, s, \sum_{j=1}^s \psi_j = 1\}$ represent the strategy sets of player I and player II , respectively. The members of S^r (S^s) of the form $\Phi = (0, 0, \dots, 1, \dots, 0)^T$ where 1 is at the i^{th} position ($\Psi = (0, 0, \dots, 1, \dots, 0)^T$ where 1 is at the j^{th} position) are called pure strategies of Player I (Player II). $\bar{L}$ is the set of symbolic proportions as defined in Definition 3.2.1, $\tilde{\mathcal{A}} = [\tilde{a}_{ij}]_{r \times s}$ being the proportional payoff matrix of player I corresponding to player II given as:

$$\tilde{\mathcal{A}} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1s} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{r1} & \tilde{a}_{r2} & \dots & \tilde{a}_{rs} \end{pmatrix}$$

where $\tilde{a}_{ij} \in \bar{L}$ represents payoff in the form of proportional 2-tuple. Basically if Player I chooses i^{th} pure strategy and Player II chooses j^{th} pure strategy then $\tilde{a}_{ij}$ is the amount paid by Player II to Player I . Moreover, $neg(\tilde{\mathcal{A}})$ comprehends the payoff matrix of player II which shows the amount paid by Player I to Player II . From the matrix $\tilde{\mathcal{A}}$, it can clearly be understood that if player I & II opt for pure strategies i and j , respectively,

then the payoff of player I would be $\tilde{a}_{ij}$ and that of player II would be $neg(\tilde{a}_{ij})$ i.e., if $\tilde{a}_{ij} = (\hbar\ell_k, (1-\hbar)\ell_{k+1})$ is the payoff of player I for some k , then payoff of player II would be $neg(\tilde{a}_{ij}) = ((1-\hbar)\ell_{g-k-1}, \hbar\ell_{g-k})$, where $g+1$ is the cardinality of the LT set L .

We next define the values of player I and player II and the value of the game in terms of symbolic proportions. The meaning of the game's solution is best recognized in terms of maxmin and minmax principles for Player I and Player II , respectively. As per these principles, both the players choose that strategy, resulting in the best of the worst outcomes. Player I (the maximizing player) opts to play an analogous strategy to the maximum of minimum benefits for his different approach. This is called the maxmin principle. In similar terms, Player II (the minimizing player) also likes to play safe, and in that case, he opts for that strategy which is analogous to the minimum of maximum losses for his different approach. This is called the minmax principle. Implementing these principles, we obtain the following values of players.

Definition 3.3.2. Let us consider $\tilde{v}^{1-}$ and $\tilde{v}^{2-}$ as the values representing the minimum gain (in terms of symbolic proportion) as received by player I and player II , respectively. Then, these values are defined in the following manner:

$$\tilde{v}^{1-} = \max_{i=1,\dots,r} \min_{j=1,\dots,s} \tilde{a}_{ij}$$

$$\tilde{v}^{2-} = \max_{j=1,\dots,s} \min_{i=1,\dots,r} neg(\tilde{a}_{ij})$$

where $\tilde{a}_{ij} \in \bar{L}$.

Note 3.3.1. The lower and upper values of the game are defined as $\tilde{v}^- = \tilde{v}^{1-}$ and $\tilde{v}^+ = neg(\tilde{v}^{2-})$, respectively. Also,

$$\begin{aligned} \tilde{v}^+ = neg(\tilde{v}^{2-}) &= neg\left(\max_{j=1,\dots,s} \min_{i=1,\dots,r} neg(\tilde{a}_{ij})\right) \\ &= \min_{j=1,\dots,s} \max_{i=1,\dots,r} (\tilde{a}_{ij}) \end{aligned}$$

The game has value $\tilde{v}$ if $\tilde{v} = \tilde{v}^+ = \tilde{v}^-$.

Remark 3.3.1. Since it can be proved that π is a linear operator and is a monotonic function, we can easily conclude that if

$$\tilde{a} = (\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1})$$

$$\tilde{b} = (\zeta \ell_{\vartheta}, (1 - \zeta) \ell_{\vartheta+1})$$

are two PLVs, then

$$(I) \quad \text{neg}(\text{neg}(\tilde{a})) = \tilde{a}$$

$$(II) \quad \min(\text{neg}(\tilde{a}), \text{neg}(\tilde{b})) = \text{neg}(\max(\tilde{a}, \tilde{b}))$$

$$(III) \quad \max(\text{neg}(\tilde{a}), \text{neg}(\tilde{b})) = \text{neg}(\min(\tilde{a}, \tilde{b}))$$

Remark 3.3.2. The maximum and minimum of two PLVs may be evaluated with the help of a comparison of proportional 2-tuples as described in Definition 3.2.4. That is, for every $\vartheta = 0, 1, \dots, g-1$,

$$\max((\hbar \ell_{\varepsilon}, (1 - \hbar) \ell_{\varepsilon+1})) = (\zeta \ell_{\vartheta}, (1 - \zeta) \ell_{\vartheta+1}) \iff (\hbar \ell_{\varepsilon}, (1 - \hbar) \ell_{\varepsilon+1}) \leq (\zeta \ell_{\vartheta}, (1 - \zeta) \ell_{\vartheta+1})$$

and, $\min((\hbar \ell_{\varepsilon}, (1 - \hbar) \ell_{\varepsilon+1})) = (\zeta \ell_{\vartheta}, (1 - \zeta) \ell_{\vartheta+1}) \iff (\hbar \ell_{\varepsilon}, (1 - \hbar) \ell_{\varepsilon+1}) \geq (\zeta \ell_{\vartheta}, (1 - \zeta) \ell_{\vartheta+1})$
 $\forall \varepsilon = 0, 1, \dots, g-1$.

Next, for the proportional linguistic values of the game defined in Definition 3.3.2, the following relation holds for any linguistic proportional game.

Theorem 3.3.1. Suppose $\tilde{v}^-$ and $\tilde{v}^+$ be the proportional lower and upper values of the game, respectively. Then $\tilde{v}^-$ and $\tilde{v}^+$ are PLVs and $\tilde{v}^- \leq \tilde{v}^+$.

Proof. Using Remark 3.3.2, it can easily be deduced that the minimum of two PLVs is further a PLV, and so is the maximum of minimum. Hence max-min of any PLV comes out to be proportional.

$$\implies \tilde{v}^- = \tilde{v}^{1-} = \max_{i=1, \dots, r} \min_{j=1, \dots, s} \tilde{a}_{ij} \text{ is a PLV.}$$

Similar argument follows for $\tilde{v}^+$.

Now, suppose $\tilde{a}_{ij} \in \bar{L}$. Then for any column j and a fixed row i , $\min_{j=1, \dots, s} \tilde{a}_{ij} \leq \tilde{a}_{ij}$ (using Comparison defined in Definition 3.2.4), thus taking maximum of both the sides of rows $i = 1, \dots, r$, we get

$$\tilde{v}^- = \max_i \min_j \tilde{a}_{ij} \leq \max_i \tilde{a}_{ij}$$

This holds for any column $j = 1, 2, \dots, s$. This means that $\tilde{v}^- \leq \min_j \max_i \tilde{a}_{ij} = \tilde{v}^+$, and hence the proof. $\square$

Definition 3.3.3. We call a particular row i^* and column j^* a saddle point in pure strategies of the game if $\tilde{a}_{ij^*} \leq \tilde{a}_{i^*j^*} \leq \tilde{a}_{i^*j}$, $\forall$ rows $i = 1, 2, \dots, r$ and columns $j = 1, 2, \dots, s$.

Lemma 3.3.1. A proportional linguistic matrix game will have a saddle point in pure strategies if and only if $\tilde{v}^- = \max_i \min_j \tilde{a}_{ij} = \min_j \max_i \tilde{a}_{ij} = \tilde{v}^+$.

Proof. By Definition 3.3.3,

$$\tilde{v}^+ = \min_j \max_i \tilde{a}_{ij} \leq \max_i \tilde{a}_{ij^*} \leq \tilde{a}_{i^*j^*} \leq \min_j \tilde{a}_{i^*j} \leq \max_i \min_j \tilde{a}_{ij} = \tilde{v}^-$$

But $\tilde{v}^- \leq \tilde{v}^+$ (Theorem 3.3.1), so we exhibit equality all round and $\tilde{v} = \tilde{v}^- = \tilde{v}^+ = \tilde{a}_{i^*j^*}$.

Further, if $\tilde{v}^- = \tilde{v}^+$, then $\min_j \max_i \tilde{a}_{ij} = \max_i \min_j \tilde{a}_{ij}$.

Suppose i^* be such that $\tilde{v}^+ = \max_j \tilde{a}_{i^*j}$ and j^* be such that $\tilde{v}^- = \min_i \tilde{a}_{ij^*}$. Then $\tilde{a}_{i^*j} \leq \tilde{v}^+ = \tilde{v}^- \leq \tilde{a}_{ij^*}$, for any $i = 1, 2, \dots, r, j = 1, 2, \dots, s$.

Moreover, taking $j = j^*$ on left and $i = i^*$ on the right, we get $\tilde{a}_{i^*j^*} = \tilde{v}^+ = \tilde{v}^-$. This suffices (i^*, j^*) to be the saddle point. $\square$

Now, we illustrate the above concept using an example.

Example 3.3.1. Suppose a builder tries to sell its two buildings I and II to a buyer. Three criteria namely Price, Architecture, and Gentry are selected for the sale. The payoffs are in terms of proportions chosen from the LT set $L = \{\ell_0 : \text{Very Low}, \ell_1 : \text{Low}, \ell_2 : \text{Average}, \ell_3 : \text{High}, \ell_4 : \text{Very High}\}$. The payoffs correspond to what the builder gets based on the above criterion. Let the payoff matrix of building I be:

Builder/Buyer		Price	Architecture	Gentry
$\tilde{\mathcal{A}} =$	Price	$(0.2\ell_2, 0.8\ell_3)$	$(0.2\ell_2, 0.8\ell_3)$	$(0.2\ell_2, 0.8\ell_3)$
	Architecture	$(0.4\ell_0, 0.6\ell_1)$	$(0.7\ell_1, 0.3\ell_2)$	$(0.2\ell_2, 0.8\ell_3)$
	Gentry	$(0.4\ell_0, 0.6\ell_1)$	$(0.1\ell_3, 0.9\ell_4)$	$(0.1\ell_3, 0.9\ell_4)$

This means that if the buyer purchases building I by seeing its price and building II based on its architecture, respectively then the builder is assured to receive a payoff which is 20% Average and 80% High of the market share.

Moreover, the payoff matrix for building II is:

Builder/Buyer		Price	Architecture	Gentry
$neg(\tilde{\mathcal{A}}) =$	Price	$(0.8\ell_1, 0.2\ell_2)$	$(0.8\ell_1, 0.2\ell_2)$	$(0.8\ell_1, 0.2\ell_2)$
	Architecture	$(0.6\ell_3, 0.4\ell_4)$	$(0.3\ell_2, 0.7\ell_3)$	$(0.8\ell_1, 0.2\ell_2)$
	Gentry	$(0.6\ell_3, 0.4\ell_4)$	$(0.9\ell_0, 0.1\ell_1)$	$(0.9\ell_0, 0.1\ell_1)$

The payoffs are interpreted in a way that if the buyer purchases building I based on Price and building II based on Architecture, respectively then the builder is assured to receive

a payoff which is 80% Low and 20% Average of the market share.

The lower value for building *I* is

$$\begin{aligned}\tilde{v}^{1-} &= \max_{i=1,2,3} \min_{j=1,2,3} (\tilde{a}_{ij}) = \max [(0.2\ell_2, 0.8\ell_3), (0.4\ell_0, 0.6\ell_1), (0.4\ell_0, 0.6\ell_1)] \\ &= (0.2\ell_2, 0.8\ell_3)\end{aligned}$$

Hence, the builder is assured of achieving at least a market share which is 20% Average and 80% High by selling building *I*.

In the similar manner, the lower value of building *II* is

$$\begin{aligned}\tilde{v}^{2-} &= \max_{i=1,2,3} \min_{j=1,2,3} (neg(\tilde{a}_{ij})) = \max [(0.8\ell_1, 0.2\ell_2), (0.8\ell_1, 0.2\ell_2), (0.9\ell_0, 0.1\ell_1)] \\ &= (0.8\ell_1, 0.2\ell_2)\end{aligned}$$

Hence, the builder will surely receive a payoff of at least 80% Low and 20% Average by selling building *II*.

Also the lower value of the game is $\tilde{v}^- = \tilde{v}^{1-} = (0.2\ell_2, 0.8\ell_3)$, and the upper value is $\tilde{v}^+ = neg(\tilde{v}^{2-}) = (0.2\ell_2, 0.8\ell_3)$. Hence, $(1, 1)$ is the saddle point and $(0.2\ell_2, 0.8\ell_3)$ is the value of the game.

Next, it is worth noting that it is not always possible to apply pure strategy in every case, or a saddle point always exists. For example, if we interchange $\tilde{a}_{21}$ with $(0.9\ell_3, 0.1\ell_4)$ in the above matrix $\tilde{\mathcal{A}}$, then $\tilde{v}^- = (0.2\ell_2, 0.8\ell_3)$ and $\tilde{v}^+ = (0.1\ell_3, 0.9\ell_4)$. So we have to apply the case where mixed strategies are involved. Hence, if the players propose mixed strategies to play the game, then their payoffs can be calculated by finding what is “expected” by the player to receive.

In the next section, we step forward to propose a Proportional Linguistic Linear Programming (PLLP) approach for solving the Proportional Matrix Game wherein players undergo mixed strategies (as well as pure strategies) to play the game.

3.3.2 Proportional Linguistic Linear Programming (PLLP) formulation of a Proportional Matrix Game

In the following section, we formulate PLLP to solve a proportional matrix game (involving games using mixed strategies) by making use of the expected payoff to evaluate the payoffs of the players.

Let $L = \{\ell_0, \ell_1, \dots, \ell_g\}$ be the LT set and $\bar{L} = \{(\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1}) : \hbar \in [0, 1] \text{ and } \varepsilon = 0, 1, \dots, g-1\}$ be the set of proportions, g being a positive integer. Consider the proportional linguistic matrix $\tilde{\mathcal{A}} = (\tilde{a}_{ij})_{r \times s}$, where each $\tilde{a}_{ij} \in \bar{L}$ represents the payoff to player I received after opting the strategy ' i ' while player II opts for strategy ' j '. Let $\Phi = (\phi_1, \phi_2, \dots, \phi_r) \in S^r$ and $\Psi = (\psi_1, \psi_2, \dots, \psi_s) \in S^s$ be the mixed strategies of player I & II , respectively. Then, in a proportional matrix game having uncertain and unbalanced payoff entries, the proportional expected payoff of players may be defined as

$$\tilde{E}_{\tilde{\mathcal{A}}}(\Phi, \Psi) = \bigoplus_{i=1}^r (\phi_i (\bigoplus_{j=1}^s \tilde{a}_{ij} \psi_j)). \quad (3.3.1)$$

Thus the expected payoff of player I corresponding to the j^{th} strategy chosen by player II is given by $\tilde{a}_{1j}\phi_1 \oplus \tilde{a}_{2j}\phi_2 \oplus \dots \oplus \tilde{a}_{rj}\phi_r$. This expression is the weighted average of all $\tilde{a}_{ij}$ which is defined as below:

Definition 3.3.4. Let $\tilde{a}_{ij} = (\hbar\ell_\varepsilon, (1-\hbar)\ell_{\varepsilon+1}) \in \bar{L}$, $\varepsilon = 0, 1, \dots, g-1$ be a proportional 2-tuple variable and $w = (w_1, w_2, \dots, w_r)$ be the associated weight vector satisfying $0 \leq w_i \leq 1, i = 1, \dots, r, \sum_{i=1}^r w_i = 1$. Then the proportional weighted average operator may be defined in the following manner:

$$\begin{aligned} PWA[\tilde{a}_{ij}, i = 1, \dots, r] &= \tilde{a}_{1j}w_1 \oplus \tilde{a}_{2j}w_2 \oplus \dots \oplus \tilde{a}_{rj}w_r \\ &= \pi^{-1} \left(\sum_{i=1}^r w_i \pi(\tilde{a}_{ij}) \right), j = 1, \dots, s \end{aligned}$$

Consequently,

$$\pi \left(\bigoplus_{i=1}^r \tilde{a}_{ij} w_i \right) = \sum_{i=1}^r w_i \pi(\tilde{a}_{ij}) \quad (3.3.2)$$

Note 3.3.2. Several operators are used to aggregate PLVs, e.g. CCV operator defined by Wang & Hao [221]. But usage of the CCV operator draws our attention towards the defuzzification of variables which is not our objective because of the information loss involved in the method. So we use the π function because here we deal with the position indices of the variables.

Proceeding towards the methodology, here, we need to evaluate the mixed strategy $\Phi \in S^r$ of player I such that the value of player I , i.e., $\tilde{v}^{1-}$ gets maximized. So

the required PLLP for player I is:

$$\begin{aligned} \max \quad & \tilde{v}^{1-} \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \bigoplus_{i=1}^r \tilde{a}_{ij} \phi_i \geq \tilde{v}^{1-} \\ \text{where} \\ \phi_i \geq 0, \sum_{i=1}^r \phi_i = 1, \tilde{a}_{ij} \in \bar{L}. \end{array} \right. \end{aligned} \quad (\text{P1})$$

As π is a monotonically increasing function, we apply π to each constraint of (P1) and then use Equation 3.3.2. Thus the strategies and minimum gain of player I can be obtained by solving the following optimization problem:

$$\begin{aligned} \max \quad & \tilde{v}^{1-} \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \sum_{i=1}^r \pi(\tilde{a}_{ij}) \phi_i \geq \pi(\tilde{v}^{1-}) \\ \text{where} \\ \phi_i \geq 0, \sum_{i=1}^r \phi_i = 1, \tilde{a}_{ij} \in \bar{L}. \end{array} \right. \end{aligned}$$

Substituting $\pi(\tilde{a}_{ij}) = a_{ij}$, and $\pi(\tilde{v}^{1-}) = v$. Here, it is clear that $\pi(\tilde{v}^{1-}) = v > 0$. So, constraints of (P1) become

$$\begin{aligned} & \sum_{i=1}^r a_{ij} \phi_i \geq v \\ \text{s.t.} \quad & \phi_i \geq 0, \sum_{i=1}^r \phi_i = 1. \end{aligned}$$

Set $X_i = \frac{\phi_i}{v}$; $i = 1, 2, \dots, r$ and $\frac{1}{v} = V$. This is summarized as a linear program:

$$\begin{aligned} \min \quad & V = \sum_{i=1}^r X_i \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \sum_{i=1}^r a_{ij} X_i \geq 1 \\ \text{where } X_i \geq 0. \end{array} \right. \end{aligned} \quad (\text{P2})$$

and then we can solve (P2) as a usual linear programming problem.

It must be noted that in all the above constraints and expressions, the variable i varies as $i = 1, 2, \dots, r$ and the variable j varies as $j = 1, 2, \dots, s$.

Similarly, the PLLP for player II would be

$$\begin{aligned} \max \quad & \tilde{v}^{2-} \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \bigoplus_{j=1}^s \text{neg}(\tilde{a}_{ij}) \psi_j \geq \tilde{v}^{2-} \\ \text{where} \\ \psi_j \geq 0, \sum_{j=1}^s \psi_j = 1, \tilde{a}_{ij} \in \bar{L}. \end{array} \right. \end{aligned} \quad (\text{P3})$$

We have to find the mixed strategy $\Psi \in S^s$ for player II . Let $\text{neg}(\tilde{v}^{2-}) = \tilde{v}^+$.

$\Rightarrow$ From the properties of negation operator, the problem becomes

$$\begin{aligned} \min \quad & \tilde{v}^+ \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \bigoplus_{j=1}^s \tilde{a}_{ij} \psi_j \leq \tilde{v}^+ \\ \text{where} \\ \psi_j \geq 0, \sum_{j=1}^s \psi_j = 1, \tilde{a}_{ij} \in \bar{L}. \end{array} \right. \end{aligned} \quad (\text{P4})$$

Now we apply π operator in (P4) and use the expression in Equation 3.3.2.

Since earlier $\pi(\tilde{v}^{1-}) = v$ and if v is the value of the game then $\pi(\tilde{v}^+) = v$.

So the problem for player II becomes

$$\begin{aligned} \min \quad & \tilde{v}^+ \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \sum_{j=1}^s \pi(\tilde{a}_{ij}) \psi_j \leq \pi(\tilde{v}^+) \\ \text{where} \\ \psi_j \geq 0, \sum_{j=1}^s \psi_j = 1, \tilde{a}_{ij} \in \bar{L}. \end{array} \right. \end{aligned}$$

Put $Y_j = \frac{\psi_j}{v}; j = 1, 2, \dots, s$ and $\pi(\tilde{a}_{ij}) = a_{ij}$, the problem for player II reduces to

$$\begin{aligned} \max \quad & V = \sum_{j=1}^s Y_j \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \sum_{j=1}^s a_{ij} Y_j \leq 1 \\ \text{where } Y_j \geq 0. \end{array} \right. \end{aligned} \quad (\text{P5})$$

So, by solving (P2) and (P5), we get the optimal solution $X = (X_1, X_2, \dots, X_r)$ and $Y = (Y_1, Y_2, \dots, Y_s)$ and the optimal value V . Hence unwinding the formulation back

to our original variables, the optimal strategies Φ and Ψ of player I and player II respectively, could be achieved and the value v of the game follows. Also, the linguistic proportional values for player I and player II would be

$$\widetilde{v}^- = \widetilde{v}^{1-} = \pi^{-1}\left(\frac{1}{V}\right)$$

and,

$$\widetilde{v}^{2-} = neg\left(\pi^{-1}\left(\frac{1}{V}\right)\right)$$

respectively. Again, the variables i & j vary as $i = 1, 2, \dots, r$ and $j = 1, 2, \dots, s$.

Note 3.3.3. It is to be noted that applying the π function directly in the payoff matrix to convert payoffs into numbers must not be a good idea because the value of α is different for each cell. Instead, to do so, we apply the function π on both sides of the PLLP (π being a monotonic function, this is justifiable).

Remark 3.3.3. Note that the linear programming problems (P2) and (P5) obtained by using a transformation function π are particular crisp problems corresponding to given proportional linguistic 2-tuples and form primal-dual pair of each other. Since it is envisaged that the translation function π is an isomorphism, henceforth, there exists a similarity relation between the numerical and symbolic proportional linguistic spaces, i.e., $\overline{L} \cong [0, g]$ and both are structurally similar sets. As the duality result holds for the crisp LPP; therefore it follows directly for PLLP as well.

Example 3.3.2. Consider a 3×3 proportional payoff matrix whose elements belongs to the set $L = \{\ell_0 : \text{Very Poor}, \ell_1 : \text{Poor}, \ell_2 : \text{Average}, \ell_3 : \text{Good}, \ell_4 : \text{Very Good}\}$ given as

$$\tilde{\mathcal{A}} = \begin{bmatrix} (0.2\ell_0, 0.8\ell_1) & (0.3\ell_1, 0.7\ell_2) & (0.3\ell_1, 0.7\ell_2) \\ (0.3\ell_1, 0.7\ell_2) & (0.4\ell_2, 0.6\ell_3) & (0.9\ell_3, 0.1\ell_4) \\ (0.2\ell_0, 0.8\ell_1) & (0.2\ell_0, 0.8\ell_1) & (0.4\ell_2, 0.6\ell_3) \end{bmatrix}$$

Let $\Phi = (\phi_1, \phi_2, \phi_3), \phi_i \geq 0, i = 1, 2, 3$ such that $\sum_{i=1}^3 \phi_i = 1$, and $\Psi = (\psi_1, \psi_2, \psi_3), \psi_j \geq 0, j = 1, 2, 3$ such that $\sum_{j=1}^3 \psi_j = 1$ be the mixed strategies for player I and player II , respectively.

We formulate the PLLP for player I as in (P1) in following manner:

$$\begin{aligned} \max \quad & \widetilde{v}^- \\ \text{s.t.} \quad & \left\{ \begin{aligned} (0.2\ell_0, 0.8\ell_1)\phi_1 \oplus (0.3\ell_1, 0.7\ell_2)\phi_2 \oplus (0.2\ell_0, 0.8\ell_1)\phi_3 &\geq \widetilde{v}^- \\ (0.3\ell_1, 0.7\ell_2)\phi_1 \oplus (0.4\ell_2, 0.6\ell_3)\phi_2 \oplus (0.2\ell_0, 0.8\ell_1)\phi_3 &\geq \widetilde{v}^- \\ (0.3\ell_1, 0.7\ell_2)\phi_1 \oplus (0.9\ell_3, 0.1\ell_4)\phi_2 \oplus (0.4\ell_2, 0.6\ell_3)\phi_3 &\geq \widetilde{v}^- \\ \phi_1 + \phi_2 + \phi_3 = 1, \phi_1, \phi_2, \phi_3 &\geq 0. \end{aligned} \right. \end{aligned}$$

Applying π function in the constraints, we get

$$\begin{aligned} 0.8\phi_1 + 1.7\phi_2 + 0.8\phi_3 &\geq \pi(\widetilde{v}^-) \\ 1.7\phi_1 + 2.6\phi_2 + 0.8\phi_3 &\geq \pi(\widetilde{v}^-) \\ 1.7\phi_1 + 3.1\phi_2 + 2.6\phi_3 &\geq \pi(\widetilde{v}^-) \\ \phi_1 + \phi_2 + \phi_3 &= 1, \phi_1, \phi_2, \phi_3 \geq 0. \end{aligned}$$

If $\pi(\widetilde{v}^-) = v$, substituting $X_i = \frac{\phi_i}{v}$ and $V = \frac{1}{v}$, the problem reduces similar to (P2) as

$$\begin{aligned} \min \quad & V = X_1 + X_2 + X_3 \\ \text{s.t.} \quad & \left\{ \begin{aligned} 0.8X_1 + 1.7X_2 + 0.8X_3 &\geq 1 \\ 1.7X_1 + 2.6X_2 + 0.8X_3 &\geq 1 \\ 1.7X_1 + 3.1X_2 + 2.6X_3 &\geq 1 \\ X_1, X_2, X_3 &\geq 0. \end{aligned} \right. \end{aligned}$$

The optimal solution of this problem is $(X_1, X_2, X_3) = (0, 0.59, 0)$ and $V = 0.59$. Hence, $(\phi_1, \phi_2, \phi_3) = (0, 1, 0)$ and $v = 1.6949$. So,

$$\widetilde{v}^- = \pi^{-1}(v) = \pi^{-1}(1.6949) = (0.3051u_1, 0.6949u_2).$$

Thus the optimal strategy for player I is $\Phi = (0, 1, 0)$ and the value is

$$\widetilde{v}^- = (0.3051\ell_1, 0.6949u\ell_2)$$

Similarly, the PLLP for player *II* as in (P4) is

$$\begin{aligned} \min \quad & \widetilde{v}^+ \\ \text{s.t.} \quad & \begin{cases} (0.2\ell_0, 0.8\ell_1)\psi_1 \oplus (0.3\ell_1, 0.7\ell_2)\psi_2 \oplus (0.3\ell_1, 0.7\ell_2)\psi_3 \leq \widetilde{v}^+ \\ (0.3\ell_1, 0.7\ell_2)\psi_1 \oplus (0.4\ell_2, 0.6\ell_3)\psi_2 \oplus (0.9\ell_3, 0.1\ell_4)\psi_3 \leq \widetilde{v}^+ \\ (0.2\ell_0, 0.8\ell_1)\psi_1 \oplus (0.2\ell_0, 0.8\ell_1)\psi_2 \oplus (0.4\ell_2, 0.6\ell_3)\psi_3 \leq \widetilde{v}^+ \\ \psi_1 + \psi_2 + \psi_3 = 1, \psi_1, \psi_2, \psi_3 \geq 0. \end{cases} \end{aligned}$$

By substitution as given in (P5), the problem becomes

$$\begin{aligned} \max \quad & V = Y_1 + Y_2 + Y_3 \\ \text{s.t.} \quad & \begin{cases} 0.8Y_1 + 1.7Y_2 + 1.7Y_3 \leq 1 \\ 1.7Y_1 + 2.6Y_2 + 3.1Y_3 \leq 1 \\ 0.8Y_1 + 0.8Y_2 + 2.6Y_3 \leq 1 \\ Y_1, Y_2, Y_3 \geq 0. \end{cases} \end{aligned}$$

The optimal solution is $(Y_1, Y_2, Y_3) = (0.59, 0, 0)$ and $V = 0.59 \implies (\psi_1, \psi_2, \psi_3) = (1, 0, 0)$ and $v = 1.6949$. So, $\widetilde{v}^+ = \pi^{-1}(v) = \pi^{-1}(1.6949) = (0.3051\ell_1, 0.6949\ell_2)$.

$\therefore$ the optimal strategy used by player *II* is $\Psi = (1, 0, 0)$ and optimal value is

$$\widetilde{v}^+ = (0.3051\ell_1, 0.6949\ell_2) = \widetilde{v}.$$

It is noteworthy that $\widetilde{v}^- = \widetilde{v}^+ = (0.3051\ell_1, 0.6949\ell_2) = \widetilde{v}$ is the value of the game.

For a better understanding of the process of evaluating a proportional linguistic matrix game, one can refer to the Figure 3.1. In the subsequent section, we apply the proposed proportional linguistic game-theoretic approach in solving a linguistic MCDM problem to validate the proposed methodology in a physical realm.

3.3.3 Proportional Matrix Game for MCDM Problems

Decision-making is a necessary process to achieve the best possible solution for a problem as per the information provided by the DMs. For example, in an MCDM problem, there are different alternatives (A_i) which are to be chosen based on some criterion (C_j) using the previously defined LT set.

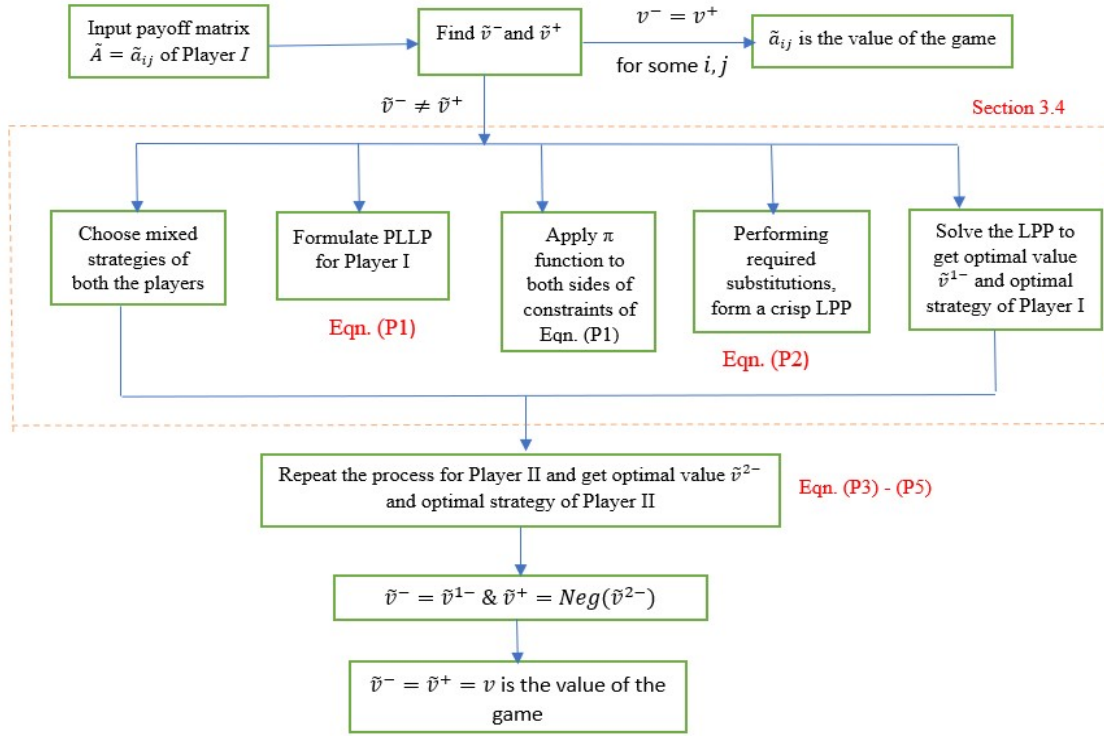


Figure 3.1: The methodology to evaluate Proportional Linguistic Matrix Game problem

As explained by Chen & Larbani in [49], the aim of the DM is to pick the best alternative from the available data in the matrix with fuzzy payoffs. The DM and the Nature act as two players of the game called player *I* and player *II*, respectively having strategies $\Phi = (\phi_1, \phi_2, \dots, \phi_r)$ and $\Psi = (\psi_1, \psi_2, \dots, \psi_s)$, where $\sum_{i=1}^r \phi_i = 1$, $\sum_{j=1}^s \psi_j = 1$ and $\phi_i, \psi_j \geq 0$. Hence, a two-player zero sum game is obtained in which $\tilde{a}_{ij}$ is treated as the payoff obtained by the DM after choosing the i^{th} strategy having probability ϕ_i and the Nature chooses the j^{th} strategy having probability ψ_j . This game problem is then solved using some models and the strategies of both the players are evaluated along with the value of the game.

In the sequence, Singh et al. [192] illustrated the applications of the linguistic matrix game approach in the MCDM problem. They aim to select the most appropriate alternative out of all based on some attributes. Analogously, we present the following example to solve an MCDM problem based on the proposed proportional linguistic matrix game approach.

Example 3.3.3. The pharmaceutical company Abbott is willing to understand the perspective of doctors in the “first-line treatment” of constipation. The study is conducted on different laxatives, namely

A_1 : Milk of Magnesia (MoM), A_2 : Lactulose, A_3 : Isphagula, A_4 : PEG

The decision of doctors is based on the following attributes:

C_1 : Safest Laxative, C_2 : Fast Acting, C_3 : Provides Overnight Relief, C_4 : Provides Long-lasting Relief, C_5 : Cost Effective, C_6 : Is a mild laxative, C_7 : Suitable for Long Term Usage, C_8 : Backed by Strong Clinical Evidence, C_9 : Non-habit forming

The doctors provide the response to each alternative based on given criteria using a predefined linguistic term set: $L = \{\ell_0 : \text{Not Effective}, \ell_1 : \text{Less Effective}, \ell_2 : \text{Average Effective}, \ell_3 : \text{Very Effective}, \ell_4 : \text{Most Effective}\}$. The following matrix having proportional linguistic payoffs is for the game problem between alternatives and criterion:

$$\tilde{\mathcal{A}} = \begin{matrix} & \begin{matrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \left[\begin{array}{ccccccccc} (\ell_0, 0) & (0.35\ell_0, 0.65\ell_1) & (0.71\ell_0, 0.29\ell_1) & (\ell_0, 0) & (0.76\ell_0, 0.24\ell_1) & (0.76\ell_0, 0.24\ell_1) & (\ell_0, 0) & (0.79\ell_0, 0.21\ell_1) & (\ell_0, 0) \\ (0.44\ell_1, 0.56\ell_2) & (0.61\ell_2, 0.39\ell_3) & (0.43\ell_1, 0.57\ell_2) & (0.35\ell_2, 0.65\ell_3) & (0.42\ell_1, 0.58\ell_2) & (0.77\ell_0, 0.23\ell_1) & (0.56\ell_1, 0.44\ell_2) & (0.66\ell_2, 0.34\ell_3) & (0.58\ell_1, 0.42\ell_2) \\ (0, \ell_4) & (\ell_0, 0) & (0.33\ell_0, 0.67\ell_1) & (0.56\ell_1, 0.44\ell_2) & (0.24\ell_2, 0.76\ell_3) & (0.22\ell_2, 0.78\ell_3) & (0.22\ell_2, 0.78\ell_3) & (0.54\ell_1, 0.46\ell_2) & (0.25\ell_2, 0.75\ell_3) \\ (\ell_0, 0) & (0.53\ell_1, 0.47\ell_2) & (0.37\ell_0, 0.63\ell_1) & (0.52\ell_1, 0.48\ell_2) & (0.8\ell_0, 0.2\ell_1) & (\ell_0, 0) & (\ell_0, 0) & (0.42\ell_1, 0.58\ell_2) & (\ell_0, 0) \end{array} \right] \end{matrix}$$

The payoffs are described so that the payoff $(0.35\ell_0, 0.65\ell_1)$ depicts that 35% of the doctors consider the laxative as non-effective while 65% doctors find it effective, but some less.

In accordance to (P2), the corresponding linear programming problem for alternatives is

$$\begin{aligned} \min \quad & V = X_1 + X_2 + X_3 + X_4 \\ \text{s.t.} \quad & \left\{ \begin{array}{l} 0X_1 + 1.56X_2 + 4X_3 + 0X_4 \geq 1 \\ 0.65X_1 + 2.39X_2 + 0X_3 + 1.47X_4 \geq 1 \\ 0.29X_1 + 1.57X_2 + 0.67X_3 + 0.63X_4 \geq 1 \\ 0X_1 + 2.65X_2 + 1.44X_3 + 1.48X_4 \geq 1 \\ 0.24X_1 + 1.58X_2 + 2.76X_3 + 0.2X_4 \geq 1 \\ 0.24X_1 + 0.23X_2 + 2.78X_3 + 0X_4 \geq 1 \\ 0X_1 + 1.44X_2 + 2.78X_3 + 0X_4 \geq 1 \\ 0.21X_1 + 2.34X_2 + 1.46X_3 + 1.58X_4 \geq 1 \\ 0X_1 + 1.42X_2 + 2.75X_3 + 0X_4 \geq 1 \\ X_1, X_2, X_3, X_4 \geq 0. \end{array} \right. \end{aligned}$$

The optimal solution of above LPP is $(0, 0.5, 0.32, 0)$, $V = 0.82$. By implementing the required computations, the optimal strategy of player I comes out to be $(\phi_1, \phi_2, \phi_3, \phi_4) = (0, 0.6095, 0.39008, 0) \approx (0, 0.6, 0.4, 0)$. Moreover, $\nu = 1.219$ and hence $\tilde{\nu}^- = (0.781\ell_1, 0.219\ell_2)$.

Similarly, for attributes, the LPP formulated using (P5) is

$$\begin{aligned} \max \quad & V = Y_1 + Y_2 + \dots + Y_9 \\ \text{s.t.} \quad & \begin{cases} 0Y_1 + 0.65Y_2 + 0.29Y_3 + 0Y_4 + 0.24Y_5 + 0.24Y_6 + 0Y_7 + 0.21Y_8 + 0Y_9 \leq 1 \\ 1.56Y_1 + 2.39Y_2 + 1.57Y_3 + 2.65Y_4 + 1.58Y_5 + 0.23Y_6 + 1.44Y_7 + 2.34Y_8 + 1.42Y_9 \leq 1 \\ 4Y_1 + 0Y_2 + 0.67Y_3 + 1.44Y_4 + 2.76Y_5 + 2.78Y_6 + 2.78Y_7 + 1.46Y_8 + 2.75Y_9 \leq 1 \\ 0Y_1 + 1.47Y_2 + 0.63Y_3 + 1.48Y_4 + 0.2Y_5 + 0Y_6 + 0Y_7 + 1.58Y_8 + 0Y_9 \leq 1 \\ Y_1, Y_2, \dots, Y_9 \geq 0. \end{cases} \end{aligned}$$

The solution obtained from this LPP is $(0, 0, 0.61, 0, 0, 0.21, 0, 0, 0)$, $V = 0.82$. The optimal strategy used by player *II* comes out to be $(\psi_1, \psi_2, \dots, \psi_9) = (0, 0, 0.7439, 0, 0, 0.2559, 0, 0, 0)$. Also, $v = 1.219$. So, $\tilde{v}^+ = (0.781\ell_1, 0.219\ell_2)$. Therefore,

$$\tilde{v}^- = \tilde{v}^+ = \tilde{v} = (0.781\ell_1, 0.219\ell_2)$$

is the value of the game.

This means that based on overnight relief and the power of laxative, 78.1% doctors find the laxative “Lactulose” followed by “Isphagula” as effective (but less) and only 21.9% doctors find it not effective out of all other laxatives in the first line treatment of constipation.

In the next section, we generalize the notion of matrix games having proportional linguistic payoffs and present the matrix games based on the information underlying an unbalanced LT set having distribution assessment payoffs. Unlike matrix games with proportional payoffs, these types of games are not restricted to the usage of just one or two terms while recording the expressions of DMs, rather facilitate the DMs to record their information in terms of whole linguistic set having proportions assigned to each linguistic term.

3.4 Linguistic Games with payoffs as Distributed Assessment Sets

This section discusses the matrix games consisting of the information from linguistic distribution assessment sets. The advantage of this method is that the

payoffs are a set that includes every single term of the original LT set with some proportions (weightage).

3.4.1 Terminologies and Definitions

Definition 3.4.1. Let us consider a two-player matrix game $\tilde{G}$ with payoffs as distributed assessment sets, where the payoff matrix of player I is given as $\tilde{A} = (\tilde{\mathbf{a}}_{ij})_{r \times s}$, whose elements $\tilde{\mathbf{a}}_{ij}$ are distributed assessment sets of a predefined LT set discussed in Definition 3.2.6. Moreover, considering only zero sum games, the payoff matrix of player II would be $\tilde{B} = \text{Neg}(\tilde{A})$, where the negation operator is same as defined earlier.

Definition 3.4.2. The amount that player I is assured to receive in the worst possible situation is

$$\tilde{v}^{1-} = \max_{i=1,2,\dots,r} \min_{j=1,2,\dots,s} \tilde{\mathbf{a}}_{ij}$$

Similarly, the amount that player II is assured to receive is

$$\tilde{v}^{2-} = \max_{j=1,2,\dots,s} \min_{i=1,2,\dots,r} \text{neg}(\tilde{\mathbf{a}}_{ij})$$

where each $\tilde{\mathbf{a}}_{ij}$ is a distribution assessment set of original LT set. The lower and upper values of the game are defined as $\tilde{v}^- = \tilde{v}^{1-}$ and $\tilde{v}^+ = \text{neg}(\tilde{v}^{2-})$, respectively. The game has value $\tilde{v}$ if $\tilde{v} = \tilde{v}^+ = \tilde{v}^-$. So the smallest maximum and the biggest minimum should be same, and the row and column i^*, j^* yielding $\tilde{\mathbf{a}}_{i^*j^*} (= \tilde{v}^+ = \tilde{v}^-)$ are optimal (a saddle point in case of pure strategies).

$$\begin{aligned} \tilde{v}^+ = \text{neg}(\tilde{v}^{2-}) &= \text{neg}\left(\max_{j=1,2,\dots,s} \min_{i=1,2,\dots,r} \text{neg}(\tilde{\mathbf{a}}_{ij})\right) \\ &= \min_{j=1,2,\dots,s} \max_{i=1,2,\dots,r} (\tilde{\mathbf{a}}_{ij}) \end{aligned}$$

Let us try to look at this with the help of an example.

Note 3.4.1. The Example 3.4.1 and Example 3.4.2 consider game problems in which the payoffs are of the form of linguistic distribution assessment sets. These distribution assessments are taken from Example 1 of Zhang et al. [262] stated as:

Let $LS = \{\ell_{-2}: \text{very poor}, \ell_{-1}: \text{poor}, \ell_0: \text{average}, \ell_1: \text{good}, \ell_2: \text{very good}\}$ be the pre-defined LT set. Suppose that m_1, m_2, m_3, m_4, m_5 represents a set of distribution assessments of

the set LS given as

$$\begin{aligned}
m_1 &= \{(\ell_{-2}, 0.2), (\ell_{-1}, 0.2), (\ell_0, 0.2), (\ell_1, 0.2), (\ell_2, 0.2)\} \\
m_2 &= \{(\ell_{-2}, 0), (\ell_{-1}, 0.3), (\ell_0, 0.7), (\ell_1, 0), (\ell_2, 0)\} \\
m_3 &= \{(\ell_{-2}, 0), (\ell_{-1}, 0), (\ell_0, 0.8), (\ell_1, 0.2), (\ell_2, 0)\} \\
m_4 &= \{(\ell_{-2}, 0.4), (\ell_{-1}, 0.4), (\ell_0, 0.2), (\ell_1, 0), (\ell_2, 0)\} \\
m_5 &= \{(\ell_{-2}, 0), (\ell_{-1}, 0), (\ell_0, 0.2), (\ell_1, 0.2), (\ell_2, 0.6)\}
\end{aligned}$$

The payoff $\tilde{a}_{12} = m_3$ demonstrates that if P_1 invests his money in F_1 and if P_2 invests her money in F_2 , then P_1 gets a payoff which is 80% Average and 20% Good (since the proportions for other linguistic terms are zero in m_3). Next,

$$\begin{aligned}
E(m_1) &= \sum_{t=-2}^2 \beta_t \ell_t \\
&= \beta_{-2} \ell_{-2} + \beta_{-1} \ell_{-1} + \beta_0 \ell_0 + \beta_1 \ell_1 + \beta_2 \ell_2 \\
&= 0.2 \times (\ell_{-2}) + 0.2 \times (\ell_{-1}) + 0.2 \times (\ell_0) + 0.2 \times (\ell_1) + 0.2 \times (\ell_2) \\
&= \ell_{-0.4-0.2+0+0.2+0.4} \text{ (using operational laws defined by Xu [244] } \mu \ell_t = \ell_{t\mu}) \\
&= \ell_0 \text{ (or precisely 0 for simplicity)}
\end{aligned}$$

Similarly,

$$E(m_2) = 0 \times \ell_{-2} + 0.3 \times \ell_{-1} + 0.7 \ell_0 + 0 \times \ell_1 + 0 \times \ell_2 = \ell_{-0.3} \text{ (or 0.3)}$$

In the similar manner, $E(m_3) = \ell_{0.2}$ (or 0.2), $E(m_4) = \ell_{-0.2}$ (or -0.2), $E(m_5) = \ell_{1.4}$ (or 1.4). As $E(m_5) > E(m_3) > E(m_1) > E(m_4) > E(m_2)$, this implies that $m_5 > m_3 > m_1 > m_4 > m_2$ (By Definition 3.2.8).

Now, we proceed towards using the given distribution assessment sets in game problems.

Example 3.4.1. Two partners P_1 and P_2 have two choices for where to invest their money, say F_1 and F_2 , where F stands for firm. Based on their choices of investing the money, we model the payoff matrix for P_1 corresponding to P_2 as:

$$\tilde{A} = \begin{matrix} & \begin{matrix} F_1 & F_2 \end{matrix} \\ \begin{matrix} F_1 \\ F_2 \end{matrix} & \begin{bmatrix} m_1 & m_3 \\ m_2 & m_1 \end{bmatrix} \end{matrix}$$

where the distribution assessments $m_i, i = 1, 2, 3$ are as defined above. The guaranteed amount of P_1 is:

$$\begin{aligned}\tilde{v}^{1-} &= \max_{i=1,2} \min_{j=1,2} (\tilde{a}_{ij}) = \max \{ \min(m_1, m_3), \min(m_2, m_1) \} \\ &= \max \{ m_1, m_2 \} \\ &= m_1\end{aligned}$$

$\therefore P_1$ receives a payoff which has equal share of 20% of all linguistic terms if he invests in firm F_1 .

Similarly, the payoff matrix for P_2 in correspondence to P_1 is given by:

$$Neg(\tilde{A}) = \begin{matrix} & \begin{matrix} F_1 & F_2 \end{matrix} \\ \begin{matrix} F_1 \\ F_2 \end{matrix} & \begin{bmatrix} Neg(m_1) & Neg(m_3) \\ Neg(m_2) & Neg(m_1) \end{bmatrix} \end{matrix}$$

The negation of distribution assessment sets m_1, m_2 and m_3 using Definition 3.2.9 are:

$$Neg(m_1) = \{(\ell_{-2}, 0.2), (\ell_{-1}, 0.2), (\ell_0, 0.2), (\ell_1, 0.2), (\ell_2, 0.2)\}$$

$$Neg(m_2) = \{(\ell_{-2}, 0), (\ell_{-1}, 0), (\ell_0, 0.7), (\ell_1, 0.3), (\ell_2, 0)\}$$

$$Neg(m_3) = \{(\ell_{-2}, 0), (\ell_{-1}, 0.2), (\ell_0, 0.8), (\ell_1, 0), (\ell_2, 0)\}$$

So by using Comparison Operator(Definition 3.2.8), we can deduce that $Neg(m_3) < Neg(m_1) < Neg(m_2)$.

Hence the assured amount received by P_2 is:

$$\begin{aligned}\tilde{v}^{2-} &= \max_{i=1,2} \min_{j=1,2} (Neg \tilde{a}_{ij}) = \max \{ Neg(m_3), Neg(m_1) \} \\ &= Neg(m_1)\end{aligned}$$

The lower value of the game is

$$\tilde{v}^- = \tilde{v}^{1-} = m_1 = \{(\ell_{-2}, 0.2), (\ell_{-1}, 0.2), (\ell_0, 0.2), (\ell_1, 0.2), (\ell_2, 0.2)\}$$

and the upper value is

$$\tilde{v}^+ = neg(\tilde{v}^{2-}) = Neg(m_1) = m_1$$

$$\therefore \tilde{v}^- = \tilde{v}^+$$

$\implies (1, 1)$ is the saddle point and $\{(\ell_{-2}, 0.2), (\ell_{-1}, 0.2), (\ell_0, 0.2), (\ell_1, 0.2), (\ell_2, 0.2)\}$ is the value of the game, i.e the players have equal chances of getting very poor, poor,

average, good and very good payoff if they both invest in firm F_1 .

Now the problem arises when $\tilde{v}^-$ is not equal to $\tilde{v}^+$. If the optimal pure strategies do not exist, then we have to mix up the choices. According to Neumann & Morgenstern [171], if mixed strategies are applied in a game, then the game would always retain a value and optimal strategies. In the case where the players apply mixed strategies, then their payoffs may be calculated by calculating what is “expected” by the player to receive. So next we define the linguistic distributed expected payoff received by the player.

Definition 3.4.3. If $\Phi = (\phi_1, \phi_2, \dots, \phi_r) \in S^r$ and $\Psi = (\psi_1, \psi_2, \dots, \psi_s) \in S^s$ are the mixed strategies corresponding to player I and player II respectively, and $\tilde{\mathbf{a}}_{ij} = m_k$ (distribution assessment sets), $k = 1, 2, \dots, r$ be the distributed payoffs of player I , then the linguistic distribution expected payoff received by player I is given as

$$\tilde{E}_{\tilde{\mathbf{A}}}(\Phi, \Psi) = \bigoplus_{i=1}^r (\phi_i (\bigoplus_{j=1}^s \tilde{\mathbf{a}}_{ij} \psi_j)). \quad (3.4.1)$$

and in similar manner, the linguistic distribution expected payoff received by player II is

$$\tilde{E}_{neg\tilde{\mathbf{A}}}(\Phi, \Psi) = \bigoplus_{i=1}^r [\phi_i (\bigoplus_{j=1}^s neg(\tilde{\mathbf{a}}_{ij}) \psi_j)]. \quad (3.4.2)$$

So now, we introduce an LDLP approach for solving matrix games with distribution assessments as payoffs.

3.4.2 Linguistic Distribution Linear Programming method for Matrix Games with Distribution Assessments as payoffs

In the following section, we formulate the LDLP method to solve matrix games (involving games using mixed strategies) by using the expected payoff to evaluate the players' payoffs.

Let $LS = \{\ell_{-t}, \dots, \ell_t\}$ be the linguistic pre-established LT set and $m = \{(\ell_k, \beta_k) | k = -t, \dots, t\}$ be the distribution assessment set. Consider the linguistic distribution matrix $\tilde{\mathbf{A}} = (\tilde{\mathbf{a}}_{ij})_{r \times s}$, where each $\tilde{\mathbf{a}}_{ij}$ represents the payoff to player I received after opting i^{th} strategy while player II opts for j^{th} strategy. Let $\Phi = (\phi_1, \phi_2, \dots, \phi_r) \in S^r$ and $\Psi = (\psi_1, \psi_2, \dots, \psi_s) \in S^s$ be the mixed strategies of player I & player II , respectively. The “expected payoff” of player I corresponding to the j^{th} strategy

chosen by player II is given by $\tilde{\mathbf{a}}_{1j}\phi_1 \oplus \tilde{\mathbf{a}}_{2j}\phi_2 \oplus \dots \oplus \tilde{\mathbf{a}}_{rj}\phi_r$. We need to evaluate the mixed strategy $\Phi \in S^r$ of player I such that the value of player I , i.e., $\tilde{v}^{1-}$ is maximized. So, the required Linguistic Distribution Linear Programming for player I is:

$$\max \quad \tilde{v}^{1-} \quad (\text{P6})$$

subject to

$$\tilde{\mathbf{a}}_{11}\phi_1 \oplus \tilde{\mathbf{a}}_{21}\phi_2 \oplus \dots \oplus \tilde{\mathbf{a}}_{r1}\phi_r \geq \tilde{v}^-$$

$$\tilde{\mathbf{a}}_{12}\phi_1 \oplus \tilde{\mathbf{a}}_{22}\phi_2 \oplus \dots \oplus \tilde{\mathbf{a}}_{r2}\phi_r \geq \tilde{v}^-$$

$\vdots$

$$\tilde{\mathbf{a}}_{1s}\phi_1 \oplus \tilde{\mathbf{a}}_{2s}\phi_2 \oplus \dots \oplus \tilde{\mathbf{a}}_{rs}\phi_r \geq \tilde{v}^-$$

$$\phi_1 + \phi_2 + \dots + \phi_r = 1, \phi_1, \phi_2, \dots, \phi_r \geq 0.$$

where each $\tilde{\mathbf{a}}_{ij}$ is linguistic distribution assessment set.

As the expectation E is monotonically increasing function, apply E to each constraint of (P6) and the constraints take the form:

$$E(\tilde{\mathbf{a}}_{11})\phi_1 + E(\tilde{\mathbf{a}}_{21})\phi_2 + \dots + E(\tilde{\mathbf{a}}_{r1})\phi_r \geq E(\tilde{v}^-)$$

$$E(\tilde{\mathbf{a}}_{12})\phi_1 + E(\tilde{\mathbf{a}}_{22})\phi_2 + \dots + E(\tilde{\mathbf{a}}_{r2})\phi_r \geq E(\tilde{v}^-)$$

$\vdots$

$$E(\tilde{\mathbf{a}}_{1s})\phi_1 + E(\tilde{\mathbf{a}}_{2s})\phi_2 + \dots + E(\tilde{\mathbf{a}}_{rs})\phi_r \geq E(\tilde{v}^-)$$

$$\phi_1 + \phi_2 + \dots + \phi_r = 1, \phi_1, \phi_2, \dots, \phi_r \geq 0.$$

$$\text{Put } E(\tilde{\mathbf{a}}_{ij}) = \mathbf{a}_{ij} \text{ and } E(\tilde{v}^{1-}) = v.$$

$\Rightarrow$ constraints of (P6) become

$$\mathbf{a}_{11}\phi_1 + \mathbf{a}_{21}\phi_2 + \dots + \mathbf{a}_{r1}\phi_r \geq v$$

$$\mathbf{a}_{12}\phi_1 + \mathbf{a}_{22}\phi_2 + \dots + \mathbf{a}_{r2}\phi_r \geq v$$

$\vdots$

$$\mathbf{a}_{1s}\phi_1 + \mathbf{a}_{2s}\phi_2 + \dots + \mathbf{a}_{rs}\phi_r \geq v$$

$$\phi_1 + \phi_2 + \dots + \phi_r = 1$$

$$\phi_1, \phi_2, \dots, \phi_r \geq 0.$$

Let $X_i = \frac{\phi_i}{v}; i = 1, 2, \dots, r$ and $\frac{1}{v} = V$. Then the problem reduces to

$$\min \quad V = X_1 + X_2 + \dots + X_r \quad (\text{P7})$$

subject to

$$\mathbf{a}_{11}X_1 + \mathbf{a}_{21}X_2 + \dots + \mathbf{a}_{r1}X_r \geq 1$$

$$\mathbf{a}_{12}X_1 + \mathbf{a}_{22}X_2 + \dots + \mathbf{a}_{r2}X_r \geq 1$$

$\vdots$

$$\mathbf{a}_{1s}X_1 + \mathbf{a}_{2s}X_2 + \dots + \mathbf{a}_{rs}X_r \geq 1$$

$$X_1, X_2, \dots, X_r \geq 0.$$

and then we may solve (P7) as usual linear programming problem.

Similarly, linguistic distribution linear programming for player *II* would be

$$\max \quad \widetilde{v}^{2-} \quad (\text{P8})$$

subject to

$$\text{neg}(\tilde{\mathbf{a}}_{11})\psi_1 \oplus \text{neg}(\tilde{\mathbf{a}}_{12})\psi_2 \oplus \dots \oplus \text{neg}(\tilde{\mathbf{a}}_{1s})\psi_s \geq \widetilde{v}^{2-}$$

$$\text{neg}(\tilde{\mathbf{a}}_{21})\psi_1 \oplus \text{neg}(\tilde{\mathbf{a}}_{22})\psi_2 \oplus \dots \oplus \text{neg}(\tilde{\mathbf{a}}_{2s})\psi_s \geq \widetilde{v}^{2-}$$

$\vdots$

$$\text{neg}(\tilde{\mathbf{a}}_{r1})\psi_1 \oplus \text{neg}(\tilde{\mathbf{a}}_{r2})\psi_2 \oplus \dots \oplus \text{neg}(\tilde{\mathbf{a}}_{rs})\psi_s \geq \widetilde{v}^{2-}$$

$$\psi_1 + \psi_2 + \dots + \psi_s = 1, \psi_1, \psi_2, \dots, \psi_s \geq 0.$$

We have to find the mixed strategy $\Psi \in S^s$ for player *II*.

Let $\text{neg}(\widetilde{v}^{2-}) = \widetilde{v}^+$. Thus, from the properties of negation operator, the problem becomes

$$\min \quad \widetilde{v}^+ \quad (\text{P9})$$

subject to

$$\tilde{\mathbf{a}}_{11}\psi_1 \oplus \tilde{\mathbf{a}}_{12}\psi_2 \oplus \dots \oplus \tilde{\mathbf{a}}_{1s}\psi_s \leq \widetilde{v}^+$$

$$\tilde{\mathbf{a}}_{21}\psi_1 \oplus \tilde{\mathbf{a}}_{22}\psi_2 \oplus \dots \oplus \tilde{\mathbf{a}}_{2s}\psi_s \leq \widetilde{v}^+$$

$\vdots$

$$\tilde{\mathbf{a}}_{r1}\psi_1 \oplus \tilde{\mathbf{a}}_{r2}\psi_2 \oplus \dots \oplus \tilde{\mathbf{a}}_{rs}\psi_s \leq \widetilde{v}^+$$

$$\psi_1 + \psi_2 + \dots + \psi_s = 1, \psi_1, \psi_2, \dots, \psi_s \geq 0.$$

Now we apply E operator in (P9). Since earlier $E(\tilde{v}^{1-}) = v$ and if v is the value of the game then $E(\tilde{v}^+) = v$. Put $Y_j = \frac{\psi_j}{v}; j = 1, 2, \dots, s$ and $E(\tilde{a}_{ij}) = a_{ij}$, the problem for player II reduces to the problem

$$\max \quad V = Y_1 + Y_2 + \dots + Y_s \quad (\text{P10})$$

subject to

$$\mathbf{a}_{11}Y_1 + \mathbf{a}_{12}Y_2 + \dots + \mathbf{a}_{1s}Y_s \leq 1$$

$$\mathbf{a}_{21}Y_1 + \mathbf{a}_{22}Y_2 + \dots + \mathbf{a}_{2s}Y_s \leq 1$$

$\vdots$

$$\mathbf{a}_{r1}Y_1 + \mathbf{a}_{r2}Y_2 + \dots + \mathbf{a}_{rs}Y_s \leq 1$$

$$Y_1, Y_2, \dots, Y_s \geq 0.$$

So, by solving (P7) and (P10), we get the optimal solution $X = (X_1, X_2, \dots, X_r)$ and $Y = (Y_1, Y_2, \dots, Y_s)$ and the optimal value V . Hence the optimal strategies Φ and Ψ of player I and player II respectively, could be achieved and the value v of the game follows. Figure 3.2 presents a better understanding of the methodology.

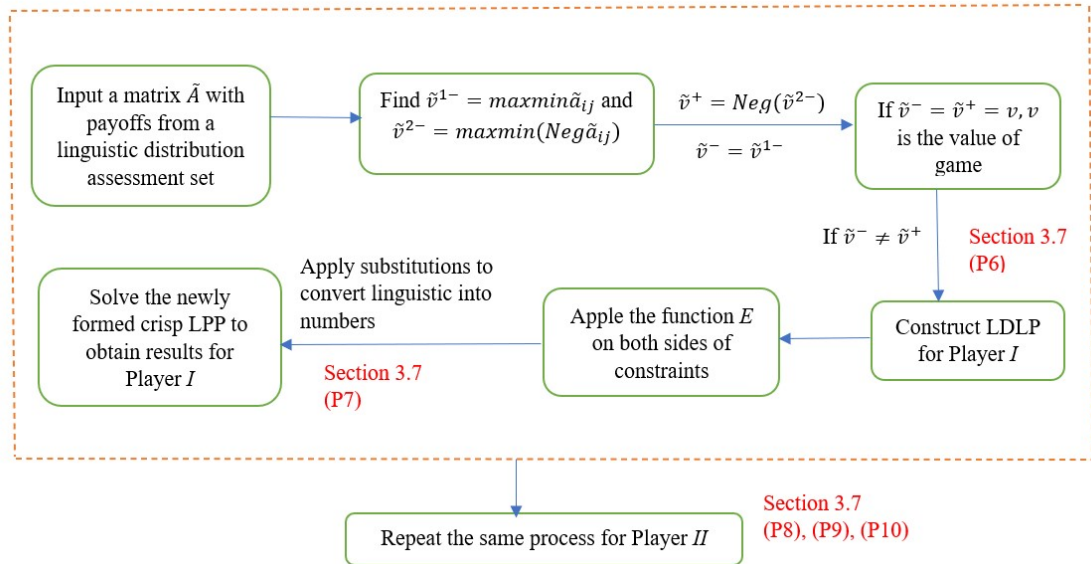


Figure 3.2: Brief methodology for solving games with distributed assessment payoffs

Note 3.4.2. As E is not a bijective function, it is not possible to always re-translate the game value into its corresponding linguistic distribution assessment. So we consider the value of the game as a number only. However, we can deduce from this approach what strategy is the best for a player to win the game.

Example 3.4.2. Consider a matrix with payoffs as distribution assessments m_i defined in Section 3.4.1

$$\tilde{A} = \begin{bmatrix} m_3 & m_5 & m_2 \\ m_1 & m_2 & m_4 \\ m_5 & m_4 & m_5 \end{bmatrix}$$

Here, $\tilde{v}^{1-} = m_4$ and $\tilde{v}^+ = m_3$. So we proceed by mixing up the choices. Let $\Phi = (\phi_1, \phi_2, \phi_3), \phi_i \geq 0, i = 1, 2, 3$ such that $\sum_{i=1}^3 \phi_i = 1$ be the mixed strategy for player I , and $\Psi = (\psi_1, \psi_2, \psi_3), \psi_j \geq 0, j = 1, 2, 3$ and $\sum_{j=1}^3 \psi_j = 1$ be mixed strategy for player II .

We formulate the LDLP for player I as in (P6) in following manner:

$$\begin{aligned} & \max \quad \tilde{v}^- \\ & \text{subject to} \\ & m_3\phi_1 \oplus m_1\phi_2 \oplus m_5\phi_3 \geq \tilde{v}^- \\ & m_5\phi_1 \oplus m_2\phi_2 \oplus m_4\phi_3 \geq \tilde{v}^- \\ & m_2\phi_1 \oplus m_4\phi_2 \oplus m_5\phi_3 \geq \tilde{v}^- \\ & \phi_1 + \phi_2 + \phi_3 = 1, \phi_1, \phi_2, \phi_3 \geq 0. \end{aligned}$$

Applying E function in the constraints, we get

$$\begin{aligned} & 0.2\phi_1 + 0\phi_2 + 1.4\phi_3 \geq E(\tilde{v}^-) \\ & 1.4\phi_1 - 0.3\phi_2 - 0.2\phi_3 \geq E(\tilde{v}^-) \\ & -0.3\phi_1 - 0.2\phi_2 + 1.4\phi_3 \geq E(\tilde{v}^-) \\ & \phi_1 + \phi_2 + \phi_3 = 1, \phi_1, \phi_2, \phi_3 \geq 0. \end{aligned}$$

Substituting $E(\tilde{v}^-) = v$, $X_i = \frac{\phi_i}{v}$ and $V = \frac{1}{v}$, the problem reduces similar to (P7) as

$$\begin{aligned} & \min \quad V = X_1 + X_2 + X_3 \\ & \text{subject to} \\ & 0.2X_1 + 0X_2 + 1.4X_3 \geq 1 \\ & 1.4X_1 - 0.3X_2 - 0.2X_3 \geq 1 \\ & -0.3X_1 - 0.2X_2 + 1.4X_3 \geq 1 \\ & X_1, X_2, X_3 \geq 0. \end{aligned}$$

The optimal solution of this problem is $(0.84, 0, 0.89)$ and $V = 1.74$. Therefore, $v = 0.5747$ (approx.) Hence the optimal strategies of player I are $(0.4827, 0, 0.51149)$. This means that player I uses row 1, 48.27% and primarily, row 3, 51.14% times to win the game.

Similarly, the LDLP for player II as in (P9) is

$$\begin{aligned}
 & \min \quad \widetilde{v}^+ \\
 & \text{subject to} \\
 & m_3\psi_1 \oplus m_5\psi_2 \oplus m_2\psi_3 \leq \widetilde{v}^+ \\
 & m_1\psi_1 \oplus m_2\psi_2 \oplus m_4\psi_3 \leq \widetilde{v}^+ \\
 & m_5\psi_1 \oplus m_4\psi_2 \oplus m_5\psi_3 \leq \widetilde{v}^+ \\
 & \psi_1 + \psi_2 + \psi_3 = 1 \\
 & \psi_1, \psi_2, \psi_3 \geq 0.
 \end{aligned}$$

By substitution as given in (P10), the problem becomes

$$\begin{aligned}
 & \max \quad V = Y_1 + Y_2 + Y_3 \\
 & \text{subject to} \\
 & 0.2Y_1 + 1.4Y_2 - 0.3Y_3 \leq 1 \\
 & 0Y_1 - 0.3Y_2 - 0.2Y_3 \leq 1 \\
 & 1.4Y_1 - 0.2Y_2 + 1.4Y_3 \leq 1 \\
 & Y_1, Y_2, Y_3 \geq 0.
 \end{aligned}$$

The optimal solution for this LPP is $(0, 0.89, 0.84)$ and $V = 1.74$

$\implies v = 0.5747$ (approx.).

Hence, the optimal strategies for player II are $(0, 0.511483, 0.4827)$.

This implies that column 2 is used 51.14% times, and column 3 is used 48.27% times by player II to win the game. Moreover, since $v = 0.5747$ for both the player, v is the value of the game.

3.5 Analysis of the Work Done

This section presents the analysis of the work done in the chapter. Subsection 3.5.1 offers the analysis of the results obtained in the previous sections, and

Subsection 3.5.2 compares the defined methodology with the existing methodologies. In Subsection 3.5.3, we discuss the computation complexity of the proposed methodology.

3.5.1 Result Analysis

In this subsection, we discuss the results of the game's value and the players' strategies obtained in earlier sections.

Lemma 3.3.1 in Section 3.3 describes the case of having a saddle point if the players in the game apply the pure strategies. Example 3.3.1 depicts the existence of saddle point in case pure strategies are played as $\tilde{v}^-$ comes out to be equal to $\tilde{v}^+$. Hence, the value of the game may be obtained as $\tilde{v} = \tilde{v}^- = \tilde{v}^+$, and the row (column) for which the value of the game is obtained is called the optimal pure strategy of player *I* (player *II*).

In case the pure strategies cannot be applied, the players use mixed strategies to evaluate the value of the game. This occurs when $\tilde{v}^- \neq \tilde{v}^+$, which is depicted in Example 3.3.2 of Section 3.3.2. For such a case, a PLLP is formulated for player *I* in equation (P1), which consists of proportional linguistic payoffs. These proportional linguistic payoffs maybe then converted into numerical terms by doing required substitutions. By doing so, the original PLLP may be converted into a usual crisp LPP that can be solved directly. The solution of the LPP maybe then re-converted into linguistic form by reverting the substitutions. This gives the values of both the players and the value of the game, and the players' strategies may be found. The same process is then repeated for player *II*. Equal values of both the players provide the value of the game.

In Section 3.4, matrix games with payoffs as distributed assessment sets that have pure strategies have been discussed. In Example 3.4.1, the matrix $\tilde{a}$ exhibits optimal pure strategies as $\tilde{v}^-$ comes out to be equal to $\tilde{v}^+$. So the value of the game $\tilde{v}$ is obtained as $\tilde{v}^- = \tilde{v}^+ = \tilde{v}$. Also, the rows and columns i, j for which $\tilde{v}^- = \tilde{v}^+$ are called the optimal strategies of player *I* corresponding to player *II*. Further, the optimal strategies of player *II* may also be obtained by repeating the same process for the matrix $Neg(\tilde{a})$ instead of $\tilde{a}$.

Example 3.4.2 in Section 3.4.2 depicts the case when the pure strategies cannot be applied, i.e., when $\tilde{v}^- \neq \tilde{v}^+$. In this case of mixed strategies, the LDLP is formed

for both players. This LDLP consists of linguistically distributed payoffs, which are then converted into numeric terms by doing required substitutions. This converts the aforementioned LDLP into a crisp LPP, which can be solved directly. The solution of this LPP provides the best strategies to be played by both players in order to win the game.

3.5.2 Comparative Analysis

This subsection compares the proposed study with the existing theories of matrix games having payoffs from an uncertain domain.

Many theories deal with matrix games having fuzzy payoffs, e.g., [20–22, 41, 65, 212, 260]. These theories discuss matrix games whose payoffs belong to a fuzzy set. Nevertheless, these theories have an associated limitation in that the payoffs may take the form of linguistic variables as well. Hence, matrix games are important to study in the realm of LT sets also. Researchers have worked in the field of linguistic matrix games and have found significant results [4, 5, 185]. However, in these methods, the linguistic payoffs are defuzzified to convert them into numeric LPPs. Defuzzification of payoffs causes the loss of information. The loss of information of payoffs is handled well by Singh & Gupta [192, 193] and Malhotra & Gupta [151] where the payoffs belong to a balanced LT set, and these payoffs are converted into crisp values using a bijective function. However, these theories are limited to a balanced LT set only. The theory defined in this chapter is believed to be the first study considering matrix games having payoffs from an unbalanced LT set. Moreover, the payoffs of balanced LT sets may also be converted to their corresponding unbalanced ones by using the inverse of the h function defined in Definition 3.2.5. Note that this can only be done for a particular h because h is different for each cell. Hence the matrix games with 2-TL payoffs as defined by Singh & Gupta [192] may also be solved by considering it as a particular case of the methodology defined in this chapter.

3.5.3 Computational Complexity

This subsection discusses the computational complexity involved in the defined methodology of unbalanced linguistic matrix games.

If we consider a game problem as a decision-making problem, then rather than

picking random weights, the weights in this method are calculated within the game problem itself (considering that the mixed strategies play the role of weights). This makes the computation of this approach more appropriate than the existing decision-making methodologies. Moreover, the PLLP and LDLP of Players I and II are first converted into usual crisp LPPs that are easy and fast to compute (manually or by using any optimization software). The result is then re-translated into the linguistic one using the bijective function. Thus, the defined methodology considers the complexity of solving game problems and computes them without making any “loss of information”.

3.6 Concluding Remarks

The payoffs based on symmetrical linguistic term sets have been successfully applied in many game problems to date. However, from a more realistic viewpoint, the payoffs of a matrix game may not always belong to an LT set where the terms are equidistant. Overcoming this limitation, we consider matrix games having payoffs belonging to an asymmetrical LT set in this chapter. In the first segment, these payoffs are expressed in ordinal linguistic information via 2-tuples, comprising two proportional linguistic terms with coefficients in $[0, 1]$. The second segment consists of payoffs comprising all the linguistic terms of the LT set with proportions assigned to all of them. This representation of payoffs is more advantageous as it assesses the payoffs consisting of linguistic terms that belong to a continuous scale. To compute such game problems, first, the case of pure strategies is discussed, where the existence of saddle points is shown. Further, when pure strategies cannot be applied, the application of mixed strategies is discussed. For this, a Proportional Linguistic Linear Programming (PLLP) (for games with proportional payoffs) and Linguistic Distributed Linear Programming (LDLP) (for games with distribution assessment payoffs) approach is introduced to obtain the optimal strategies for players I and II . The value of such a game, where the conventional requirement of equidistant linguistic terms may not be required, is also discussed. These approaches are then illustrated with the help of some examples. The suggested approach provides an integrated structure as it can be accepted to solve unbalanced linguistic matrix games and MCDM problems.

Although our approach is advantageous in solving matrix games where the data

is asymmetrical or unbalanced, giving rise to unbalanced linguistic matrix games, it has a limitation associated with it. Concerning the proportional linguistic term set, the symbolic proportions are allocated only to two consecutive terms of a linguistic term set. The distribution assessment set took care of this limitation. However, recommendations for future work include the development of a bijective function in place of the existing function ' E ' so that the value of the game may be obtained in terms of the linguistic distribution assessment set only. In the future, we look forward to extending our methodology to solve unbalanced linguistic matrix games overcoming the current limitation and enhancing the applicability of linguistic matrix games in this ambiguous world.

Chapter 4

A Novel Cloud Model based on a Multiplicative Unbalanced Linguistic Term Set

The cloud decision support model is an effective technique for describing an ambiguous mapping between qualitative notions and their quantitative manifestations. However, the existing research on cloud models is based on a balanced linguistic term set and cannot address information in an unbalanced linguistic form. In such sets, the basic cognitive units, i.e., the linguistic terms, are equidistant from each other. Nevertheless, real-world problems are not restricted to symmetric linguistic sets only. Henceforth, an accurate decision model is required to acquire a good solution to thoroughly capture the intrinsic fuzziness and vagueness of decision information. In this direction, the proposed work in this chapter¹ aims to introduce a cloud model based on multiplicative unbalanced linguistic term sets. To match our research requirements, we initially propose a multiplicative unbalanced point-based cloud model. The innovation of the model overcomes the inherent limitation of the existing cloud and is competent in producing new solutions according to the cloud model theory. This cloud model can synthetically capture the vagueness and fuzziness of information based on predefined unbalanced linguistic term sets. Furthermore, a generalized multiplicative unbalanced linguistic

¹The work presented in this chapter is based on the research paper, “A Novel Cloud Model based on a Multiplicative Unbalanced Linguistic Term Set”, *The Journal of Supercomputing*, Springer, 79, 16378–16408 (2023). <https://doi.org/10.1007/s11227-023-05295-z>. (SCIE, Impact Factor : 3.3)

scale function (MULSF) is introduced. The proposed generalized function creates a map between the linguistic data and numerical interval values. Based on this rational conversion method, interval cloud variables are transformed, which overcomes the loss of information existing in the point-based cloud models where linguistic variables are transformed into a specific numerical value. The process is then used to demonstrate the accuracy and efficacy of the suggested model using a numerical example, which is then followed by a comparison study. The overall results typically show that the cloud model has excellent performance and provides better solutions.

4.1 Introduction

Several decision analysis problems under uncertain conditions encounter the challenge of ultimately representing the opinions and preferences of DMs. Given the vague information incorporated within those problems or the high computational cost required to express precise numerical data, the DMs do not describe their evaluations in quantitative descriptors but rather in linguistic form. The requirement for linguistic data is excellently suited because it is simple to distinguish uncertainty with a non-probabilistic nature. As a result, numerous researchers have researched it [155,156]. As the basis of human thought and the embodiment of human intellect, natural languages are crucial to understanding difficulties. In the present specialized literature on linguistic decision analysis and CW-related approaches, the utility of linguistic ideas [257,258] implies a broad outlook.

Nevertheless, as the complications within the practical decision-making issues increase, it is arduous to reflect the qualitative information in a balanced linguistic term set. Taking investment returns as an example where investors care about the distribution of investment return data. Empirically, the investment return data follows an asymmetric distribution scheme. This is because investment performance is usually skewed by periods of high market volatility or unusual fiscal and monetary policies during which returns growth are abnormally high or low. Considering the irregularity of the data, it is envisioned that the existing linguistic models are not capable enough to represent asymmetric information effectively. To resolve this issue, a series of research is conducted, for instance, [1,18,38,39,67,76,97,221,262,269] to proficiently describe qualitative concept having an irregular spacing of linguistic labels. Henceforth, the developed content related to unbalanced linguistic information has made a significant impact on linguistic decision-making theories. As a popular tool for modeling qualitative assessment information, the theories [135–141] allow the DMs to work under uncertain linguistic environments. The use of uncertain linguistic information enriches the flexibility of DMs in decision-making [126,130,197,200] scenarios. Considering the existing literature on unbalanced LCMs, it is identified that the representation of unbalanced information is presented in terms of a uniformly distributed set of linguistic terms. A new consensus-reaching approach for MCGDM with multi-granular unbalanced linguistic information, for instance,

is developed by Zhang et al. [265] that simultaneously considers the adjustment amount and limited confidence levels of DMs. To do this, the authors first use the model provided in [67] to determine the numerical scales of multi-granular unbalanced linguistic terms. Then they convert these terms into linguistic 2-tuples defined on the same balanced linguistic term set. A clustering approach based on LDAs is subsequently developed to categorize DMs in the paper [266], which converts all multi-granular unbalanced hesitant fuzzy linguistic data into LDAs defined on a balanced linguistic word set. The results are then used to compute the overall ratings for each cluster. In this way, the accurate sense of “unbalanced” is not well represented. To remedy this, Malhotra & Gupta [153] introduced a novel 2-tuple multiplicative unbalanced linguistic computational model addressing a substantial amount of linguistic information in terms of a globally inconsistent multiplicative unbalanced linguistic scale set. This set is primarily based on the notion of minimum distance measure. It is anticipated that under complicated decision-making circumstances, the proposed model is computationally less expensive and reduces time complexity.

Addressing decision-making issues with linguistic information to apprehend the uncertainty has received much interest. However, with rapid societal development, linguistic decision-making problems have become increasingly complex and challenging. In such a scenario, the cognitive cloud models proposed by Li et al. [111, 124, 125], are lucrative for defining linguistic labels as they can synthetically explain the notion of randomness and fuzziness of the decision information simultaneously. Since the introduction of cloud models, several researchers have devoted their efforts to enhance cloud specialized studies such as integrated cloud [123], risk assessment [198, 199], water quality assessment [214] along with more wide-ranging applications [127, 132, 144, 160, 161, 222, 227, 232]. The experts introduce some methods for applying cloud models to MCDM [101, 145, 146, 220, 223, 226]. The cloud models outline a linguistic concept using three numerical properties, namely *EX* (expectation), *EN* (entropy), and *HE* (hyper entropy). To realise the bidirectional cognitive transformation of a notion [142, 216, 217], two cloud transformations, forward cloud transformation and backward cloud transformation, are used. In addition to propose a 2nd-order generic forward normal cloud transformation technique, Wang et al. [215] provided a generic cloud model that established a connection between the normal cloud and the normal distribution.

This transformation algorithm reveals that the 2nd- order normal cloud [218] and the normal distribution are both special cases of the 2nd- order generic normal cloud. A linguistic assessment scale is defined by Wang et al. [223] to perform the transformation among the clouds and the linguistic variables. Although it is envisioned that the existing cloud models can retain the original information as much as possible, they still cannot create a good representation for the concepts where linguistic labels are asymmetrically distributed. To resolve this issue, Wang et al. [231] proposed an extended cloud model, namely the Asymmetric Trapezoidal Clouds (ATCs) model. The *EN* and *HE* of the ATCs are different on the left and right sides in this model, thus the authors gave expectation as an interval rather than a precise number. The authors claim that ATCs are excellent quantitative information representations. However, unlike unbalanced linguistic assessments, ATCs are not promptly used in decision-making due to their non-existence in a practical scenario. Consequently, the authors link ATCs to unbalanced linguistic distribution assessments (ULDAs). The authors also provided an extended linguistic scale function (LSF), which indicates variations in spacing rather than symmetrical spacing among linguistic terms, to map linguistic terms to interval values. The expectation interval value of ATCs can be computed using the extended LSF. Also, a method is suggested for calculating the entropy and hyper-entropy values of ATCs developed using the 3σ concept.

Throughout the past, researchers working on cloud models representing balanced concepts were acknowledged for enhancing the literature on cloud models. As a kind of uncertain information representation in complex decision-making issues, these models are not competent to express unbalanced concepts well. In light of the latter point, a number of computationally challenging unbalanced LCMs have been proposed in the literature; as a result, solving complex decision problems becomes more time-consuming. Henceforth, the primary purpose of this current study is to develop an accurate and innovative cloud decision support model to address unbalanced linguistic concepts by emphasizing the succeeding momentous research gaps.

- The ULDA utilized in the existing ATC model is primarily based on the hierarchical unbalanced LCM which is a very complex and computationally expensive approach.
- The existing theories apply to additive linguistic data only.

- It is crucial to choose a string of appropriate linguistic cardinalities in real fuzzy decision-making problems using linguistic variables. To the best of our knowledge, the majority of the existing literature uses linguistic cardinality with a uniformly distributed scale pattern, which means that any two neighbouring linguistic cardinalities have the same level of granularity. A symmetrically distributed LT set, thus, provides the representation of the unbalanced linguistic judgements. As a result, the uneven linguistic information lacks its literal meaning.
- The use of the translation function to convert a balanced LT set into a ULTS leads to the complexity of the model.

To address above mentioned practical issues, in our research, we provide two kinds of cloud models, i.e. point based and generalized interval cloud models where linguistic information is defined on a multiplicative scale set. The advantage of using a multiplicative scale set is that the set perfectly maintains unbalanced information throughout the process. As a result, it is computationally cheaper and simpler than other LCMs like [67,155]. Further, the proposed model manages the linguistic information effortlessly, reducing the time and cost complexity. Additionally, the innovative model proposed in this chapter reduces the risk of information loss occurring in the process of unbalanced decision-making by contributing to the existing research as follows:

- The research gap is filled by introducing a multiplicative unbalanced linguistic point-based and interval cloud model that makes use of an unbalanced LT set which is computationally less expensive than other existing computational models.
- Extended multiplicative unbalanced LSFs have been developed to transform an unbalanced linguistic term into an interval value. The so-developed extended multiplicative unbalanced linguistic scale function efficiently embodies the differences in spacing of linguistic terms, both in terms of representation and meaning.
- A method based on the 3σ principle is provided to determine entropy and hyper entropy of the extended multiplicative unbalanced interval cloud model. The entropy and hyper entropy thus defined differ from the ones defined in [223] as the former is based on multiplicative unbalanced linguistic data.

- Based on the concept of generalized multiplicative unbalanced linguistic interval cloud, our research develops the comparison method to rank the cloud variables.
- The proposed theory contributes to the existing decision-making theories [109, 220, 251] by solving a decision-making real-life problem.

Finally, the chapter is structured in the following manner. Section 4.2 introduces some basic concepts related to normal cloud model. In Section 4.3, a method for transformation between the unbalanced linguistic terms and clouds is proposed. Section 4.4 introduces the generalized multiplicative unbalanced interval cloud model. In Section 4.5, some numerical example along with comparison analysis is given to demonstrate the consistency of the defined model. Section 4.6 offers the conclusions of the chapter.

4.2 Preliminaries

This chapter uses some basic normal cloud models along with an unbalanced 2-tuple LCM. For reviewing the normal cloud model as defined by Li et al. [111], one can refer to Section 1.5.5, and for understanding the basic concept of unbalanced 2-tuple LCM, refer to Section 1.5.4 of the introductory chapter 1. The present section briefly reviews the transformation between linguistic variables and clouds presented by Wang et al. in [223]. The challenging problem of transforming the linguistic variables into normal clouds was first dealt with by Wang et al. [220] based on the golden ratio. However, this method had a limitation associated that it could transform an LT set possessing five labels only. To overcome this limitation, Wang et al. [223] introduced a new method as given below.

Definition 4.2.1. [223] Let $LT = \{\ell_\varepsilon | \varepsilon = -g, \dots, 0, \dots, g, g \in N^*\}$ be a finite SSLTS with odd cardinality, where N^* = the set of positive integers, ℓ_i denotes the value of a linguistic variable, ℓ_{-g} and ℓ_g are the lower and upper limits of LT set. The LSF running the mapping from ℓ_ε to θ_ε ($\varepsilon = -g, \dots, 0, \dots, g$) is given as:

$$f : \ell_\varepsilon \rightarrow \theta_\varepsilon$$

$$\theta_\varepsilon = \begin{cases} \frac{b^g - b^{-\varepsilon}}{2b^g - 2}, -g \leq \varepsilon \leq 0 \\ \frac{b^g + b^\varepsilon - 2}{2b^g - 2}, 0 < \varepsilon \leq g \end{cases} \quad (4.2.1)$$

Here, $\theta_\varepsilon \in [0, 1]$ is a numeric value. Also, the value of b can be calculated with the help of experiments. Based on various experimental studies, the value of b is mostly obtained in the interval $[1.36, 1.4]$. The procedure for transforming linguistic labels into $2g + 1$ number of clouds is given as follows.

Assume the SSLTS to be $LT = \{\ell_\varepsilon | \varepsilon = -g, \dots, 0, \dots, g, g \in N^*\}$, and a valid universe $[X_{\min}, X_{\max}]$ as provided by the experts. When the cloud model is used to express linguistic values, a corresponding cloud variable $Y_\varepsilon(EX_\varepsilon, EN_\varepsilon, HE_\varepsilon)$ is formulated for each arbitrary $\ell_\varepsilon \in LT$. The following is the process for creating cloud variables.

1. Calculate θ_ε . Map ℓ_ε to θ_ε using Equation (4.2.1).
2. Calculate EX_ε as $EX_\varepsilon = X_{\min} + \theta_\varepsilon(X_{\max} - X_{\min})$.
3. Calculate EN_ε as

$$EN_{-\varepsilon} = EN_\varepsilon = \begin{cases} \frac{(\theta_{|\varepsilon-1|} + \theta_{|\varepsilon|} + \theta_{|\varepsilon+1|})(X_{\max} - X_{\min})}{9} & 0 < |\varepsilon| \leq g-1 \\ \frac{(\theta_{|\varepsilon|} + \theta_{|\varepsilon-1|})(X_{\max} - X_{\min})}{6} & |\varepsilon| = g \\ \frac{(\theta_\varepsilon + 2\theta_{\varepsilon+1})(X_{\max} - X_{\min})}{9} & \varepsilon = 0 \end{cases}$$

4. Calculate HE_ε .

Due to $EN'_\varepsilon \sim N(EN_\varepsilon, HE_\varepsilon^2)$, the hyper entropy must follow the 3σ principle of the normal distribution curve. Hence, $HE_{-\varepsilon} = HE_\varepsilon = (EN'^+ - EN_\varepsilon)/3$, where $EN'^+ = \max_k \{EN'_k\}$. Here,

$$EN'_\varepsilon = \begin{cases} \frac{(1-\theta_\varepsilon)(X_{\max} - X_{\min})}{3} & -g \leq \varepsilon \leq 0 \\ \frac{\theta_\varepsilon(X_{\max} - X_{\min})}{3} & 0 < \varepsilon \leq g \end{cases}$$

Next, we introduce the asymmetric trapezoidal cloud (ATC) method as defined by Wang et al. [231] in which the authors describe the values of expectation, entropy, and hyper entropy as interval values instead of specific ones. Moreover, this model bridges ULDA (common information representations in group decision-making) and ATCs. Also, the ATC model overcomes the limitations of the normal cloud model.

Definition 4.2.2. [231] The ATC model is denoted by F and is given by

$$F = (\underline{HE}, \underline{EN}, \underline{EX}, \overline{EX}, \overline{EN}, \overline{HE})$$

The expectation interval $[\underline{EX}, \overline{EX}]$, the left half cloud $(\underline{HE}, \underline{EN}, \underline{EX}, 0, 0)$, and the right half cloud $(0, 0, \overline{EX}, \overline{EN}, \overline{HE})$ constitute this model. $\underline{EX}, \underline{EN}, \underline{HE}$ and $\overline{EX}, \overline{EN}, \overline{HE}$ are the expectation, entropy and hyper entropy of the left and right half cloud, respectively. Also, the range of the qualitative notions is explained by the interval $[\underline{EX}, \overline{EX}]$ explains the range of the qualitative notions. The level of uncertainty of a qualitative concept is shown by the $\underline{EN}$ and $\overline{EN}$, respectively, while the entropies' level of uncertainty is shown by $\underline{HE}$ and $\overline{HE}$.

Compared with the normal cloud model, the ATC model has novel advantages instinctively represented in Figure 4.1. For detailed knowledge of the ATC model,

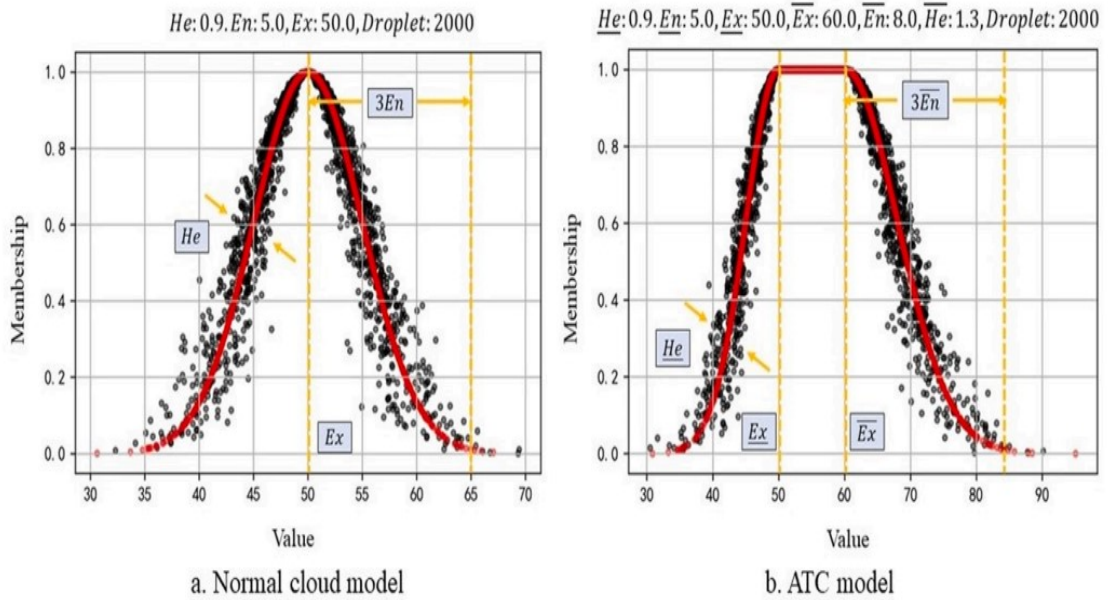


Figure 4.1: Comparison between normal cloud model and ATC model [231]

one can refer to the paper [231].

It is well known that operations cannot be carried out directly when numbers combine with linguistic terms. So there is a need to define LSFs that assign different semantic values to linguistic terms under certain situations. Wang et al. [222] defined different LSFs for different cases, but their research includes a transformation from a linguistic term to a single number. However, it does not sufficiently outline the uncertain information in some conditions. Hence to avoid this information loss, Wang et al. [232] defined some LSFs that convert linguistic variables to interval numbers. These LSFs are described below.

Definition 4.2.3. [232] Let $\mathbb{L} = \{\ell_\varepsilon | \varepsilon = 0, 1, \dots, 2g, g \in \mathbb{N}\}$ be an LT set and $\ell_\varepsilon \in \mathbb{L}$ be a linguistic variable. When $[\underline{\theta}_\varepsilon, \bar{\theta}_\varepsilon]$ is an interval fuzzy number, the LSF runs a mapping given as:

$$f: \ell_\varepsilon \rightarrow [\underline{\theta}_\varepsilon, \bar{\theta}_\varepsilon], \varepsilon = 0, 1, \dots, 2g$$

$[\underline{\theta}_\varepsilon, \bar{\theta}_\varepsilon]$ satisfies the relationship $0 \leq \underline{\theta}_0 \leq \bar{\theta}_0 \leq \underline{\theta}_1 \leq \bar{\theta}_1 \leq \dots \leq \underline{\theta}_{2g} \leq \bar{\theta}_{2g} \leq 1$. To meet this property, the mapping rule is improved and shown below:

$$\begin{aligned} H: \ell_0 &\rightarrow [\underline{\theta}_0, \bar{\theta}_0]; \\ \ell_\varepsilon &\rightarrow [\max\{\bar{\theta}_{\varepsilon-1}, \underline{\theta}_\varepsilon\}, \bar{\theta}_\varepsilon], \varepsilon = 1, 2, \dots, g; \\ \ell_\varepsilon &\rightarrow [\underline{\theta}_\varepsilon, \min\{\bar{\theta}_\varepsilon, \underline{\theta}_{\varepsilon+1}\}], \varepsilon = g+1, \dots, 2g-1; \\ \ell_{2g} &\rightarrow [\underline{\theta}_{2g}, \bar{\theta}_{2g}]. \end{aligned}$$

Hence the improved LSF is :

$$LSF: H_2(\ell_\varepsilon) = [\underline{\theta}_\varepsilon, \bar{\theta}_\varepsilon]$$

$$\underline{\theta}_\varepsilon = \begin{cases} \frac{(b^g - b^{g-\varepsilon})}{2(b^g - 1)}, & \varepsilon = 0, 1, \dots, g \\ \frac{(b^g + b^{\varepsilon-g} - 2 - \lambda_3(\varepsilon - g))}{2(b^g - 1)}, & \varepsilon = g+1, \dots, 2g \end{cases} \quad (4.2.2)$$

$$\bar{\theta}_\varepsilon = \begin{cases} \frac{(b^g - b^{(g-\varepsilon)} + \lambda_4(g-\varepsilon))}{2(b^g - 1)}, & \varepsilon = 0, 1, \dots, g \\ \frac{(b^g + b^{\varepsilon-g} - 2)}{2(b^g - 1)}, & \varepsilon = g+1, \dots, 2g \end{cases} \quad (4.2.3)$$

where λ_3 and λ_4 are adjustable parameters.

Based on the above mentioned LSF, the transformation method is conducted to map linguistic information into numerical values. For detailed understanding of the given transformation procedure, one may refer to the corresponding paper [231].

In the practical world scenario, several situations exist involving imprecise information. The uncertainty of such information is too complicated to explain with the help of quantitative expressions, but linguistic descriptors describe reality much better. However, the transformation between quantitative and qualitative concept values has always been a hurdle in developing linguistic decision-making methods. Modern researchers need a perfect way for CW. Our next section introduces a novel transformation method between the quantitative and unbalanced qualitative concepts.

4.3 Method for transformation between multiplicative unbalanced linguistic variables and clouds

The present section introduces a novel multiplicative unbalanced LSF to transform multiplicative unbalanced linguistic variables into point based unbalanced clouds.

Definition 4.3.1. Let $\mathbb{L}\mathbb{S} = \{t_{a^\varepsilon} | \varepsilon = -n, \dots, 0, \dots, m\}$, $a > 1$ (n and m may or may not be equal) be any finite ordered multiplicative ULTS, and a valid universe $[X_{\min}, X_{\max}]$. When the defined cloud model is applied to represent the multiplicative unbalanced linguistic variables, the corresponding clouds are represented as follows:

$$t_{a^{-n}} \rightarrow (EX_{a^{-n}}, EN_{a^{-n}}, HE_{a^{-n}})$$

$$\vdots$$

$$t_{a^0} \rightarrow (EX_{a^0}, EN_{a^0}, HE_{a^0})$$

$$\vdots$$

$$t_{a^m} \rightarrow (EX_{a^m}, EN_{a^m}, HE_{a^m})$$

Definition 4.3.2. Let $\mathbb{L}\mathbb{S} = \{t_{a^\varepsilon} | \varepsilon = -n, \dots, 0, \dots, m\}$, $a > 1$ (n and m may or may not be equal) be any finite ordered multiplicative ULTS. The LSF converting a linguistic term t_{a^ε} to a certain number δ_{a^ε} , ($\varepsilon = -n, \dots, 0, \dots, m$) can be expressed as follows:

$$\tilde{\delta}: t_{a^\varepsilon} \rightarrow \delta_{a^\varepsilon} (\varepsilon = -n, \dots, 0, \dots, m)$$

where

$$\delta_{a^\varepsilon} = \begin{cases} \left(\frac{b^n - b^{-\varepsilon}}{2b^n - 2} \right) \frac{1}{\log a}, & -n \leq \varepsilon \leq 0 \\ \left(\frac{b^n + b^\varepsilon - 2}{2b^n - 2} \right) \frac{1}{\log a}, & 0 < \varepsilon \leq n \end{cases} \quad (4.3.1)$$

Here, $\delta_{a^\varepsilon} \in [0, 1/\log a] \subseteq [0, 1]$ is a numeric value. The value of the parameter b can be obtained through experiments such as $b \in [1.36, 1.4]$ [16].

We next describe the following theorem giving the specific properties of the above-mentioned multiplicative unbalanced LSF.

Theorem 4.3.1. Based on the multiplicative unbalanced linguistic scale function (MULSF), the following properties hold:

1. If $-n \leq \varepsilon \leq 0$, then $0 \leq \delta_{a^\varepsilon} \leq \frac{1}{2\log a}$
2. If $0 < \varepsilon \leq n$, then $\frac{1}{2\log a} < \delta_{a^\varepsilon} \leq \frac{1}{\log a}$
3. If $\varepsilon + \iota = 0$, then $\delta_{a^\varepsilon} + \delta_{a^\iota} = \frac{1}{\log a}$.

Proof. 1. For given $-n \leq \varepsilon \leq 0$, we have $\delta_{a^\varepsilon} = \left(\frac{b^n - b^{-\varepsilon}}{2b^n - 2} \right) \frac{1}{\log a}$.

By setting $\varepsilon = -n$ in the above equation, we have

$$\delta_{a^{-n}} = \left(\frac{b^n - b^n}{2b^n - 2} \right) \frac{1}{\log a} = 0 \text{ as } n \rightarrow \infty$$

By setting $\varepsilon = 0$, we have

$$\delta_{a^0} = \left(\frac{b^n - b^0}{2b^n - 2} \right) \frac{1}{\log a} = \frac{1}{2\log a} \text{ as } n \rightarrow \infty$$

Next, by setting $-n < \varepsilon < 0$, we have

$$\delta_{a^\varepsilon} = \left(\frac{b^n - b^{-\varepsilon}}{2b^n - 2} \right) \frac{1}{\log a}$$

Taking limits on both sides, we get

$$\lim_{n \rightarrow \infty} \delta_{a^\varepsilon} = \lim_{n \rightarrow \infty} \left(\frac{b^n - b^{-\varepsilon}}{2b^n - 2} \right) \frac{1}{\log a} = \frac{1}{2\log a}$$

Therefore, the limiting value of the function δ_{a^ε} is obtained as

$$\lim_{n \rightarrow \infty} \delta_{a^\varepsilon} = \begin{cases} 0, & \varepsilon = -n \\ \frac{1}{2\log a}, & -n < \varepsilon \leq 0 \end{cases}$$

2. Based on the Definition 4.3.2, for $0 < \varepsilon \leq n$, we have $\delta_{a^\varepsilon} = \left(\frac{b^n + b^\varepsilon - 2}{2b^n - 2} \right) \frac{1}{\log a}$.

By setting $\varepsilon = 0$,

$$\delta_{a^0} = \left(\frac{b^n + 1 - 2}{2b^n - 2} \right) \frac{1}{\log a} = \frac{1}{2\log a} \text{ as } n \rightarrow \infty.$$

By setting $\varepsilon = n$

$$\delta_{a^n} = \left(\frac{b^n + b^n - 2}{2b^n - 2} \right) \frac{1}{\log a} = \frac{1}{\log a} \text{ as } n \rightarrow \infty.$$

By setting $0 < \varepsilon < n$, then

$$\delta_{a^\varepsilon} = \left(\frac{b^n + b^\varepsilon - 2}{2b^n - 2} \right) \frac{1}{\log a}$$

On rearranging the terms and taking limit on both sides, we get $\lim_{n \rightarrow \infty} \delta_{a^\varepsilon} = \frac{1}{2 \log a}$.

3. Based on the Definition [4.3.2](#),

$$\delta_{a^\varepsilon} = \begin{cases} \left(\frac{b^n - b^{-\varepsilon}}{2b^n - 2} \right) \frac{1}{\log a}, & -n \leq \varepsilon \leq 0 \\ \left(\frac{b^n + b^\varepsilon - 2}{2b^n - 2} \right) \frac{1}{\log a}, & 0 < \varepsilon \leq n \end{cases}$$

For $\varepsilon + \iota = 0$, we have $\iota = -\varepsilon$. Therefore,

$$\delta_{a^\varepsilon} + \delta_{a^\iota} = \left(\frac{b^n - b^{-\varepsilon}}{2b^n - 2} \right) \frac{1}{\log a} + \left(\frac{b^n + b^\iota - 2}{2b^n - 2} \right) \frac{1}{\log a}$$

Adding and subtracting the term $b^{-\varepsilon}$ in the above expression, and by rearranging the terms, we obtain

$$\delta_{a^\varepsilon} + \delta_{a^\iota} = \frac{1}{\log a}$$

Hence the proof. □

We will next introduce the process of generating cloud droplets based on the rational transformation method of unbalanced linguistic variables into clouds.

Definition 4.3.3. Let $t_{a^\varepsilon} \in \mathbb{L}\mathbb{S}$ be any arbitrary unbalanced linguistic term corresponding to the cloud drop $z_{a^\varepsilon}(EX_{a^\varepsilon}, EN_{a^\varepsilon}, HE_{a^\varepsilon})$. Then the working cloud generation procedure is given as follows:

Step 1: Calculate δ_{a^ε} using Equation [4.3.1](#).

Step 2: Calculate EX_{a^ε} as $EX_{a^\varepsilon} = X_{\min} + \delta_{a^\varepsilon}(X_{\max} - X_{\min})$.

Step 3: Calculate EN_{a^ε} .

Let us represent a cloud droplet as (x, y) , since $x \sim N(EX_{a^\varepsilon}, EN_{a^\varepsilon}^2)$, then as per the 3σ principle of the normal distribution curve, we have

$$3EN_{a^\varepsilon}' = \max(X_{\max} - EX_{a^\varepsilon}, EX_{a^\varepsilon} - X_{\min})$$

Because of $EN' \sim N(EN, HE^2)$, EN_{a^ε} may be considered the mean value of EN'_{a^ε} corresponding to the clouds in consideration and their adjacent clouds.

$$EN_{a^{-\varepsilon}} = EN_{a^\varepsilon} = \frac{EN'_{a^{\varepsilon-1}} + EN'_{a^\varepsilon} + EN'_{a^{\varepsilon+1}}}{3 \log a} \text{ for } |\varepsilon| = 0, 1, \dots, n-1, \quad (4.3.2)$$

and for $|\varepsilon| = n$,

$$EN_{a^{-n}} = EN_{a^n} = \frac{EN'_{a^n} + EN'_{a^{n-1}}}{2 \log a} \quad (4.3.3)$$

These calculations may be rewritten as the following formulas:

$$EN'_{a^\varepsilon} = \begin{cases} \frac{(1-\delta_{a^\varepsilon})(X_{\max}-X_{\min})}{3 \log a}, & -n \leq \varepsilon \leq 0 \\ \frac{\delta_{a^\varepsilon}(X_{\max}-X_{\min})}{3 \log a}, & 0 < \varepsilon \leq n \end{cases} \quad (4.3.4)$$

$$EN_{a^{-\varepsilon}} = EN_{a^\varepsilon} = \begin{cases} \frac{(\delta_{a^{|\varepsilon-1|}} + \delta_{a^{|\varepsilon|}} + \delta_{a^{|\varepsilon+1|}})(X_{\max}-X_{\min})}{9 \log a}, & 0 < |\varepsilon| < n-1 \\ \frac{(\delta_{a^{|\varepsilon|}} + \delta_{a^{|\varepsilon-1|}})(X_{\max}-X_{\min})}{6 \log a}, & |\varepsilon| = n \\ \frac{(\delta_{a^\varepsilon} + 2\delta_{a^{\varepsilon+1}})(X_{\max}-X_{\min})}{9 \log a}, & \varepsilon = 0 \end{cases} \quad (4.3.5)$$

Step 4: Calculate HE_{a^ε} .

Due to $EN'_{a^\varepsilon} \sim N(EN_{a^\varepsilon}, HE_{a^\varepsilon}^2)$, the hyper entropy satisfies the 3σ principle.

In particular,

$$HE_{a^{-\varepsilon}} = HE_{a^\varepsilon} = \frac{EN'^+ - EN_{a^\varepsilon}}{3 \log a} \quad (4.3.6)$$

where, $EN'^+ = \sup_k \{EN'_k\}$.

Next, we present the working example to explain the methodology mentioned above.

Example 4.3.1. Consider the universe $[X_{\min}, X_{\max}] = [0, 9]$ and let the ULTS be given as

$\mathbb{LS} = \{t_{2^{-4}} : \text{None}, t_{2^{-3}} : \text{extremely poor}, t_{2^{-2}} : \text{very poor}, t_{2^{-1}} : \text{poor}, t_{2^0} : \text{avg}, t_{2^1} : \text{good},$

$t_{2^2} : \text{very good}, t_{2^3} : \text{extremely good}, t_{2^4} : \text{excellent}\}$

The procedure of transforming unbalanced linguistic variables into clouds is presented in the following manner:

1. Based on Definition [4.3.2](#), setting $b = 1.4$, the value for $\delta_{2^\varepsilon} (i = -4, -3, \dots, 0, 1, 2, \dots, 4)$

can be computed as

$$\delta_{2^{-4}} = 0 \quad \delta_{2^{-3}} = 0.6416 \quad \delta_{2^{-2}} = 1.0998 \quad \delta_{2^{-1}} = 1.4271 \quad \delta_{2^0} = 1.6609$$

$$\delta_{2^1} = 1.8948 \quad \delta_{2^2} = 2.2221 \quad \delta_{2^3} = 2.6804 \quad \delta_{2^4} = 3.3219$$

2.

$$Ex_{2^{-4}} = 0.0625 \quad Ex_{2^{-3}} = 10.2875 \quad Ex_{2^{-2}} = 17.5910 \quad Ex_{2^{-1}} = 22.8078 \quad Ex_{2^0} = 26.5341$$

$$Ex_{2^1} = 30.2604 \quad Ex_{2^2} = 35.4772 \quad Ex_{2^3} = 42.7808 \quad Ex_{2^4} = 53.0057$$

3.

$$En'_{2^{-4}} = 1.5992 \quad En'_{2^{-3}} = 0.5732 \quad En'_{2^{-2}} = -0.1596 \quad En'_{2^{-1}} = -0.6831 \quad En'_{2^0} = -1.0570$$

$$En'_{2^1} = 3.0302 \quad En'_{2^2} = 3.5536 \quad En'_{2^3} = 4.2865 \quad En'_{2^4} = 5.3125$$

4.

$$En_{2^{-4}} = 4.7995 \quad En_{2^{-3}} = 4.042 \quad En_{2^{-2}} = 3.3791 \quad En_{2^{-1}} = 2.9056 \quad En_{2^0} = 0.6840$$

$$En_{2^1} = 2.9056 \quad En_{2^2} = 3.3791 \quad En_{2^3} = 4.042 \quad En_{2^4} = 4.7995$$

5.

$$He_{2^{-4}} = 0.0515 \quad He_{2^{-3}} = 0.1275 \quad He_{2^{-2}} = 0.1940 \quad He_{2^{-1}} = 0.2415 \quad He_{2^0} = 0.4644$$

$$He_{2^1} = 0.2415 \quad He_{2^2} = 0.1940 \quad He_{2^3} = 0.1275 \quad He_{2^4} = 0.0515$$

Thus, the 9-asymmetric linguistic clouds obtained are given as follows:

$$\begin{aligned} t_{2^{-4}} &\rightarrow (0.0625, 4.7995, 0.0515), t_{2^{-3}} \rightarrow (10.2875, 4.042, 0.1275), t_{2^{-2}} \rightarrow (17.5910, 3.3791, 0.1940), \\ t_{2^{-1}} &\rightarrow (22.8078, 2.9056, 0.2415), t_{2^0} \rightarrow (26.5341, 0.684, 0.4644), t_{2^1} \rightarrow (30.2604, 2.9056, 0.2415), \\ t_{2^2} &\rightarrow (35.4772, 3.3791, 0.1940), t_{2^3} \rightarrow (42.7808, 4.042, 0.1275), t_{2^4} \rightarrow (53.0057, 4.7995, 0.0515). \end{aligned}$$

Figure 4.2 presents the graphical representation of the obtained nine asymmetric clouds.

Here, EN and HE reflect the extent of dispersion of the cloud droplets.

Note 4.3.1. It should be noted that the point-based cloud generating procedure described above can be used in the scenario $n \neq m$ if dummy variables are added to the multiplicative ULTS in a way that $n = m$.

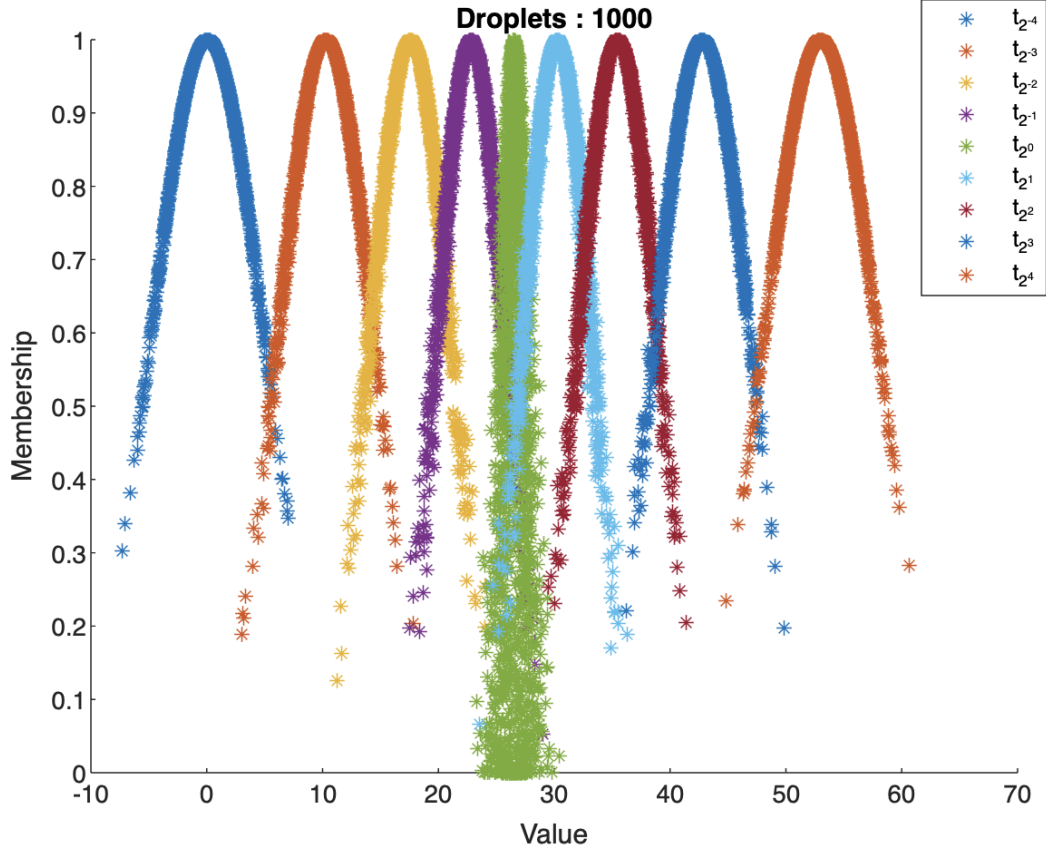


Figure 4.2: Evaluation information for asymmetric point based cloud model with respect to unbalanced linguistic terms

Although the above-defined methodology is a well-defined model; it has some limitations associated, as the expected value of the unbalanced linguistic term is a single numeric value. This causes information loss in representing the semantics of linguistic terms to an extent. Thus, to overcome this information loss, we define a generalized multiplicative unbalanced interval cloud model in the next section, to effectively describe unbalanced linguistic terms that entail interval concepts instead of specific ones.

4.4 Generalised Multiplicative Unbalanced interval Cloud Model

In this section, we first develop a generalized MULSF to transform an unbalanced linguistic variable into an interval numerical value. Then, we provide a detailed procedure to generate a generalized multiplicative unbalanced interval cloud, followed by suggesting a novel interval score function to rank interval

clouds.

Definition 4.4.1. Let $\mathbb{LS} = \{t_{a^\varepsilon} | \varepsilon = -n, -(n-1), \dots, 0, \dots, m, a > 1, n, m \in \mathbb{Z}^+\}$ (where n and m may or may not be equal) be a finitely ordered ULTS. Then the extended MULSF can be conducted as a mapping from unbalanced linguistic label t_{a^ε} to interval numbers $[\underline{\delta}_{a^\varepsilon}, \bar{\delta}_{a^\varepsilon}]$ and the mapping can be expressed as below

$$\tilde{\mathbb{G}}: t_{a^\varepsilon} \rightarrow [\underline{\delta}_{a^\varepsilon}, \bar{\delta}_{a^\varepsilon}], \varepsilon = -n, \dots, 0, \dots, m$$

where, $[\underline{\delta}_{a^\varepsilon}, \bar{\delta}_{a^\varepsilon}] \in [0, 1/\log a] \subseteq [0, 1]$.

Usually, the correspondence between unbalanced linguistic term t_{a^ε} and the subscript a^ε , $a > 1$ is strictly increasing, therefore $[\underline{\delta}_{a^\varepsilon}, \bar{\delta}_{a^\varepsilon}]$ needs to satisfy

$$0 \leq \underline{\delta}_{a^{-n}} < \bar{\delta}_{a^{-n}} \leq \dots \leq \underline{\delta}_{a^0} < \bar{\delta}_{a^0} \leq \dots \leq \underline{\delta}_{a^m} < \bar{\delta}_{a^m} \leq 1/\log a$$

To strictly meet this property, an improved mapping rule is defined below

$$\begin{aligned} \tilde{\mathbb{G}}: t_{a^{-n}} &\rightarrow [\underline{\delta}_{a^{-n}}, \bar{\delta}_{a^{-n}}]; \\ t_{a^\varepsilon} &\rightarrow [\sup\{\bar{\delta}_{a^{\varepsilon-1}}, \underline{\delta}_{a^\varepsilon}\}, \bar{\delta}_{a^\varepsilon}], \varepsilon = -n, \dots, 0; \\ t_{a^\varepsilon} &\rightarrow [\underline{\delta}_{a^\varepsilon}, \inf\{\bar{\delta}_{a^\varepsilon}, \underline{\delta}_{a^{\varepsilon+1}}\}], \varepsilon = 1, 2, \dots, m-1; \\ t_{a^m} &\rightarrow [\underline{\delta}_{a^m}, \bar{\delta}_{a^m}] \end{aligned}$$

Here, t_{a^0} is the mid-term of the set $\mathbb{LS}$ and $I(t_{a^\varepsilon}) = a^\varepsilon$.

The preceding mapping rules state that $\tilde{\mathbb{G}}$ is a strictly and monotonically increasing function with respect to a^ε . Based on the knowledge of function translation, the extended MULSF is represented as follows:

$$MULSF: \tilde{\mathbb{G}}(t_{a^\varepsilon}) = [\underline{\delta}_{a^\varepsilon}, \bar{\delta}_{a^\varepsilon}]$$

where,

$$\underline{\delta}_{a^\varepsilon} = \begin{cases} \frac{1}{\log a} \left| \frac{b^{\#\mathbb{LS}_{left}-b(\#\mathbb{LS}_{left}-a^\varepsilon)}}{2(b^{\#\mathbb{LS}_{left}}-1)} \right|, & -n \leq \varepsilon \leq 0 \\ \frac{1}{\log a} \left| \frac{b^{\#\mathbb{LS}_{right}+b(a^\varepsilon+C-\#\mathbb{LS}_{right})-2-\lambda_3(a^\varepsilon+C-\#\mathbb{LS}_{right})}}{2(b^{\#\mathbb{LS}_{right}}-1)} \right|, & 0 < \varepsilon \leq m \end{cases} \quad (4.4.1)$$

And,

$$\bar{\delta}_{a^\varepsilon} = \begin{cases} \frac{1}{\log a} \left| \frac{b^{\#\mathbb{L}\mathbb{S}_{left}} - b^{(\#\mathbb{L}\mathbb{S}_{left} - a^\varepsilon) + \lambda_4(\#\mathbb{L}\mathbb{S}_{left} - a^\varepsilon)}}{2(b^{\#\mathbb{L}\mathbb{S}_{left}} - 1)} \right|, & -n \leq \varepsilon \leq 0 \\ \frac{1}{\log a} \left| \frac{b^{\#\mathbb{L}\mathbb{S}_{right}} + b^{(a^\varepsilon + C - \#\mathbb{L}\mathbb{S}_{right})}}{2(b^{\#\mathbb{L}\mathbb{S}_{right}} - 1)} \right|, & 0 < \varepsilon \leq m \end{cases} \quad (4.4.2)$$

The meaning of other symbols remains the same, where λ_3 and λ_4 are adjustable parameters. Further, $C = |\#\mathbb{L}\mathbb{S}_{right} - \#\mathbb{L}\mathbb{S}_{left}|$. We define the adjustable parameters λ_3 and λ_4 for the sake of calculation convenience as follows:

$$\lambda_3 = \frac{1}{a^\varepsilon + C - \#\mathbb{L}\mathbb{S}_{right} + 1}$$

and,

$$\lambda_4 = \frac{1}{\#\mathbb{L}\mathbb{S}_{left} - a^\varepsilon + 1}$$

For a fit with the psychological value during the evaluation process, *MULSF* makes use of the bidirectional rise in the geometric progression of the scale value to accurately depict the cognitive behaviour of experts. In this situation, the absolute deviation between adjacent linguistic subscripts increases with an extension from the central portion of linguistic terms to both ends. After considerable experiments, parameter ' b ' is demonstrated to belong to the interval $[1.36, 1.4]$ [16]. Without loss of generality, b is taken to be equal to 1.4 throughout the chapter for numerical computation.

Next, we proceed to define the concept of the multiplicative unbalanced interval cloud model.

Definition 4.4.2. The multiplicative unbalanced interval cloud (MUIC) model, denoted by $\mathbb{C}$ is defined as

$$\mathbb{C} = (\underline{HE}, \underline{EN}, \underline{EX}, \overline{EX}, \overline{EN}, \overline{HE})$$

The MUIC model involves three numerical sections, i.e. the expectation interval $[\underline{EX}, \overline{EX}]$, left half cloud $(\underline{HE}, \underline{EN}, \underline{EX}, 0, 0)$ and right half cloud $(0, 0, \overline{EX}, \overline{EN}, \overline{HE})$. Here, $\underline{EX}$, $\underline{EN}$, $\underline{HE}$ are the expectation, entropy, and hyper entropy of the left half cloud, also known as lower expectation, lower entropy, and lower hyper entropy, respectively. On the other hand, $\overline{EX}$, $\overline{EN}$, $\overline{HE}$ are the expectation, entropy, and hyper entropy of the right half cloud, also called upper expectation, upper entropy, and upper hyper entropy, respectively.

We next proceed to present the procedure proficient in converting unbalanced linguistic term into unbalanced linguistic interval cloud variables based on the aforementioned MULSF. Let $\mathbb{LS} = \{t_{a^\varepsilon} | \varepsilon = -n, \dots, 0, 1, \dots, m \text{ and } a > 1, n, m \in \mathbb{Z}^+\}$ be a multiplicative ULTS, where t_{a^ε} is plausible unbalanced linguistic term. The relationship between the unbalanced linguistic term and MUIC is presented in the following steps.

Step 1: Initially calculate $[\underline{\delta}_{a^\varepsilon}, \overline{\delta}_{a^\varepsilon}]$ using *MULSF* defined earlier, over the universe $[X_{\min}, X_{\max}]$.

Step 2: Determine the expectation interval $[\underline{EX}_{a^\varepsilon}, \overline{EX}_{a^\varepsilon}]$ where $\underline{EX}_{a^\varepsilon}$ and $\overline{EX}_{a^\varepsilon}$ are calculated as

$$\begin{aligned}\underline{EX}_{a^\varepsilon} &= X_{\min} + \underline{\delta}_{a^\varepsilon} (X_{\max} - X_{\min}) \\ \overline{EX}_{a^\varepsilon} &= X_{\min} + \overline{\delta}_{a^\varepsilon} (X_{\max} - X_{\min})\end{aligned}\quad (4.4.3)$$

Step 3: Determine the entropy $\underline{EN}_{a^\varepsilon}$ and $\overline{EN}_{a^\varepsilon}$. Based on the 3σ principle,

$$\begin{aligned}\underline{EN}_{a^{-n}} &= 0 \\ \underline{EN}_{a^\varepsilon} &= \left| \frac{\underline{EX}_{a^\varepsilon} - \overline{EX}_{a^{\varepsilon-1}}}{3 \log a} \right|, \quad -(n-1) \leq \varepsilon \leq m \\ \overline{EN}_{a^\varepsilon} &= \left| \frac{\underline{EX}_{a^{\varepsilon+1}} - \overline{EX}_{a^\varepsilon}}{3 \log a} \right|, \quad -n \leq \varepsilon \leq m-1 \\ \overline{EN}_{a^m} &= 0\end{aligned}\quad (4.4.4)$$

Step 4: Determine the hyper entropies $\underline{HE}_{a^\varepsilon}$ and $\overline{HE}_{a^\varepsilon}$. Since $EN' \sim N(EN, HE^2)$, HE also satisfies the 3σ principle. Hence the hyper entropies $\underline{HE}_{a^\varepsilon}$ and $\overline{HE}_{a^\varepsilon}$ of MUIC may be obtained using the formulas:

$$\begin{aligned}\underline{HE}_{a^{-n}} &= 0 \\ \underline{HE}_{a^\varepsilon} &= \frac{\max\{\max\{\underline{EN}_{a^\varepsilon}\} - \underline{EN}_{a^\varepsilon}, \underline{EN}_{a^\varepsilon} - \min\{\underline{EN}_{a^\varepsilon}\}\}}{3 \log a}, \quad -(n-1) \leq \varepsilon \leq m \\ \overline{HE}_{a^\varepsilon} &= \frac{\max\{\max\{\overline{EN}_{a^\varepsilon}\} - \overline{EN}_{a^\varepsilon}, \overline{EN}_{a^\varepsilon} - \min\{\overline{EN}_{a^\varepsilon}\}\}}{3 \log a}, \quad -n \leq \varepsilon \leq m-1 \\ \overline{HE}_{a^m} &= 0\end{aligned}\quad (4.4.5)$$

In literature, Li et al. [111] proposed the concept of variance of a normal cloud. Analogously, we introduce the notion of variance for MUIC.

Definition 4.4.3. Let $A = (\underline{HE}, \underline{EN}, \underline{EX}, \overline{EX}, \overline{EN}, \overline{HE})$ be a MUIC with c generated interval cloud droplets, the variance of c interval cloud droplets can be calculated as follows:

$$var_{Int} = [\underline{EN}, \overline{EN}] \times [\underline{EN}, \overline{EN}] + [\underline{HE}, \overline{HE}] \times [\underline{HE}, \overline{HE}]$$

then, the standard deviation of c interval cloud droplets is

$$sd_{Int} = \sqrt{[\underline{EN}, \overline{EN}] \times [\underline{EN}, \overline{EN}] + [\underline{HE}, \overline{HE}] \times [\underline{HE}, \overline{HE}]}$$

Here, the standard deviation sd_{Int} estimates the dispersal of interval cloud droplets that can be utilized to address the existing uncertainty and alteration of an unbalanced qualitative concept. A low sd_{Int} specifies that an unbalanced linguistic label tends to be relatively stable. In contrast, a high sd_{Int} implies the existence of a high level of uncertainty and alteration about the linguistic label. In this scenario, the relationship between $EX_{Int} = [\underline{EX}, \overline{EX}]$ and sd_{Int} can be considered as the expected utility and the utility dispersion in the utility theory [30]. Then the preference made for a choice can be presented by a linear trade-off between expected utility and utility dispersal. Motivated by the notion, we define an interval score function below to rank MUICs.

Definition 4.4.4. Given a MUIC $\mathbb{C} = (\underline{HE}, \underline{EN}, \underline{EX}, \overline{EX}, \overline{EN}, \overline{HE})$, an interval score function can be stated as

$$sc_{Int} = EX_{Int} - \chi \cdot sd_{Int}$$

where $\chi \in [0, 1]$ reflects a coefficient to capture the attitude toward the standard deviation.

The χ value is estimated based on the real-world decision-making scenario. If $\chi = 0$, it means that the DMs simply take into account the EX_{Int} and ignore uncertainty and informational modification. The monotonicity property is satisfied by the defined interval score function. When compared to EX_{Int} , the score function increases, but when compared to EN_{Int} or HE_{Int} , it decreases.

Next, based on Definition 4.4.4, the following method can be proposed to rank two MUICs.

Definition 4.4.5. Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be any two MUICs, then

- If $sc_{Int1} > sc_{Int2}$, then $\mathbb{C}_1 > \mathbb{C}_2$;
- If $sc_{Int1} = sc_{Int2}$, then $\mathbb{C}_1 = \mathbb{C}_2$;
- If $sc_{Int1} < sc_{Int2}$, then $\mathbb{C}_1 < \mathbb{C}_2$.

Note 4.4.1. It is noted that the inequality stated in the above definition is crisply defined. Henceforth, the case of a nested interval is neglected.

4.5 Illustrative example and comparative analysis

This section demonstrates the complete process of transforming unbalanced linguistic variables into MUICs via numerical illustration. Subsequently, the feasibility and rationality of the proposed methodology is given via comparative analyses from existing studies in the literature.

4.5.1 Illustrative Example

Here, we propose a practical decision-making example to demonstrate the feasibility and advantages of the developed methodology for both the cases, $n = m$ and $n \neq m$.

Example 4.5.1. We first consider the case where $n = m$, i.e., $\mathbb{L}\mathbb{S} = \{t_{a(-n)}, \dots, t_{a^0}, \dots, t_{a(n)}\}$. Suppose there is a DM who is willing to invest his sum of money in IT companies by carefully observing the average closing stock values for the past 5 consecutive year, i.e., 2017-21. There are five possible alternatives DM is considering for investing the money. Table 4.1 shows the alternatives of the IT companies. Since the stock values of companies get

IT1	IT2	IT3	IT4	IT5
Infosys	Wipro	TCS	HCL Technologies	Larsen & Toubro Infotech

Table 4.1: Five IT Companies considered for investing purpose

affected by linguistic factors. Henceforth, it facilitates to evaluate stock prices for the IT companies in terms of the predefined unbalanced linguistic scale set given as: $\mathbb{L}\mathbb{S} = \{t_{2-2} : \text{Very Low (VL)}, t_{2-1} : \text{Low (L)}, t_{2^0} : \text{Average (Avg)}, t_{2^1} : \text{High (H)}, t_{2^2} : \text{Very High (VH)}\}$. The variations incurred in the companies' average stock values for the past five years are expressed in an unbalanced linguistic scale set $\mathbb{L}\mathbb{S}$. In this scenario, analyzing the uncertainty and fluctuation of stock values expressed by the dispersion of cloud drops

is significant. Henceforth, we utilize our methodology to transform unbalanced linguistic variables into clouds so that the preference for selecting a particular alternative is possible. To do this, we follow similar steps 1 – 4 as given in the last section.

Based on the defined methodology in Steps, 1 – 4, the assessment information presented by MUICs is shown in Table 4.2 and their graphical representation is illustrated in Figure 4.3 and 4.4. Next, based on the Definition 4.4.4, the scoring values obtained are given as follows: $sc_{Int1} = [1.3675, 1.7094]$, $sc_{Int2} = [-1.2956, 1.8665]$, $sc_{Int3} = [3.0111, 4.9861]$, $sc_{Int4} = [4.5111, 5.9685]$ and $sc_{Int5} = [10.8145, 12.0274]$. According to the score values, the ranking scheme of the alternatives can be fixed, and they are obtained as follows: $\mathbb{C}_5 > \mathbb{C}_4 > \mathbb{C}_3 > \mathbb{C}_1$. Hence, IT5 has a stable cloud and is preferred over other alternatives.

Linguistic Terms	MUICs
t_{2-2}	$\mathbb{C}_1 = (0, 0, 1.3675, 6.873, 4.7032, 2.1314)$
t_{2-1}	$\mathbb{C}_2 = (2.1314, 3.2918, 2.626, 7.817, 4.7032, 3.6454)$
t_{2^0}	$\mathbb{C}_3 = (1.563, 0.9584, 4.8445, 9.171, 3.2918, 2.5841)$
t_{2^1}	$\mathbb{C}_4 = (2.1314, 0.9584, 8.3055, 8.3055, 2.7785, 2.5841)$
t_{2^2}	$\mathbb{C}_5 = (0, 0, 10.8145, 16.611, 2.7785, 3.6454)$

Table 4.2: The assessment representation of the linguistic terms by MUICs for case $n = m$

Example 4.5.2. Next, we discuss the case where $n \neq m$ i.e., $\mathbb{LS} = \{t_{a^{(-n)}}, \dots, t_{a^0}, \dots, t_{a^{(m)}}\}$. Consider a multiplicative unbalanced set $\mathbb{LS} = \{t_{a^{-2}} : \text{Terrible}, t_{a^{-1}} : \text{Poor}, t_{a^0} : \text{Middle}, t_{a^1} : \text{Satisfactory}, t_{a^2} : \text{Good}, t_{a^3} : \text{Excellent}\}$. In this particular instance, $\#\mathbb{LS}_{left} = n = 2$, $\#\mathbb{LS}_{right} = m = 3$. Let $a = 2$, $b = 1.4$, $[X_{\min}, X_{\max}] = [0, 7]$. The assessment representation of linguistic terms by MUICs is given in Table 4.3 and their graphical representation is illustrated in Figure 4.5 and 4.6.

Linguistic Terms	MUICs
t_{2-2}	$\mathbb{C}_1 = (0, 0, 1.9145, 9.6222, 6.5853, 31.234)$
t_{2-1}	$\mathbb{C}_2 = (5.1017, 4.6069, 3.6757, 10.9424, 6.5853, 31.234)$
t_{2^0}	$\mathbb{C}_3 = (3.6166, 1.3411, 6.7823, 12.838, 4.6069, 33.425)$
t_{2^1}	$\mathbb{C}_4 = (3.6166, 1.3411, 11.627, 11.627, 2.14, 36.1572)$
t_{2^2}	$\mathbb{C}_5 = (2.7319, 2.14, 13.5597, 18.0264, 34.79, 36.1572)$
t_{2^3}	$\mathbb{C}_6 = (3.6454, 0, 49.4424, 55.1565, 34.79, 0)$

Table 4.3: The assessment representation of the linguistic terms by MUICs for the case $n \neq m$

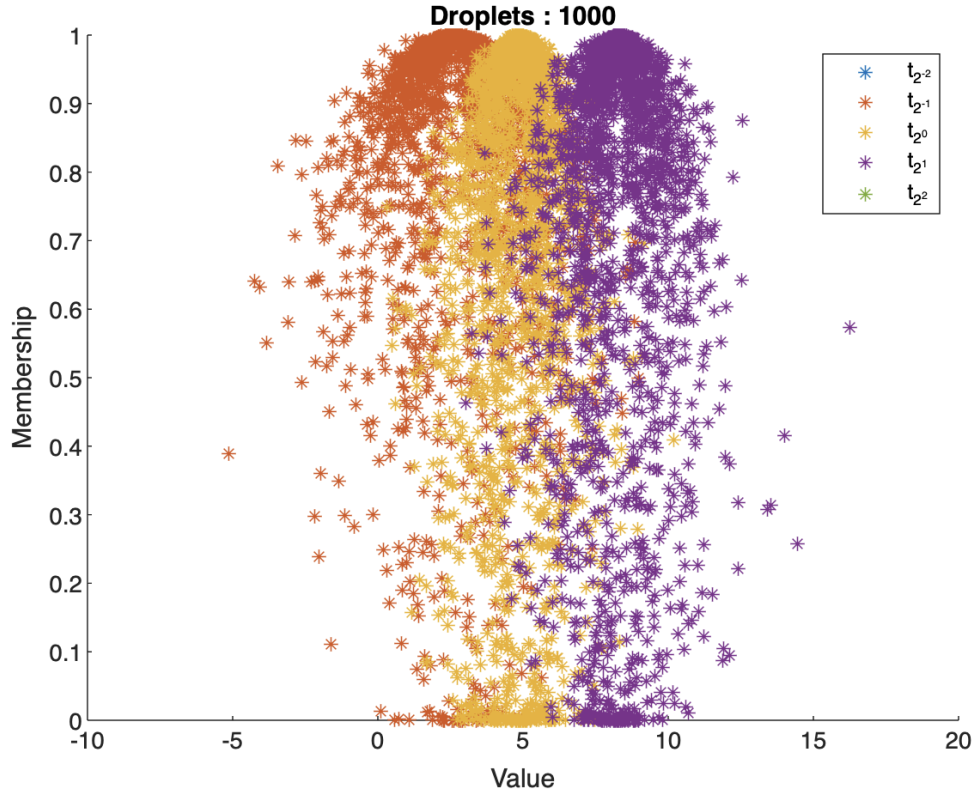


Figure 4.3: Evaluation information for asymmetric lower half cloud with respect to unbalanced linguistic terms for Case $n = m$.

4.5.2 Comparative Analysis

Here, we analyze the numerical results presented in the previous subsection and demonstrate the viability and superiority of the proposed cloud model by comparing it to the existing studies from [231] and [101].

To confirm that the ranking outcomes are comparable, numerical example 4.5.2 given in the previous subsection is taken from the paper [231]. Since the final ranking outcome obtained is the same as that of the ATC model [231], the proposed model offers analogous results. However, the difference lies in the information representation and the evaluation method to solve decision problems. The unbalanced linguistic computational model put forth by Herrera et al. [97] serves as the foundation for the model proposed by Wang et al. [231]. The scale values could be mapped to a number of hierarchically constructed uniformly distributed scale sets, which is effective for unbalanced linguistic cardinalities. Then, in information aggregation procedures, the uniformly dispersed scale values are used to replace the original unbalanced ones. This whole process altogether in-

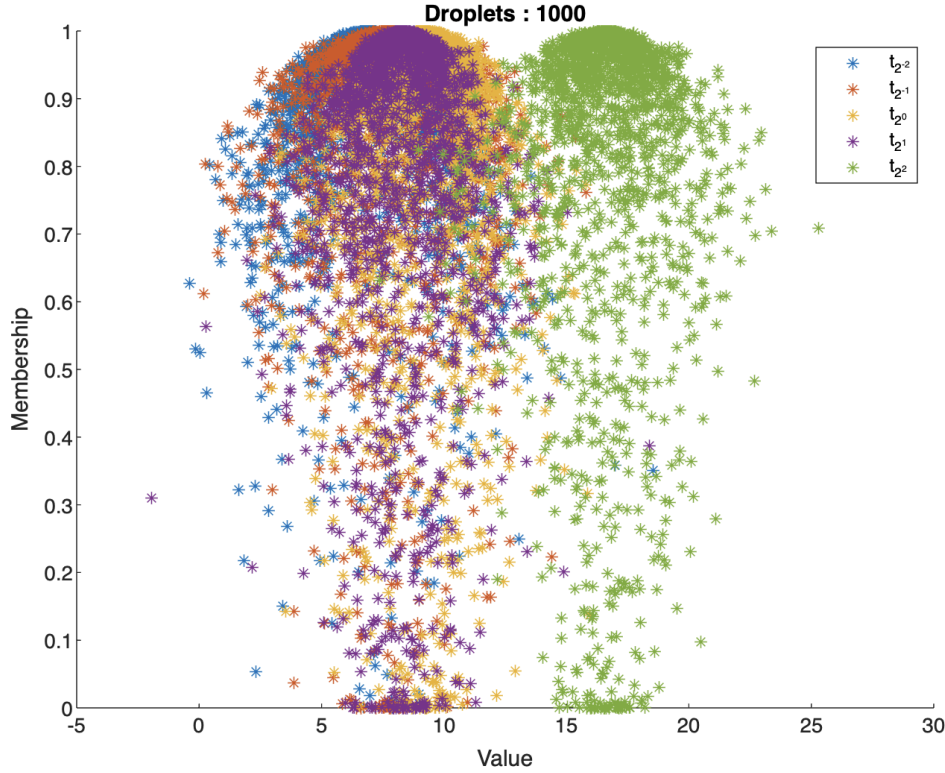


Figure 4.4: Evaluation information for asymmetric upper half cloud with respect to unbalanced linguistic terms for Case $n = m$.

creases the complexity of the ATC model. Moreover, the unbalanced linguistic information presented in the model is given by a symmetrically distributed LT set. Consequently, the literal meaning of the ULTS is missing. On the contrary, the superiority of our work lies in the fact that it develops a new concept, namely a multiplicative ULCM, to overcome the limitation of the existing cloud model as a qualitative information representation model. Our proposed model is based on the multiplicative ULCM developed by Malhotra & Gupta [153], which is computationally less expensive and preserves the literal meaning of the unbalanced LT set. In addition, the existing ATC cloud model has difficulty describing Type II unbalanced linguistic concepts; however, our proposed model is suitable for both Type I and Type II ULTSs. The MULSF proposed in our work is based on a multiplicative scale set and improves the related mapping rules. Moreover, the existing cloud model [231] has difficulty in describing a continuous concept. Taking the concept of investment return, where the data distribution of investment return follows an irregular pattern, our proposed model is more suitable for description as it preserves the meaning and prevents information loss.

In literature, the Fuzzy Trapezium Cloud model [101] is a subjective and effective

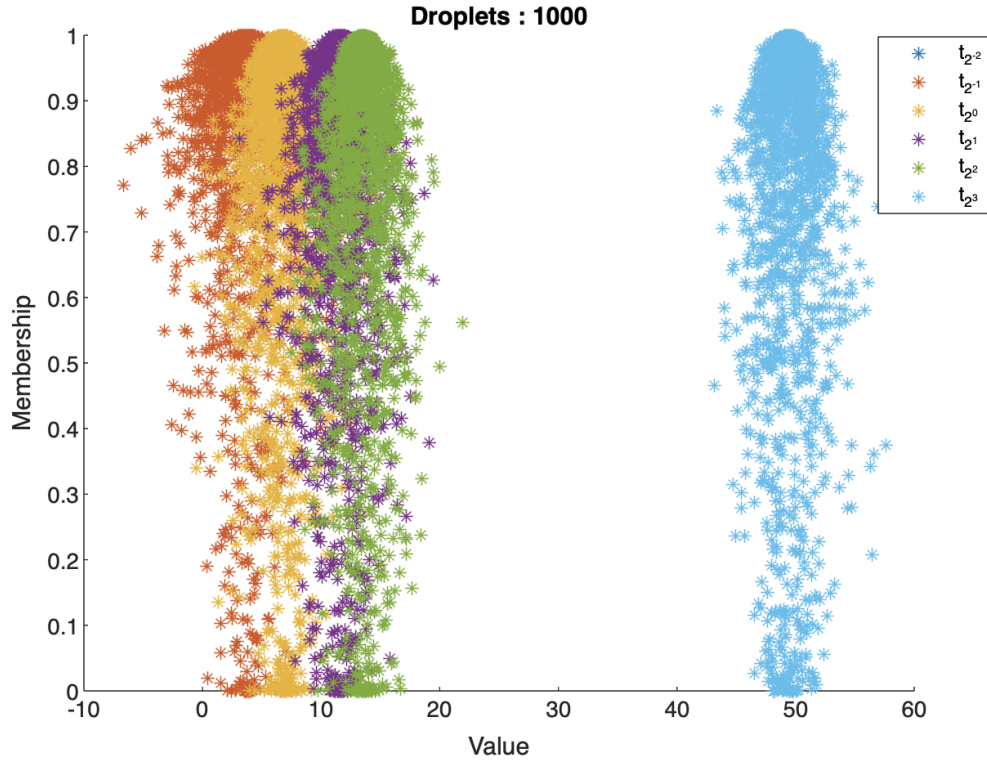


Figure 4.5: Evaluation information for asymmetric lower half cloud with respect to unbalanced linguistic terms for Case $n \neq m$.

cloud model that transforms different types of linguistic fuzzy sets and trapezium clouds. However, this model has a limitation that it is based on an LT set which is a purely balanced and symmetric term set. On the contrary, our model works for unbalanced linguistic data along with balanced linguistic data. Furthermore, the ULTS used in this chapter is not transformed from a balanced linguistic set. This property reduces the time and evaluation steps of the transforming method. Taking the example picked in the Fuzzy Trapezium Cloud model [101], the battlefield situation is analyzed and expressed in terms of different levels. Further, these levels correspond to linguistic terms that are symmetrically distributed. Nevertheless, in a real-life scenario, this situation might occur irregularly, and the level of performance may not be symmetrically distributed. Hence multiplicative unbalanced information could provide improved results.

Apart from the examples mentioned above, there are various real-life examples such as picking a multiplicative unbalanced cloud model in paper [144] for improving power grid failures analysis of the shortage of power supply and the pressure of dual control of energy consumption in [146] etc., provides significant results. Finally, on calculating the complexity of the proposed model given in

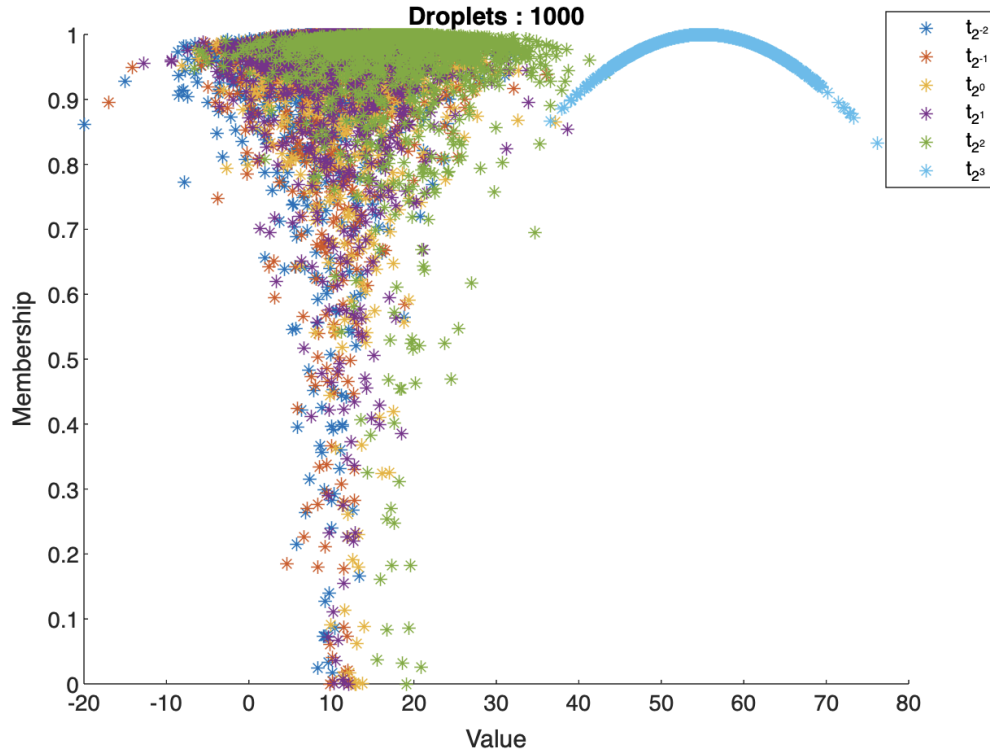


Figure 4.6: Evaluation information for asymmetric upper half cloud with respect to unbalanced linguistic terms for Case $n \neq m$.

Step	Complexity
δ_{a^ε}	$O(n+m+1)$
EX_{a^ε}	$O(n+m+1)$
EN_{a^ε}	$O(n+m+1)$
HE_{a^ε}	$O(n+m+1)$

where $n+m+1$ is the cardinality of the multiplicative ULTS.

Table 4.4: Computational complexity of the defined methodology

Table 4.4, it is foreseen that the complexity of the defined model is linear, i.e., $4O(n+m+1) \equiv O(n+m+1)$ whereas other existing cloud model [231] has a non-linear complexity. Hence our model is applicable to handle complex decision problems.

4.6 Conclusions

The normal cloud model is an excellent tool for converting qualitative concepts to quantitative information in linguistic decision-making problems. But the existing cloud models either cannot adequately describe the linguistic concepts or include information loss in the conversion process. Our work defines a compre-

hensive information representation model, the MUIC model, that can handle the multiplicative unbalanced data well. This model forms a bridge enabling efficient conversions between unbalanced linguistic information and clouds. A MULSF has been defined for this purpose. However, the LSFs existing in the literature map only a linguistic term to a particular specific value, thus creating information loss in the transformation process. To overcome this defect, we have defined a generalized MULSF that can convert the unbalanced linguistic data into an interval rather than a specific point. Thus the advantage of a MUIC model is that it overcomes the loss of information persisting in converting the unbalanced linguistic data to a specific value. Rather it maps an unbalanced linguistic term to a continuous interval. The generalised MULSF is thus stated to efficiently incorporate the disparities in spacing of linguistic terms in terms of representation and meaning by making a mapping from unbalanced linguistic terms to interval values. Also, a method based on the 3σ principle is proposed to determine the values of entropy and hyper entropy of the generalized MUIC. Lastly, this research provides a comparison method to rank the MUICs. The defined methodology has been implicated practically with the help of some examples.

Our method effectively addresses the linguistic imbalance, but it also has some limitations. It is noted that one must add a dummy variable to the set $\mathbb{L}$ to proceed with calculations for the point-based unbalanced cloud model for the case where the ULTS has an unequal number of linguistic terms spread on each side of the central term. Some MCDM problems may not have an accurate solution after adding this dummy variable. Therefore, further extension of this model is required to get over the corresponding limitation.

Chapter 5

Probabilistic Multiplicative Unbalanced Linguistic Game using Linguistic Cloud model

In many game theoretic environments, the precise, crisp values of payoffs are not always easy to gather as real-life decision-making problems are not inevitably precise and symmetrically distributed in nature. Probabilistic linguistic term sets (PLTSs) are regarded as a suitable tool for enunciating evaluators' complex linguistic perceptions more accurately within the intricate qualitative setting. Hence there is a need to explore matrix games having unbalanced probabilistic linguistic information, i.e., the matrix games in which the players provide or receive information that is of unbalanced probabilistic nature. To fill this research gap, this chapter¹ proposes a concept of probabilistic multiplicative unbalanced linguistic (PM-UL) game in which the players express their responses in terms of PM-UL term. This type of PM-UL game is a useful technique for multiple criteria analysis. Further, to thoroughly capture the uncertainty involved in the payoffs of such games, they are converted to probabilistic unbalanced linguistic clouds (PULCs) that describe the uncertainties of qualitative payoffs by implementing the uncertain transformation between PM-UL payoffs and quantitative values. The PULCs are represented by three numerical characteristics, i.e., expectation, entropy, and

¹The content of this chapter is based on the research paper, "Probabilistic Multiplicative Unbalanced Linguistic Game using Linguistic Cloud model", *The Journal of Supercomputing*, Springer (2024). <https://doi.org/10.1007/s11227-024-06240-4> (SCIE, Impact Factor : 3.3)

hypertrophy. Hence this model overcomes the inherent limitation of existing linguistic games where only the fuzziness of concepts is evaluated by providing a PULC strategy and PULC value of the game that involve the fuzziness as well as randomness as uncertainty prevailing in them. This procedure overcomes the loss of information occurring in the process of transformation of payoff values into quantitative ones. The defined methodology is implemented on a real-life working problem to demonstrate its effect and applicability in decision-making circumstances.

5.1 Introduction

The ambiguity present in a concept makes it difficult to assess a decision-making problem in a particular way. Hence it is necessary to consider the uncertainty pertaining to a DM's judgment to solve a problem underlying in a fuzzy environment. Although fuzzy games have provided phenomenal results in the literature, these are still incapable of addressing games having qualitative information as the best expression in strategic forms occurs in the form of natural language only. Thus linguistic variables [257] play an important role in real-life decision-making. However, to deal with the ambiguity involved in qualitative concepts, it is sometimes not possible to express the information with the help of a single particular term, or there are probabilities of the occurrence of a qualitative term attached to them. For example, if $\mathbb{LT} = \{\text{Good}, \text{Very Good}\}$ is a linguistic term set, then it might be possible that the player expresses his linguistic information as "50% chances of being good", then we might say that the linguistic term "Good" has a probability attached as "Good(0.5)". For expressing such type of information, there is a need to use PLTSs introduced in [174]. The PLTS enables the players or DMs to evaluate using numerous linguistic terms rather than a single term more efficiently by giving each term a weight through probability. Much research has been carried out based on PLTSs [12, 104, 238, 252, 254] as these types of sets allow the DMs to use more than one linguistic term with associated probabilities. Mi et al. [162] put forward this extensive and comprehensive way of expressing information in matrix games taking PLTS as input of matrix games. However, this PLTS consists of symmetrically distributed linguistic terms only. It is well known that real-life decision-making problems are not just based on symmetric data sets. Rather, a practical decision-making problem considers terms that are irregularly distributed. In order to deal with unbalanced linguistic information, Han et al. [87] established a probabilistic unbalanced linguistic term set (PULTS), in which the authors proposed a computational model based on copula and co-copula. However, an additive scale set is still used to represent the data. Therefore, it does not retain its literal meaning. This is the first research gap in the literature that is addressed in the present study.

To handle unbalanced linguistic probabilistic information, Malhotra & Gupta [154] introduced a probabilistic multiplicative ULTS (PM-ULTS) that takes into account

the irregular distribution of probabilistic linguistic terms and is sufficient enough to deal with all types of ULTSs. Further, they have solved matrix games with payoffs regarding PM-ULTS. However, their methodology is also limited in handling only one kind of uncertainty, i.e., fuzziness. However, the two most significant uncertainties in natural language are randomness and fuzziness, which have received much attention in artificial intelligence. It is typically exceedingly challenging to ascertain precise information or define the bounds of extensions if we only take fuzziness into account. For instance, there is no specific value that can be used to describe the concept's distinct border when we define the linguistic term "old". This is the second research gap addressed in the current study. In general, it is always possible for a linguistic variable to take on distinct meanings for different people, and even for the same person, to have different interpretations in different circumstances due to the abstractness and generality of notions. As a result, some notions frequently include randomness as well. Many linguistic terms have both randomness and fuzziness at the same time, despite the fact that they are of very different kinds. For instance, the linguistic terms "about", "maybe", "probably", etc., although representing random thoughts, also exhibit fuzziness due to the vague boundary of the extension, despite the fact that people disregard this when choosing the terms for emotional and mental expression. In light of this, it seems more practical for a formal representation model of concepts to simultaneously account for both randomness and fuzziness. Li et al. [111] presented a cloud model that can synthetically express the concept of randomness and fuzziness of decision information simultaneously to formally characterize the underlying relationship between randomness and fuzziness and effectively integrate both uncertainties. The cloud models use EX (expectation), EN (entropy), and HE (hypertrophy) to illustrate a linguistic concept. Since the development of cloud models, numerous academics have dedicated their efforts to advance cloud-specific studies [127, 132, 160, 161, 220, 223, 227, 228]. Although existing cloud models are believed to be able to preserve the original information, they are still unable to accurately represent labels that are asymmetrically distributed. To solve this problem, Wang et al. [231] proposed an Asymmetric Trapezoidal Clouds (ATCs) model as a solution to this problem. The EN and HE of the ATCs differ on the left and right sides in this model, and the authors expressed the expectation as an interval rather than a specific number. However, The ATC model is stipulated on a very complex computational model. Moreover, no concept of cloud

is explored to eradicate the randomness and fuzziness of the payoffs. Hence, in the current chapter, we solve two-person matrix games having payoffs in the form of PM-ULTS by dealing with both the randomness and fuzziness of the linguistic concepts. Further, the operational laws used to aggregate the payoffs in the paper [154] are based on probabilistic linguistic terms, while in this chapter, well-defined cloud aggregation operators are used that are already well-established in the literature and do not follow any complex representation. This is the third research gap to be addressed in the current chapter.

To fill all the discussed research gaps, the following contributions are offered in this chapter:

- The matrix games having payoffs in the form of PM-ULTS are considered and solved using the defined methodology. These PM-ULTSs can handle all types of regular and irregular distribution of linguistic concepts.
- The defined methodology is able to deal with the most important uncertainties, i.e., randomness and fuzziness underlying qualitative information.
- An unbalanced linguistic cloud model is used to transform the probabilistic multiplicative data into its numerical form without making any significant loss of information. The PM-UL variables that are used as payoffs of a game problem are transformed into PULCs to effectively capture the uncertainty of the probabilistic linguistic payoff.
- The dispersion of cloud drops is measured by standard deviation S.D. defined in the chapter, which suggests the relative stability distribution of the linguistic term. Also, the score function is defined to compare different PULCs.
- To address the practical issues, the defined approach is then applied to address matrix games having PM-UL payoffs. The values of the players are defined in terms of PULCs. Also, the proposed method provides more accurate results by providing the values and strategies of the game in terms of expectation, entropy, and hyperentropy. The expectation is the most representative and typical sample of the qualitative concept, while entropy represents the randomness and fuzziness extent of the cloud drop. Hyperentropy reflects the uncertainty degree of entropy.

The remaining chapter unfolds as follows. Section 5.2 reviews the basic concept of PM-ULTS that is required in the chapter ahead. Section 5.3 transforms PM-UL variables into PULCs and operational laws. In Section 5.4, the application of the PULC model in two-person constant sum matrix games is illustrated by first defining a PM-ULC game (PM-ULCG) along with the methodology for solving PM-ULCG followed by a numerical problem. Comparative analysis and discussion are provided in Section 5.5. Section 5.6 concludes the chapter along with the future scope.

5.2 Probabilistic Multiplicative Unbalanced Linguistic Term Set (PM-ULTS): Basic Concepts [154]

This section reviews some basic concepts of the PM-ULTS that consists of unbalanced linguistic terms having probabilities associated to them.

Definition 5.2.1. [154] Let $\mathbb{LS} = \{t_{a^\varepsilon} | \varepsilon = -n, \dots, 0, \dots, m\}$ be a ULTS, where n and m may or may not be equal, $a > 1$. Then a PM-ULTS is defined as

$$\mathbb{LS}(p) = \{t_{a^\varepsilon}^k(p^k) | k = 1, 2, \dots, \#\mathbb{LS}(p)\}$$

where $t_{a^\varepsilon}^k \in \mathbb{LS}$, $p^k \geq 0$, $\sum_{k=1}^{\#\mathbb{LS}(p)} p^k$ may or may not be equal to 1 and $\#\mathbb{LS}(p)$ is the cardinality of all the given distinct multiplicative unbalanced linguistic terms in $\mathbb{LS}(p)$. The term $t_{a^\varepsilon}^k(p^k)$ is interpreted as unbalanced linguistic term $t_{a^\varepsilon}^k$ associated with the probability p^k .

Note 5.2.1. If $\sum_{k=1}^{\#\mathbb{LS}(p)} p^k < 1$ or $\sum_{k=1}^{\#\mathbb{LS}(p)} p^k > 1$, then the probability information within the set is incomplete and we normalize the incomplete PM-ULTS to transform the set into complete PM-ULTS using the following process.

1. Normalize the probability. If $\sum_{k=1}^{\#\mathbb{LS}(p)} p^k \neq 1$, then the given PM-ULTS is defined as $\widetilde{\mathbb{LS}}(p) = \{t_{a^\varepsilon}^k(\bar{p}^k) | \sum_{k=1}^{\#\mathbb{LS}(p)} \bar{p}^k = 1\}$, where $\bar{p}^k = \frac{p^k}{\sum_{k=1}^{\#\mathbb{LS}(p)} p^k}$, $t_{a^\varepsilon} \in \mathbb{LS}$, $p^k \geq 0$, $k = 1, \dots, \#\mathbb{LS}(p)$;
2. Normalize the granularity. If $\mathbb{LS}_1(p)$ and $\mathbb{LS}_2(p)$ are two PM-ULTS. If $\#\mathbb{LS}_1(p) < \#\mathbb{LS}_2(p)$, add $\#\mathbb{LS}_2(p) - \#\mathbb{LS}_1(p)$ multiplicative unbalanced linguistic terms to $\mathbb{LS}_1(p)$

Definition 5.2.2. [154] Let $\mathbb{LS}(p) = \{t_{a^\varepsilon}^k(\bar{p}^k) | k = 1, \dots, \#\mathbb{LS}(p)\}$ be a normalized PM-ULTS. The negation of $\mathbb{LS}(p)$ is defined as $Neg(\mathbb{LS}(p)) = \{neg(t_{a^\varepsilon}^k)(\bar{p}^k) | k = 1, \dots, \#\mathbb{LS}(p)\}$.

Definition 5.2.3. [154] Let $\mathbb{LS}_1(p_1) = \{t_{a^\varepsilon}^k(p^k) | k = 1, 2, \dots, \#\mathbb{LS}_1(p_1)\}$ and $\mathbb{LS}_2(p_2) = \{t_{a^\vartheta}^k(p^k) | k = 1, 2, \dots, \#\mathbb{LS}_2(p_2)\}$, $\varepsilon, \vartheta = -n, \dots, m$ be any two ordered PM-ULTS. Then the operational laws are given as follows:

1. $\mathbb{LS}_1(p_1) \oplus \mathbb{LS}_2(p_2) = \{t_{a^\varepsilon \oplus a^\vartheta}^k(\bar{p}_1^k \oplus \bar{p}_2^k) | k = 1, 2, \dots, \#\mathbb{LS}_1(p_1)\}$
2. $\mathbb{LS}_1(p_1) \otimes \mathbb{LS}_2(p_2) = \{t_{a^\varepsilon \otimes a^\vartheta}^k(\bar{p}_1^k \otimes \bar{p}_2^k) | k = 1, 2, \dots, \#\mathbb{LS}_1(p_1)\}$
3. Let $\lambda \in (0, 1)$ be any arbitrary scalar, $\lambda \odot \mathbb{LS}(p) = \{t_{\lambda \odot a^\varepsilon}^k(\bar{p}^k) | k = 1, 2, \dots, \#\mathbb{LS}(p)\}$
4. $(\mathbb{LS}(p))^\lambda = \{t_{a^{\lambda\varepsilon}}^k(\bar{p}^k) | k = 1, 2, \dots, \#\mathbb{LS}(p)\}$

Apart from the above defined basic concept, this chapter makes use of some basic concepts of matrix games, defined earlier in Section 1.5.1 of the introductory chapter. Also, the current chapter implements the theory of unbalanced linguistic cloud model proposed in Chapter 4 of this thesis to solve matrix games that have probabilistic multiplicative unbalanced linguistic payoffs. Based on these theories, the definition of PM-UL value of a game follows:

Definition 5.2.4. Suppose v is the numerical value of the game, then the corresponding PM-UL value can be transformed using the following procedure:

1. First, find $\Omega^{-1}(v) \sim (t_{a^\varepsilon}, \gamma)$.
2. The probabilistic value would be $v(p)$ given by $v(p) = \{t_{a^\varepsilon}^k(\theta) | k = 1, \dots, \#\mathbb{LS}(p)\}$

$$\text{where, } \theta = \begin{cases} 1 & \text{if } \gamma = 0 \\ \gamma + 0.5 & \text{otherwise} \end{cases}$$

5.3 Transformation of PM-UL variables to PULCs and operational laws

In this section, we present how to transform a PM-UL variable to its corresponding PULC, and a numerical example is shown. Further, the operational laws and score function of PULCs are defined.

5.3.1 Method for transformation of probabilistic unbalanced linguistic variables to probabilistic unbalanced linguistic clouds (PULCs)

Definition 5.3.1. Let $\mathbb{LS} = \{t_{a^\varepsilon} | \varepsilon = -n, \dots, 0, \dots, m\}$ be a ULTS where n and m may or may not be equal and $a > 1$. Suppose p_ε be any probability associated with the unbalanced linguistic term t_{a^ε} such that $\mathbb{LS}(p) = \{t_{a^\varepsilon}(p_\varepsilon) | \varepsilon = -n, \dots, m\}$ denotes the linguistic set representing linguistic terms having attached probabilities.

Consider a valid universe $X = [X_{\min}, X_{\max}]$. Now to represent the linguistic terms of the set $\mathbb{LS}(p)$, the procedure is as given below.

- First transform the linguistic terms into their corresponding cloud using the method given in Definition 4.3.2 and 4.3.3.
- Now consider the attached probabilities with the linguistic terms by multiplying these probabilities with the above obtained cloud. Hence, a cloud corresponding to a linguistic term $t_{a^\varepsilon}(p_\varepsilon)$ is given as $\mathbb{Y}_\varepsilon(p_\varepsilon EX_{a^\varepsilon}, p_\varepsilon EN_{a^\varepsilon}, p_\varepsilon HE_{a^\varepsilon})$.

Note. The above mentioned procedure is used in the scenario $n = m$. In case $n \neq m$, then we add a dummy linguistic term with probability '0' in $\mathbb{LS}(p)$ so that $n = m$, where the linguistic term is added on the side of $\min(m, n)$.

Example 5.3.1. Given the universe $[X_{\min}, X_{\max}] = [0, 7]$, let $\mathbb{LS} = \{t_{2^{-2}} : \text{Terrible}, t_{2^{-1}} : \text{Poor}, t_{2^0} : \text{Middle}, t_{2^1} : \text{Satisfactory}, t_{2^2} : \text{Good}, t_{2^3} : \text{Excellent}\}$.

As $n \neq m$, we add a dummy linguistic term on the left hand side of $\mathbb{LS}$ with probability '0'. Consider $\mathbb{LS}(p) = \{t_{2^{-3}}(0), t_{2^{-2}}(0), t_{2^{-1}}(0.1), t_{2^0}(0.4), t_{2^1}(0), t_{2^2}(0.2), t_{2^3}(0.3)\}$.

By following the steps mentioned above, the cloud is generated as follows.

1. Setting $b = 1.4$, first we calculate δ_{a^ε} for each t_{a^ε} .

$$\delta_{2^{-3}} = 0 \quad \delta_{2^{-2}} = 0.7467 \quad \delta_{2^{-1}} = 1.2801 \quad \delta_{2^0} = 1.6611 \quad \delta_{2^1} = 2.0421 \quad \delta_{2^2} = 2.5755 \quad \delta_{2^3} = 3.3222$$

2. $EX_{2^{-3}} = 0 \quad EX_{2^{-2}} = 5.22718 \quad EX_{2^{-1}} = 8.9607 \quad EX_{2^0} = 11.6277 \quad EX_{2^1} = 14.2947 \quad EX_{2^2} = 18.0285 \quad EX_{2^3} = 23.2557$

3. $EN'_{2^{-3}} = 7.7519 \quad EN'_{2^{-2}} = 1.9632 \quad EN'_{2^{-1}} = -2.1715 \quad EN'_{2^0} = -5.125 \quad EN'_{2^1} = 15.8308$
 $EN'_{2^2} = 19.9652 \quad EN'_{2^3} = 25.7538$
 $\therefore \max EN'_{a^\varepsilon} = 25.7538$

Next, $EN_{2^{-3}} = EN_{2^3} = 22.8595$ $EN_{2^{-2}} = EN_{2^2} = 20.5164$ $EN_{2^{-1}} = EN_{2^1} = 16.2241$ $EN_{2^0} = 14.8457$

4. $HE_{2^{-3}} = HE_{2^3} = 20.9258$ $HE_{2^{-2}} = HE_{2^2} = 23.9762$ $HE_{2^{-1}} = HE_{2^1} = 26.7236$ $HE_{2^0} = 21.944$

Hence the probabilistic multiplicative unbalanced clouds corresponding to each linguistic term of the set $\mathbb{LS}(p)$ are given as:

$t_{2^{-3}}(0) \rightarrow (0, 0, 0)$ $t_{2^{-2}}(0) \rightarrow (0, 0, 0)$ $t_{2^{-1}}(0.1) \rightarrow (0.89607, 1.6224, 2.6723)$ $t_{2^0}(0.4) \rightarrow (4.651, 5.9382, 8.777)$ $t_{2^1}(0) \rightarrow (0, 0, 0)$ $t_{2^2}(0.2) \rightarrow (3.6057, 4.1032, 4.7952)$ $t_{2^3}(0.3) \rightarrow (6.9767, 6.8578, 6.2777)$

These probabilistic multiplicative unbalanced linguistic clouds are represented in Figure 5.1. In the graph, the dispersing extent of cloud drops is depicted by entropy and the hy-

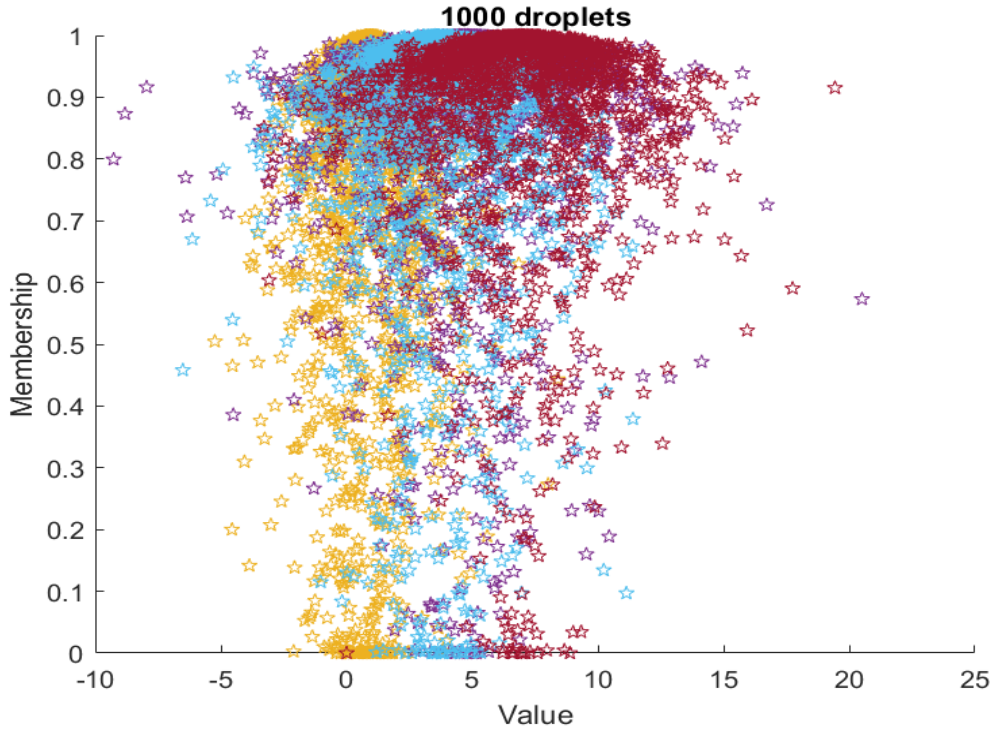


Figure 5.1: Representation of probabilistic multiplicative unbalanced linguistic clouds with respect to linguistic terms having attached probabilities based on 1000 droplets

perentropy EN and HE , respectively. It is also the measurement of fuzziness, representing the scope of the universe that can be accepted by the concept.

5.3.2 Operations and score functions of Probabilistic Unbalanced Linguistic Clouds

In this section, we define some operational laws for PULCs based on the operators defined for normal clouds.

Definition 5.3.2. Given the PULCs $\mathbb{Y}_\varepsilon = (p_\varepsilon EX_{a^\varepsilon}, p_\varepsilon EN_{a^\varepsilon}, p_\varepsilon HE_{a^\varepsilon})$, the basic operations are developed as follows:

1. $\sum_{\varepsilon=-n}^m (p_\varepsilon EX_{a^\varepsilon}, p_\varepsilon EN_{a^\varepsilon}, p_\varepsilon HE_{a^\varepsilon}) = \left(\sum_{\varepsilon=-n}^m p_\varepsilon EX_{a^\varepsilon}, \sqrt{\sum_{\varepsilon=-n}^m (p_\varepsilon EN_{a^\varepsilon})^2}, \sqrt{\sum_{\varepsilon=-n}^m (p_\varepsilon HE_{a^\varepsilon})^2} \right)$
2. $\lambda(p_\varepsilon EX_{a^\varepsilon}, p_\varepsilon EN_{a^\varepsilon}, p_\varepsilon HE_{a^\varepsilon}) = (\lambda p_\varepsilon EX_{a^\varepsilon}, \lambda p_\varepsilon EN_{a^\varepsilon}, \lambda p_\varepsilon HE_{a^\varepsilon})$, where λ is any real scalar.
3. Let $\mathbb{Y}_\varepsilon(p_\varepsilon EX_{a^\varepsilon}, p_\varepsilon EN_{a^\varepsilon}, p_\varepsilon HE_{a^\varepsilon})$, $\varepsilon = -n, \dots, m$ be a collection of PULCs with associated weights $\omega_\varepsilon \in [0, 1]$ and $\sum_{\varepsilon=-n}^m \omega_\varepsilon = 1$. Then their aggregated value with use of the weighted average (WA) operator is still a cloud and is denoted as follows:

$$WA(Y_{-n}, \dots, Y_m) = \left(\sum_{\varepsilon=-n}^m \omega_\varepsilon p_\varepsilon EX_{a^\varepsilon}, \sqrt{\sum_{\varepsilon=-n}^m \omega_\varepsilon (p_\varepsilon EN_{a^\varepsilon})^2}, \sqrt{\sum_{\varepsilon=-n}^m \omega_\varepsilon (p_\varepsilon HE_{a^\varepsilon})^2} \right)$$

The score function is a crucial tool for obtaining the ranking results during the decision-making process. For the normal clouds, a number of score functions have previously been defined. Here, we define the score function for PULCs. Before presenting the new function, we first introduce an important concept, the variance of a normal cloud, which is developed by Li et al. [162].

Definition 5.3.3. Let $(p_\varepsilon EX_{a^\varepsilon}, p_\varepsilon EN_{a^\varepsilon}, p_\varepsilon HE_{a^\varepsilon})$ be a PULC with $n + m + 1$ generated cloud drops, and the variance of these cloud drops can be computed as

$$V = (p_\varepsilon EN_{a^\varepsilon})^2 + (p_\varepsilon HE_{a^\varepsilon})^2$$

Hence, the standard deviation follows $S.D. = \sqrt{V}$.

The standard deviation ($S.D.$), which can be used to express the ambiguity and fluctuation of a quantitative notion, is used to measure the dispersion of cloud drops. A low standard deviation indicates that the distribution of the linguistic term is reasonably steady. A high standard deviation suggests stability even when the

linguistic concept has a high degree of ambiguity and volatility. In this case, the relationship between EX and $S.D.$ can be seen as the expected utility and utility dispersion of the utility theory. Therefore, the choice for an option (alternative) can be expressed as a linear trade-off between expected utility and utility dispersion. In response to this approach, we construct a score formula below to compare PULCs.

Definition 5.3.4. Given a PULC $(p_\varepsilon EX_{a^\varepsilon}, p_\varepsilon EN_{a^\varepsilon}, p_\varepsilon HE_{a^\varepsilon})$, then a score function of PULC is defined as

$$\hat{S} = p_\varepsilon EX_{a^\varepsilon} - \chi \cdot S.D.$$

where $S.D.$ is the standard deviation of a PULC, $\chi \in [0, 1]$ refers to a coefficient to capture the attitude towards the standard deviation. The value of χ can be chosen based on actual decision-making circumstances. $\chi = 0$ signifies that the DMs do not care about the uncertainty and fluctuation of the assessment information and simply take into account the average level, which is the expectation EX . The greater the value of χ , the greater the weight given to $S.D.$

The score function $\hat{S}$ is monotonically increasing w.r.t. EX while monotonically decreasing w.r.t. EN and HE . On the basis of score function, two PULCs may be compared in the following manner.

Definition 5.3.5. Let $\mathbb{Y}_1$ and $\mathbb{Y}_2$ be two PULCs, and $\hat{S}_1$ and $\hat{S}_2$ denote their score functions, respectively, then

- If $\hat{S}_1 > \hat{S}_2$, then we prefer $\mathbb{Y}_1$ over $\mathbb{Y}_2$ and denote it as $\mathbb{Y}_1 > \mathbb{Y}_2$.
- $\hat{S}_1 = \hat{S}_2$ represents that both the PULCs are equally preferred, and are denoted as $\mathbb{Y}_1 = \mathbb{Y}_2$.
- $\max\{\mathbb{Y}_\varepsilon | \varepsilon = -n, \dots, m\} = \mathbb{Y}_{\max}$ such that $\hat{S}(\mathbb{Y}_{\max}) \geq \hat{S}(\mathbb{Y}_\varepsilon) \forall \varepsilon$.
- $\min\{\mathbb{Y}_\varepsilon | \varepsilon = -n, \dots, m\} = \mathbb{Y}_{\min}$ such that $\hat{S}(\mathbb{Y}_{\min}) \leq \hat{S}(\mathbb{Y}_\varepsilon) \forall \varepsilon$.

Example 5.3.2. Consider the problem given in Example [5.3.1](#), the set consisting of linguistic terms with attached probabilities is $\mathbb{L}\mathbb{S}(p) = \{t_{2^{-3}}(0), t_{2^{-2}}(0), t_{2^{-1}}(0.1), t_{2^0}(0.4), t_{2^1}(0), t_{2^2}(0.2), t_{2^3}(0.3)\}$. The PULCs obtained from these probabilistic unbalanced linguistic terms are

$$\mathbb{Y}_1(0, 0, 0) \quad \mathbb{Y}_2(0, 0, 0) \quad \mathbb{Y}_3(0.89607, 1.6224, 2.6723) \quad \mathbb{Y}_4(4.651, 5.9382, 8.777)$$

$$\mathbb{Y}_5(0,0,0) \quad \mathbb{Y}_6(3.6057,4.1032,4.7952) \quad \mathbb{Y}_7(6.9767,6.8578,6.2777)$$

Using the definition of score function, we find scores of these PULCs for some random values of χ and provide their rankings in Table 5.1. It can be observed that there are different rankings for different values of χ depending upon the DM's choice of giving weightage to expectation and standard deviation.

Value of χ	$\hat{S}(\mathbb{Y}_1)$	$\hat{S}(\mathbb{Y}_2)$	$\hat{S}(\mathbb{Y}_3)$	$\hat{S}(\mathbb{Y}_4)$	$\hat{S}(\mathbb{Y}_5)$	$\hat{S}(\mathbb{Y}_6)$	$\hat{S}(\mathbb{Y}_7)$	Score Ranking	PULC Ranking
0	0	0	0.896	4.651	0	3.605	6.277	$\hat{S}(\mathbb{Y}_7) > \hat{S}(\mathbb{Y}_4) > \hat{S}(\mathbb{Y}_6) > \hat{S}(\mathbb{Y}_3) > \hat{S}(\mathbb{Y}_1) = \hat{S}(\mathbb{Y}_2) = \hat{S}(\mathbb{Y}_5)$	$\mathbb{Y}_7 > \mathbb{Y}_4 > \mathbb{Y}_6 > \mathbb{Y}_3 > \mathbb{Y}_1 = \mathbb{Y}_2 = \mathbb{Y}_5$
0.1	0	0	0.583	3.591	0	2.974	6.046	$\hat{S}(\mathbb{Y}_7) > \hat{S}(\mathbb{Y}_4) > \hat{S}(\mathbb{Y}_6) > \hat{S}(\mathbb{Y}_3) > \hat{S}(\mathbb{Y}_1) = \hat{S}(\mathbb{Y}_2) = \hat{S}(\mathbb{Y}_5)$	$\mathbb{Y}_7 > \mathbb{Y}_4 > \mathbb{Y}_6 > \mathbb{Y}_3 > \mathbb{Y}_1 = \mathbb{Y}_2 = \mathbb{Y}_5$
0.2	0	0	0.27	2.531	0	2.343	5.117	$\hat{S}(\mathbb{Y}_7) > \hat{S}(\mathbb{Y}_4) > \hat{S}(\mathbb{Y}_6) > \hat{S}(\mathbb{Y}_3) > \hat{S}(\mathbb{Y}_1) = \hat{S}(\mathbb{Y}_2) = \hat{S}(\mathbb{Y}_5)$	$\mathbb{Y}_7 > \mathbb{Y}_4 > \mathbb{Y}_6 > \mathbb{Y}_3 > \mathbb{Y}_1 = \mathbb{Y}_2 = \mathbb{Y}_5$
0.3	0	0	-0.041	1.471	0	1.712	4.187	$\hat{S}(\mathbb{Y}_7) > \hat{S}(\mathbb{Y}_6) > \hat{S}(\mathbb{Y}_4) > \hat{S}(\mathbb{Y}_1) = \hat{S}(\mathbb{Y}_2) = \hat{S}(\mathbb{Y}_5) > \hat{S}(\mathbb{Y}_3)$	$\mathbb{Y}_7 > \mathbb{Y}_6 > \mathbb{Y}_4 > \mathbb{Y}_1 = \mathbb{Y}_2 = \mathbb{Y}_5 > \mathbb{Y}_3$
0.4	0	0	-0.354	0.412	0	1.081	3.257	$\hat{S}(\mathbb{Y}_7) > \hat{S}(\mathbb{Y}_6) > \hat{S}(\mathbb{Y}_4) > \hat{S}(\mathbb{Y}_1) = \hat{S}(\mathbb{Y}_2) = \hat{S}(\mathbb{Y}_5) > \hat{S}(\mathbb{Y}_3)$	$\mathbb{Y}_7 > \mathbb{Y}_6 > \mathbb{Y}_4 > \mathbb{Y}_1 = \mathbb{Y}_2 = \mathbb{Y}_5 > \mathbb{Y}_3$
0.5	0	0	-0.667	-0.647	0	0.45	2.328	$\hat{S}(\mathbb{Y}_7) > \hat{S}(\mathbb{Y}_6) > \hat{S}(\mathbb{Y}_1) = \hat{S}(\mathbb{Y}_2) = \hat{S}(\mathbb{Y}_5) > \hat{S}(\mathbb{Y}_4) > \hat{S}(\mathbb{Y}_3)$	$\mathbb{Y}_7 > \mathbb{Y}_6 > \mathbb{Y}_1 = \mathbb{Y}_2 = \mathbb{Y}_5 > \mathbb{Y}_4 > \mathbb{Y}_3$
0.6	0	0	-0.979	-1.707	0	-0.18	1.398	$\hat{S}(\mathbb{Y}_7) > \hat{S}(\mathbb{Y}_1) = \hat{S}(\mathbb{Y}_2) = \hat{S}(\mathbb{Y}_5) > \hat{S}(\mathbb{Y}_6) > \hat{S}(\mathbb{Y}_3) > \hat{S}(\mathbb{Y}_4)$	$\mathbb{Y}_7 > \mathbb{Y}_1 = \mathbb{Y}_2 = \mathbb{Y}_5 > \mathbb{Y}_6 > \mathbb{Y}_3 > \mathbb{Y}_4$
0.7	0	0	-1.292	-2.766	0	-0.812	0.468	$\hat{S}(\mathbb{Y}_7) > \hat{S}(\mathbb{Y}_1) = \hat{S}(\mathbb{Y}_2) = \hat{S}(\mathbb{Y}_5) > \hat{S}(\mathbb{Y}_6) > \hat{S}(\mathbb{Y}_3) > \hat{S}(\mathbb{Y}_4)$	$\mathbb{Y}_7 > \mathbb{Y}_1 = \mathbb{Y}_2 = \mathbb{Y}_5 > \mathbb{Y}_6 > \mathbb{Y}_3 > \mathbb{Y}_4$
0.8	0	0	-1.604	-3.826	0	-1.443	-0.461	$\hat{S}(\mathbb{Y}_1) = \hat{S}(\mathbb{Y}_2) = \hat{S}(\mathbb{Y}_5) > \hat{S}(\mathbb{Y}_7) > \hat{S}(\mathbb{Y}_6) > \hat{S}(\mathbb{Y}_3) > \hat{S}(\mathbb{Y}_4)$	$\mathbb{Y}_1 = \mathbb{Y}_2 = \mathbb{Y}_5 > \mathbb{Y}_7 > \mathbb{Y}_6 > \mathbb{Y}_3 > \mathbb{Y}_4$
0.9	0	0	-1.917	-4.886	0	-2.074	-1.39	$\hat{S}(\mathbb{Y}_1) = \hat{S}(\mathbb{Y}_2) = \hat{S}(\mathbb{Y}_5) > \hat{S}(\mathbb{Y}_7) > \hat{S}(\mathbb{Y}_3) > \hat{S}(\mathbb{Y}_6) > \hat{S}(\mathbb{Y}_4)$	$\mathbb{Y}_1 = \mathbb{Y}_2 = \mathbb{Y}_5 > \mathbb{Y}_7 > \mathbb{Y}_3 > \mathbb{Y}_6 > \mathbb{Y}_4$
1	0	0	-2.239	-5.946	0	-2.705	-2.32	$\hat{S}(\mathbb{Y}_1) = \hat{S}(\mathbb{Y}_2) = \hat{S}(\mathbb{Y}_5) > \hat{S}(\mathbb{Y}_3) > \hat{S}(\mathbb{Y}_7) > \hat{S}(\mathbb{Y}_6) > \hat{S}(\mathbb{Y}_4)$	$\mathbb{Y}_1 = \mathbb{Y}_2 = \mathbb{Y}_5 > \mathbb{Y}_3 > \mathbb{Y}_7 > \mathbb{Y}_6 > \mathbb{Y}_4$

Table 5.1: PULC rankings based on scores for different values of χ

Now, from the aspect of information representation form, the PM-UL information is a flexible tool for CW. Also the unbalanced linguistic cloud model has provided wonderful results to deal with uncertainty of a linguistic concept. But in the framework of game theory, it is still not investigated how to solve a game problem whose payoffs are of the form PM-UL by capturing the randomness and fuzziness involved in the payoffs. In this regard, Section 5.4 presents the definition of a PM-UL game (PM-ULG) and a methodology is given to solve such type of game by converting its payoffs to PULCs and by treating it as a PM-UL cloud game (PM-ULCG).

5.4 Two-Person Probabilistic Multiplicative Unbalanced Linguistic Cloud Games (PM-ULCG)

In uncertain situations, the probabilistic unbalanced linguistic matrix game with payoffs as clouds is introduced in this section to model uncertain cloud payoff values. The payoff matrix with probabilistic unbalanced linguistic information can be denoted as:

Player <i>I</i>	Player <i>II</i>			
	Strategy 1	Strategy 2	...	Strategy <i>s</i>
Strategy 1	$\tilde{a}_{11}$	$\tilde{a}_{12}$	...	$\tilde{a}_{1s}$
Strategy 2	$\tilde{a}_{21}$	$\tilde{a}_{12}$	...	$\tilde{a}_{2s}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Strategy <i>r</i>	$\tilde{a}_{r1}$	$\tilde{a}_{r2}$	...	$\tilde{a}_{rs}$

where each $\tilde{a}_{ij} \in \mathbb{LS}(p)$ is a probabilistic unbalanced linguistic term.

5.4.1 PM-ULCG: Definition and Meaning

A two-person constant sum probabilistic multiplicative unbalanced linguistic cloud game is defined as $(S^r, S^s, \mathbb{LS}(p), \tilde{\mathbb{A}})$ where S^r and S^s are the strategy sets of Player *I* and Player *II*, respectively. $\mathbb{LS}(p)$ is the set of PM-UL terms having associated probabilities. Now to deal with the uncertainties like randomness and fuzziness underlying the PM-UL terms $\tilde{a}_{ij}$, we first convert the payoffs in the form of PULC. Hence finally, $\tilde{\mathbb{A}}$ is the payoff matrix of Player *I* corresponding to Player *II* given as

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1s} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{r1} & \tilde{a}_{r2} & \cdots & \tilde{a}_{rs} \end{pmatrix}$$

where each $\tilde{a}_{ij} = (p_{\varepsilon}EX_{a^{\varepsilon}}, p_{\varepsilon}EN_{a^{\varepsilon}}, p_{\varepsilon}HE_{a^{\varepsilon}})$ is a PULC. On the contrary, $neg(\tilde{\mathbb{A}})$ is the payoff matrix of Player *II*, where $neg(\tilde{\mathbb{A}}) = neg(\tilde{a}_{ij})_{r \times s}$, where $neg(\tilde{a}_{ij})$ is as defined in Definition [5.3.2](#).

Conventionally, Player *I* is the row player and wants to maximize his payoff, while column player *II* wants to minimize row player's payoff so as to maximize his own payoff. The rows are pure strategies for Player *I* and columns for Player *II*.

$$v^- = \max_{i=1, \dots, r} \min_{j=1, \dots, s} \tilde{a}_{ij}$$

called the lower value of the game, and is the minimum amount that Player *I* is guaranteed to receive. Here $\tilde{a}_{ij}$ is a PM-UL term.

On the other hand,

$$\omega^- = \max_{i=1, \dots, r} \min_{j=1, \dots, s} neg(\tilde{a}_{ij})$$

is the minimum amount that Player *II* is guaranteed to receive. Obviously,

$$v^+ = \text{neg}(\omega^-) = \min_{j=1,\dots,s} \max_{i=1,\dots,r} \tilde{a}_{ij}$$

is the upper value of the game and is the guaranteed maximum amount that Player *II* can loose. The game is said to have a value $v = (EX_{a^\varepsilon}, EN_{a^\varepsilon}, HE_{a^\varepsilon})$ if $v^- = v^+ = v$, also called the saddle point in pure strategies. However, in most two-person zero sum games a saddle point in pure strategies will not exist because that would interpret that the players must always play in a similar manner. But that is not always possible as in such case, the opposing player will be able to figure out the player's strategy in advance.

Definition 5.4.1. A mixed cloud strategy is a vector $\varrho = (\varrho_1, \varrho_2, \dots, \varrho_r)$ for Player *I* and $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_s)$ for Player *II*, where

$$\varrho_i \geq 0, \varrho_i = (EX_{a^\varepsilon}, EN_{a^\varepsilon}, HE_{a^\varepsilon})_i \text{ so that } \sum_{i=1}^r \varrho_i = 1 \implies \sum_{i=1}^r (EX_{a^\varepsilon})_i = 1, \sqrt{\sum_{i=1}^r (EN_{a^\varepsilon})_i^2} = 1, \sqrt{\sum_{i=1}^r (HE_{a^\varepsilon})_i^2} = 1$$

and,

$$\varphi_j \geq 0, \varphi_j = (EX_{a^\theta}, EN_{a^\theta}, HE_{a^\theta})_j \text{ so that } \sum_{j=1}^s \varphi_j = 1 \implies \sum_{j=1}^s (EX_{a^\theta})_j = 1, \sqrt{\sum_{j=1}^s (EN_{a^\theta})_j^2} = 1, \text{ and,}$$

$$\sqrt{\sum_{j=1}^s (HE_{a^\theta})_j^2} = 1$$

The set of mixed strategies having k components is denoted as

$$S^k = \left\{ (z_1, z_2, \dots, z_k) | z_p \geq 0, \sum_{p=1}^k (EX_{a^\varepsilon})_p = 1, \sqrt{\sum_{p=1}^k (EN_{a^\varepsilon})_p^2} = 1, \text{ and } \sqrt{\sum_{p=1}^k (HE_{a^\varepsilon})_p^2} = 1 \right\}$$

such that $\varrho \in S^r$ and $\varphi \in S^s$.

Since it is mentioned that a PM-ULCG has a saddle point in pure strategies if $v^- = v^+$. However, in practical real life scenarios, this is not always the case that saddle point exists. Hence in that case, the following inequality holds in general.

Theorem 5.4.1. Suppose v^- and v^+ be the lower and upper values of a PM-ULCG respectively, and let both the values exist. Then $v^- \leq v^+$.

Proof. For any column j , we know that for any fixed row i , by Definition 5.3.5 $\min_{j=1,\dots,s} \tilde{\mathbf{a}}_{ij} \leq \tilde{\mathbf{a}}_{ij}$. Taking max over $i = 1, \dots, r$ both the sides,

$$v^- = \max_{i=1,\dots,r} \min_{j=1,\dots,s} \tilde{\mathbf{a}}_{ij} \leq \max_{i=1,\dots,r} \tilde{\mathbf{a}}_{ij}$$

This is true for any column $j = 1, \dots, s$. The LHS is just a number independent of i and j , and is smaller than RHS for any j . This means that $v^- \leq \min_{j=1,\dots,s} \max_{i=1,\dots,r} \tilde{\mathbf{a}}_{ij} = v^+$. This proves the theorem. $\square$

In crisp games, the expected payoff of the player is presented in the form of mathematical expectation; but the uncertainty pertaining in the PM-ULCG does not allow the expected payoffs to be defined simply as in crisp games. Thus it is necessary to define PM-UL cloud expected payoffs.

Definition 5.4.2. Let $\varrho \in S^r$ and $\varphi \in S^s$ be the mixed strategies associated with Player *I* and Player *II* respectively. Then the PM-UL cloud expected payoff of Player *I* is defined as:

$$E(\varrho, \varphi) = \bigoplus_{i=1}^r \left(\varrho_i \left(\bigoplus_{j=1}^s \tilde{\mathbf{a}}_{ij} \varphi_j \right) \right)$$

where $\tilde{\mathbf{a}}_{ij} = (p_\varepsilon EX_{a^\varepsilon}, p_\varepsilon EN_{a^\varepsilon}, p_\varepsilon HE_{a^\varepsilon})$, $\varepsilon = -n, \dots, m$ is a PULC.

$$\Rightarrow E(\varrho, \varphi) = \left(\sum_{i=1}^r \sum_{j=1}^s \varrho_i \varphi_j p_\varepsilon EX_{a^\varepsilon}, \sqrt{\sum_{i=1}^r \sum_{j=1}^s \varrho_i \varphi_j (p_\varepsilon EN_{a^\varepsilon})^2}, \sqrt{\sum_{i=1}^r \sum_{j=1}^s \varrho_i \varphi_j (p_\varepsilon HE_{a^\varepsilon})^2} \right)$$

and this expected payoff is further a PULC. Similarly, the PM-UL cloud expected payoff of Player *II* is given by $Neg(E(\varrho, \varphi))$.

5.4.2 PM-ULCG: Methodology and Solution

This section discusses the probabilistic unbalanced linguistic cloud linear programming (PULCLP) method for solving matrix games problems having payoffs in the form of PM-UL terms.

Suppose $\mathbb{LS}(p) = \{t_{a^\varepsilon}(p_\varepsilon) | \varepsilon = -n, \dots, m\}$ where n and m are positive integers and $a > 1$ be the linguistic set having PM-UL terms with probabilities associated to them; $\tilde{\mathbf{A}} = [\tilde{\mathbf{a}}_{ij}]_{r \times s}$ be the PULC matrix so that each $\tilde{\mathbf{a}}_{ij}$ represents a PULC written as $\tilde{\mathbf{a}}_{ij} = (p_\varepsilon EX_{a^\varepsilon}, p_\varepsilon EN_{a^\varepsilon}, p_\varepsilon HE_{a^\varepsilon})_{ij}$. Let v^- refers to the lower value

of the game corresponding to Player *I* and let $\varrho \in S^r$ be the mixed strategies of Player *I*. Now, the PM-UL cloud expected payoff of Player *I* for s^{th} strategy of Player *II* is the weighted average of $\tilde{\alpha}_{ij}$, i.e. $\bigoplus_{j=1}^s (\varrho_i \tilde{\alpha}_{ij})$ where $\tilde{\alpha}_{ij} = (p_\varepsilon EX_{a^\varepsilon}, p_\varepsilon EN_{a^\varepsilon}, p_\varepsilon HE_{a^\varepsilon})_{ij}$, $\varepsilon = -n, \dots, m$ is a PULC such that $i = 1, 2, \dots, r$. Analogously, v^+ is the maximum expected payoff of Player *II*, $\varphi \in S^s$ is its mixed strategy, and its expected payoff against Player *I*'s r^{th} strategy is given by $\bigoplus_{i=1}^r (\varphi_i \tilde{\alpha}_{ij})$ where $j = 1, 2, \dots, s$.

For finding the expected payoffs and values of both the players in terms of PULCs, we first convert each PM-UL payoff to a PULC. Hence the PULCLP for Player *I* can be given as

$$\begin{aligned} \max \quad & v^- \\ \text{s.t.} \quad & \begin{cases} \bigoplus_{i=1}^r \tilde{\alpha}_{ij} \varrho_i \geq v^- \quad j = 1, 2, \dots, s \\ \text{where } \varrho_i \geq 0, \sum_{i=1}^r \varrho_i = 1. \end{cases} \end{aligned} \quad (\text{P1})$$

and for Player *II*, the problem is

$$\begin{aligned} \max \quad & \omega^- \\ \text{s.t.} \quad & \begin{cases} \bigoplus_{j=1}^s \text{neg}(\tilde{\alpha}_{ij}) \varphi_j \geq \omega^- \quad i = 1, 2, \dots, r \\ \text{where } \varphi_j \geq 0, \sum_{j=1}^s \varphi_j = 1. \end{cases} \end{aligned} \quad (\text{P2})$$

In above problems, each $\tilde{\alpha}_{ij}$ is a PULC. Further, simplifying (P2) and using the properties of negation operator, the problem for Player *II* becomes

$$\begin{aligned} \min \quad & v^+ \\ \text{s.t.} \quad & \begin{cases} \bigoplus_{j=1}^s \tilde{\alpha}_{ij} \varphi_j \leq v^+ \quad i = 1, 2, \dots, r \\ \text{where } \varphi_j \geq 0, \sum_{j=1}^s \varphi_j = 1. \end{cases} \end{aligned} \quad (\text{P3})$$

A unified mechanism for solving a PM-ULCG for Player *I* is given below in the form of a computation procedure.

Step 1: First normalize the payoff matrix for the row player (i.e. normalize the payoff entries row-wise) so that the sum of probabilities in each row is 1.

Step 2: Convert each payoff to a PULC using the procedure given in Section 5.3.1

to deal with uncertainty in a better way.

Step 3: Since each payoff is a PULC now, suppose the expected value of Player I be $v^- = (v_{EX}^-, v_{EN}^-, v_{HE}^-)$ such that the PULCLP becomes

$$\begin{aligned} \max \quad & v^- \implies (\max v_{EX}^-, \min v_{EN}^-, \min v_{HE}^-) \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \bigoplus_{i=1}^r \tilde{\alpha}_{ij} \varrho_i \geq v^- \quad j = 1, 2, \dots, s \\ \text{where } \varrho_i \geq 0, \sum_{i=1}^r \varrho_i = 1. \end{array} \right. \end{aligned}$$

Step 4: From here, the PM-ULCG splits into three individual games, namely the expectation game, entropy game, and the hyperentropy game given individually as follows:

Expectation Game

$$\begin{aligned} \max \quad & v_{EX}^- \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \sum_{i=1}^r (p_{\varepsilon} EX_{a^{\varepsilon}})_{ij} \varrho_i \geq v_{EX}^- \quad j = 1, 2, \dots, s \\ \text{where } \varrho_i \geq 0, \sum_{i=1}^r \varrho_i = 1. \end{array} \right. \end{aligned}$$

Entropy Game

$$\begin{aligned} \min \quad & v_{EN}^- \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \sqrt{\sum_{i=1}^r [(p_{\varepsilon} EN_{a^{\varepsilon}})_{ij}]^2} \varrho_i \leq v_{EN}^- \quad j = 1, 2, \dots, s \\ \text{where } \varrho_i \geq 0, \sum_{i=1}^r \varrho_i = 1. \end{array} \right. \end{aligned}$$

Hyperentropy Game

$$\begin{aligned} \min \quad & v_{HE}^- \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \sqrt{\sum_{i=1}^r [(p_{\varepsilon} HE_{a^{\varepsilon}})_{ij}]^2} \varrho_i \leq v_{HE}^- \quad j = 1, 2, \dots, s \\ \text{where } \varrho_i \geq 0, \sum_{i=1}^r \varrho_i = 1. \end{array} \right. \end{aligned}$$

Step 5: Proceeding further,

Expectation Game

Setting $X_i = \frac{\varrho_i}{v_{EX}^-}$ and letting $V_{EX} = \frac{1}{v_{EX}^-}$, the transformed linear programming

problem becomes

$$\begin{aligned} \min \quad & V_{EX} = \sum_{i=1}^r X_i \\ \text{s.t.} \quad & \begin{cases} \sum_{i=1}^r (p_{\varepsilon} E X_{a^{\varepsilon}})_{ij} X_i \geq 1, j = 1, 2, \dots, s \\ \text{where } X_i \geq 0 \end{cases} \end{aligned}$$

Entropy Game

Squaring both sides of the constraints and letting $(v_{EN}^-)^2 = v_{EN}'$. Further setting $Y_i = \frac{q_i}{v_{EN}'}$, and letting $V_{EN}' = \frac{1}{v_{EN}'}$. $v_{EN}^- = \sqrt{v_{EN}'} = \frac{1}{\sqrt{V_{EN}'}}$. Hence, $\min v_{EN}^- \equiv \max \sqrt{V_{EN}'}$. V_{EN}' being a positive quantity, we maximize V_{EN}' instead of $\sqrt{V_{EN}'}$ and square the value later on. Thus, the final entropy linear programming problem becomes

$$\begin{aligned} \max \quad & V_{EN}' = \sum_{i=1}^r Y_i \\ \text{s.t.} \quad & \begin{cases} \sum_{i=1}^r [(p_{\varepsilon} E N_{a^{\varepsilon}})_{ij}]^2 Y_i \leq 1, j = 1, 2, \dots, s \\ \text{where } Y_i \geq 0 \end{cases} \end{aligned}$$

Hyperentropy Game

Similar to entropy game, squaring both sides and letting $(v_{HE}^-)^2 = v_{HE}'$. Further setting $Z_i = \frac{q_i}{v_{HE}'}$, and suppose $V_{HE}' = \frac{1}{v_{HE}'}$. $v_{HE}^- = \sqrt{v_{HE}'} = \frac{1}{\sqrt{V_{HE}'}}$. Thus, $\min v_{HE}^- \equiv \max \sqrt{V_{HE}'}$. V_{HE}' being a positive quantity, we maximize V_{HE}' instead of $\sqrt{V_{HE}'}$ and square the value later on. Thus, the final hyperentropy linear programming problem becomes

$$\begin{aligned} \max \quad & V_{HE}' \\ \text{s.t.} \quad & \begin{cases} \sum_{i=1}^r [(p_{\varepsilon} H E_{a^{\varepsilon}})_{ij}]^2 Z_i \leq 1, j = 1, 2, \dots, s \\ \text{where } Z_i \geq 0 \end{cases} \end{aligned}$$

Step 6: After the formulation of final LPPs each for expectation, entropy and hyperentropy, we solve them to get the PULC strategies and PULC value of each game. Combining all three, we get

Lower value of the game: $v^- = (v_{EX}^-, v_{EN}^-, v_{HE}^-)$ and mixed strategies q_i for each game such that $\sum_{i=1}^r q_{iEX} = 1$, $\sqrt{\sum_{i=1}^r q_{iEN}^2} = 1$, $\sqrt{\sum_{i=1}^r q_{iHE}^2} = 1$.

Step 7: Substituting back to the original variables, we get the final results. Also, the final strategy to be played by the player may be decided by using score of each strategy. The higher the score, the better (optimum) is the strategy.

The exact same procedure is to be followed for Player *II*, where the payoffs are $neg(\tilde{\alpha}_{ij})$. The expected value of Player *II* would be $v^+ = (v_{EX}^+, v_{EN}^+, v_{HE}^+)$ such that as per (P3), the PULCLP becomes

$$\begin{aligned} \min \quad & v^+ \implies (\min v_{EX}^+, \max v_{EN}^+, \max v_{HE}^+) \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \bigoplus_{j=1}^s \tilde{\alpha}_{ij} \varphi_j \leq v^+ \quad i = 1, 2, \dots, r \\ \text{where } \varphi_j \geq 0, \sum_{j=1}^s \varphi_j = 1. \end{array} \right. \end{aligned}$$

and thus, the PULCLP splits into three problems given as:

Expectation Game

$$\begin{aligned} \min \quad & v_{EX}^+ \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \sum_{j=1}^s (p_{\varepsilon} EX_{a^{\varepsilon}})_{ij} \varphi_j \leq v_{EX}^+ \quad i = 1, 2, \dots, r \\ \text{where } \varphi_j \geq 0, \sum_{j=1}^s \varphi_j = 1. \end{array} \right. \end{aligned}$$

Entropy Game

$$\begin{aligned} \max \quad & v_{EN}^+ \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \sqrt{\sum_{j=1}^s [(p_{\varepsilon} EN_{a^{\varepsilon}})_{ij}]^2} \varphi_j \geq v_{EN}^+ \quad i = 1, 2, \dots, r \\ \text{where } \varphi_j \geq 0, \sum_{j=1}^s \varphi_j = 1. \end{array} \right. \end{aligned}$$

Hyperentropy Game

$$\begin{aligned} \max \quad & v_{HE}^+ \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \sqrt{\sum_{j=1}^s [(p_{\varepsilon} HE_{a^{\varepsilon}})_{ij}]^2} \varphi_j \geq v_{HE}^+ \quad i = 1, 2, \dots, r \\ \text{where } \varphi_j \geq 0, \sum_{j=1}^s \varphi_j = 1. \end{array} \right. \end{aligned}$$

Then solving just as in Step 5 mentioned above, we combine the strategies and values obtained from all three games so that we get upper value of the game, i.e., $v^+ = (v_{EX}^+, v_{EN}^+, v_{HE}^+)$.

We illustrate the defined methodology using an example.

5.4.3 Numerical Illustration

Example 5.4.1. An investor wishes to invest his money in the stocks of some IT companies. He does so by analysing the average stock closing values for the past 6 years 2017-2022. The possible alternatives under consideration are the five IT companies: Infosys, Wipro, TCS, HCL Technologies, and Larsen & Toubro Infotech. The following linguistic term set involves such linguistic variables that represent the stock values: $\mathbb{L}\mathbb{S} = \{t_{2-2} : \text{Very Low (VL)}, t_{2-1} : \text{Low (L)}, t_{20} : \text{Average (Avg)}, t_{21} : \text{High (H)}, t_{22} : \text{Very High (VH)}\}$. The Table 5.2 provides the values of stocks of the IT companies for the financial years 2017-2022. This data is the NSE data taken from Money Control <https://www.moneycontrol.com/>. Based on the values listed in Table 5.2, the variations

	2017	2018	2019	2020	2021	2022
Infosys	521.03	658.95	731.15	1255.8	1887.75	1508.2
Wipro	235.69	248.14	245.8	386.25	715.35	392.75
TCS	1350.6	1893.05	2161.7	2862.75	3738.35	3256.7
HCL Technologies	445.25	482.18	568.1	946.15	1319.1	1039.3
Larsen & Toubro Infotech	1118.85	1731.2	1750.5	3659.5	7332	2085.8

Table 5.2: Closing Stock prices of IT companies for different financial years

that occurred in the stock values of different IT companies are evaluated in terms of PM-ULTS. Thus, the probabilistic unbalanced linguistic stock data for the years 2017-2022 for the above-mentioned IT companies is provided in the following “IT company versus year” matrix $\tilde{\mathbb{A}}$ given as:

	2017	2018	2019	2020	2021	2022
Infosys	$t_{2-1}(0.13)$	$t_{2-1}(0.194)$	$t_{2-1}(0.225)$	$t_{20}(0.194)$	$t_{21}(0.225)$	$t_{21}(0.032)$
Wipro	$t_{2-2}(0.118)$	$t_{2-2}(0.118)$	$t_{2-2}(0.118)$	$t_{2-1}(0.118)$	$t_{2-1}(0.41)$	$t_{2-1}(0.118)$
TCS	$t_{20}(0.28)$	$t_{21}(0.28)$	$t_{22}(0.04)$	$t_{22}(0.08)$	$t_{22}(0.16)$	$t_{22}(0.16)$
HCL Technologies	$t_{2-1}(0.125)$	$t_{2-1}(0.167)$	$t_{2-1}(0.208)$	$t_{20}(0.041)$	$t_{20}(0.292)$	$t_{20}(0.167)$
Larsen & Toubro Infotech	$t_{20}(0.114)$	$t_{21}(0.114)$	$t_{21}(0.144)$	$t_{22}(0.114)$	$t_{22}(0.286)$	$t_{21}(0.228)$

The payoff $\tilde{a}_{ij}$ of Player I represents the response to strategy i of Player I for the case when Player II selects strategy j in order to win the game. Also, a probabilistic linguistic payoff signifies that any strategy played by a player can be evaluated on an unbalanced linguistic scale.

Next, we apply our methodology to deal with PM-UL information to analyse which

company has had a higher growth in share market from financial year 2017 to 2022. On the basis of this analysis, one can interpret which company to invest in, for future purpose. Now first we normalize the payoff matrix. So, the normalized payoff matrix is given as :

$$\tilde{\mathbf{A}} = \begin{matrix} & \begin{matrix} 2017 & 2018 & 2019 & 2020 & 2021 & 2022 \end{matrix} \\ \begin{matrix} \text{Infosys} \\ \text{Wipro} \\ \text{TCS} \\ \text{HCL Technologies} \\ \text{Larsen \& Toubro Infotech} \end{matrix} & \left[\begin{matrix} t_{2^{-1}}(0.4) & t_{2^{-1}}(0.6) & t_{2^{-1}}(0.7) & t_{2^0}(0.6) & t_{2^1}(0.7) & t_{2^1}(0.1) \\ t_{2^{-2}}(0.2) & t_{2^{-2}}(0.2) & t_{2^{-2}}(0.2) & t_{2^{-1}}(0.2) & t_{2^{-1}}(0.7) & t_{2^{-1}}(0.2) \\ t_{2^0}(0.7) & t_{2^1}(0.7) & t_{2^2}(0.1) & t_{2^2}(0.2) & t_{2^2}(0.4) & t_{2^2}(0.4) \\ t_{2^{-1}}(0.3) & t_{2^{-1}}(0.4) & t_{2^{-1}}(0.5) & t_{2^0}(0.1) & t_{2^0}(0.7) & t_{2^0}(0.4) \\ t_{2^0}(0.4) & t_{2^1}(0.4) & t_{2^1}(0.5) & t_{2^2}(0.4) & t_{2^2}(1) & t_{2^1}(0.8) \end{matrix} \right] \end{matrix}$$

The ambiguity and volatility of stock values are represented by the dispersion of cloud droplets. So we convert the probabilistic unbalanced linguistic terms to PULCs so as to choose a specific alternative. Hence we use the procedure given in Section 5.3.1 to convert the payoffs to PULCs and thus the PULC payoff matrix is given as:

$$\tilde{\mathbf{A}}_{PULC} = \begin{matrix} & \begin{matrix} 2017 & 2018 & 2019 & 2020 & 2021 & 2022 \end{matrix} \\ \begin{matrix} \text{Infosys} \\ \text{Wipro} \\ \text{TCS} \\ \text{HCL Technologies} \\ \text{Larsen \& Toubro Infotech} \end{matrix} & \left[\begin{matrix} (0.6298, 1.76033, 0.6987) & (0.9399, 2.6269, 1.0427) & (1.0901, 3.0467, 1.2093) & (1.6112, 2.2799, 1.42706) & (2.6473, 3.0467, 1.2093) & (0.3765, 0.4333, 0.172) \\ (0, 1.854, 0.3506) & (0, 1.854, 0.3506) & (0, 1.854, 0.3506) & (0.5717, 1.5978, 0.6342) & (1.9864, 5.5518, 2.203) & (0.5717, 1.5978, 0.6342) \\ (2.3255, 3.2907, 2.0596) & (3.2944, 3.7914, 1.505) & (0.6284, 0.6284, 0.1188) & (1.2569, 1.2569, 0.2376) & (2.5139, 2.5139, 0.4753) & (2.5139, 2.5139, 0.4753) \\ (0.6056, 1.6926, 0.6718) & (0.8091, 2.2613, 0.8976) & (1.0077, 2.8165, 1.118) & (0.3405, 0.4818, 0.3015) & (2.4252, 3.4317, 2.1479) & (1.387, 1.9626, 1.2284) \\ (0.9468, 1.3397, 0.8385) & (1.3413, 1.5436, 0.6127) & (1.6943, 1.9499, 0.774) & (1.7911, 1.7911, 0.3387) & (4.4936, 4.4936, 0.8497) & (2.6826, 3.0873, 1.2255) \end{matrix} \right] \end{matrix}$$

Solving this game problem for Player I , i.e. for “IT companies”, we get three separate games as expectation game, entropy game and hyperentropy game given as :

Expectation Game

$$\begin{aligned} \max \quad & v_{EX}^- \\ \text{s.t.} \quad & \left\{ \begin{array}{l} 0.6298\varrho_1 + 0\varrho_2 + 2.3255\varrho_3 + 0.6056\varrho_4 + 0.9468\varrho_5 \geq v_{EX}^- \\ 0.9399\varrho_1 + 0\varrho_2 + 3.2944\varrho_3 + 0.8091\varrho_4 + 1.3413\varrho_5 \geq v_{EX}^- \\ 1.0901\varrho_1 + 0\varrho_2 + 0.6284\varrho_3 + 1.0077\varrho_4 + 1.6943\varrho_5 \geq v_{EX}^- \\ 1.6112\varrho_1 + 0.5717\varrho_2 + 1.2569\varrho_3 + 0.3405\varrho_4 + 1.7911\varrho_5 \geq v_{EX}^- \\ 2.6473\varrho_1 + 1.9864\varrho_2 + 2.5139\varrho_3 + 2.4252\varrho_4 + 4.4936\varrho_5 \geq v_{EX}^- \\ 0.3765\varrho_1 + 0.5717\varrho_2 + 2.5139\varrho_3 + 1.387\varrho_4 + 2.6826\varrho_5 \geq v_{EX}^- \\ \text{where } \varrho_i \geq 0, \sum_{i=1}^5 \varrho_i = 1 \end{array} \right. \end{aligned}$$

Setting $X_i = \frac{\varrho_i}{v_{EX}^-}$, the problem becomes

$$\begin{aligned} \min \quad & V_{EX} = X_1 + X_2 + X_3 + X_4 + X_5 \\ \text{s.t.} \quad & \left\{ \begin{array}{l} 0.6298X_1 + 0X_2 + 2.3255X_3 + 0.6056X_4 + 0.9468X_5 \geq 1 \\ 0.9399X_1 + 0X_2 + 3.29448X_3 + 0.8091X_4 + 1.3413X_5 \geq 1 \\ 1.0901X_1 + 0X_2 + 0.6284X_3 + 1.0077X_4 + 1.6943X_5 \geq 1 \\ 1.6112X_1 + 0.5717X_2 + 1.2569X_3 + 0.3405X_4 + 1.7911X_5 \geq 1 \\ 2.6473X_1 + 1.9864X_2 + 2.5139X_3 + 2.4252X_4 + 4.4936X_5 \geq 1 \\ 0.3765X_1 + 0.5717X_2 + 2.5139X_3 + 1.387X_4 + 2.6826X_5 \geq 1 \\ \text{where } X_i \geq 0, i = 1, \dots, 5 \end{array} \right. \end{aligned}$$

The solution of above LPP is:

$$X_1 = 0, X_2 = 0, X_3 = 0.223, X_4 = 0, X_5 = 0.508, V_{EX} = 0.731$$

Substituting back to the original variables, we get the optimal solution of the expectation game problem as : $v_{EX}^- = \frac{1}{V_{EX}} = 1.36$, $\varrho_1 = 0$, $\varrho_2 = 0$, $\varrho_3 = 0.305$, $\varrho_4 = 0$, $\varrho_5 = 0.695$.

Hence the strategies are $(0, 0, 0.305, 0, 0.695)$ and expectation value is 1.36.

Next,

Entropy Game

$$\begin{aligned} \min \quad & v_{EN}^- \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \sqrt{(1.7603)^2 \varrho_1 + (1.854)^2 \varrho_2 + (3.2907)^2 \varrho_3 + (1.6926)^2 \varrho_4 + (1.3397)^2 \varrho_5} \leq v_{EN}^- \\ \sqrt{(2.6269)^2 \varrho_1 + (1.854)^2 \varrho_2 + (3.7914)^2 \varrho_3 + (2.2613)^2 \varrho_4 + (1.5436)^2 \varrho_5} \leq v_{EN}^- \\ \sqrt{(3.0467)^2 \varrho_1 + (1.854)^2 \varrho_2 + (0.6285)^2 \varrho_3 + (2.8165)^2 \varrho_4 + (1.95)^2 \varrho_5} \leq v_{EN}^- \\ \sqrt{(2.2799)^2 \varrho_1 + (1.5978)^2 \varrho_2 + (1.2569)^2 \varrho_3 + (0.482)^2 \varrho_4 + (1.791)^2 \varrho_5} \leq v_{EN}^- \\ \sqrt{(3.0467)^2 \varrho_1 + (5.5518)^2 \varrho_2 + (2.5139)^2 \varrho_3 + (3.4317)^2 \varrho_4 + (4.4936)^2 \varrho_5} \leq v_{EN}^- \\ \sqrt{(0.4333)^2 \varrho_1 + (1.5978)^2 \varrho_2 + (2.5139)^2 \varrho_3 + (1.9626)^2 \varrho_4 + (3.0873)^2 \varrho_5} \leq v_{EN}^- \\ \text{where } \varrho_i \geq 0, i = 1, \dots, 5, \sum_{i=1}^5 \varrho_i = 1 \end{array} \right. \end{aligned}$$

Squaring both sides, and letting $(v_{EN}^-)^2 = v_{EN}'$, the entropy game converts to the pro-

gramming problem given as:

$$\begin{aligned} \min \quad & \sqrt{v_{EN}'} \equiv \min v_{EN}' \\ \text{s.t.} \quad & \left\{ \begin{array}{l} 3.0987q_1 + 3.4374q_2 + 10.8287q_3 + 2.8649q_4 + 1.7948q_5 \leq v_{EN}' \\ 6.9q_1 + 3.437q_2 + 14.375q_3 + 5.113q_4 + 2.382q_5 \leq v_{EN}' \\ 9.282q_1 + 3.437q_2 + 0.395q_3 + 7.933q_4 + 3.802q_5 \leq v_{EN}' \\ 5.198q_1 + 2.553q_2 + 1.579q_3 + 0.232q_4 + 3.208q_5 \leq v_{EN}' \\ 9.282q_1 + 30.822q_2 + 6.319q_3 + 11.653q_4 + 20.192q_5 \leq v_{EN}' \\ 0.187q_1 + 2.553q_2 + 6.319q_3 + 3.851q_4 + 9.531q_5 \leq v_{EN}' \\ \text{where } q_i \geq 0, i = 1, \dots, 5, \sum_{i=1}^5 q_i = 1 \end{array} \right. \end{aligned}$$

Setting $Y_i = \frac{q_i}{v_{EN}'}$ and $V_{EN}' = \frac{1}{v_{EN}'}$, hence the final entropy linear programming problem comes out as

$$\begin{aligned} \max \quad & V_{EN}' = Y_1 + Y_2 + Y_3 + Y_4 + Y_5 \\ \text{s.t.} \quad & \left\{ \begin{array}{l} 3.0987Y_1 + 3.4374Y_2 + 10.8287Y_3 + 2.8649Y_4 + 1.7948Y_5 \leq 1 \\ 6.9Y_1 + 3.437Y_2 + 14.375Y_3 + 5.113Y_4 + 2.382Y_5 \leq 1 \\ 9.282Y_1 + 3.437Y_2 + 0.395Y_3 + 7.933Y_4 + 3.802Y_5 \leq 1 \\ 5.198Y_1 + 2.553Y_2 + 1.579Y_3 + 0.232Y_4 + 3.208Y_5 \leq 1 \\ 9.282Y_1 + 30.822Y_2 + 6.319Y_3 + 11.653Y_4 + 20.192Y_5 \leq 1 \\ 0.187Y_1 + 2.553Y_2 + 6.319Y_3 + 3.851Y_4 + 9.531Y_5 \leq 1 \\ \text{where } Y_i \geq 0, i = 1, \dots, 5 \end{array} \right. \end{aligned}$$

Solving the above LPP, the solution comes out to be

$$Y_1 = 0.08968, Y_2 = 0, Y_3 = 0.0265, Y_4 = 0, Y_5 = 0$$

Also, $V_{EN}' = 0.1162$. Thus, $v_{EN}^- = 2.93$ and the strategies are $(0.7717, 0, 0.2282, 0, 0)$.

Further,

Hyperentropy Game

$$\begin{aligned}
 & \min \quad \nu_{HE}^- \\
 & \text{s.t.} \quad \left\{ \begin{aligned}
 & \sqrt{(0.6987)^2 \varrho_1 + (0.3506)^2 \varrho_2 + (2.0596)^2 \varrho_3 + (0.6718)^2 \varrho_4 + (0.8385)^2 \varrho_5} \leq \nu_{HE}^- \\
 & \sqrt{(1.0427)^2 \varrho_1 + (0.3506)^2 \varrho_2 + (1.505)^2 \varrho_3 + (0.8976)^2 \varrho_4 + (0.6127)^2 \varrho_5} \leq \nu_{HE}^- \\
 & \sqrt{(1.2093)^2 \varrho_1 + (0.3506)^2 \varrho_2 + (0.1188)^2 \varrho_3 + (1.118)^2 \varrho_4 + (0.774)^2 \varrho_5} \leq \nu_{HE}^- \\
 & \sqrt{(1.427)^2 \varrho_1 + (0.6342)^2 \varrho_2 + (0.2376)^2 \varrho_3 + (0.3015)^2 \varrho_4 + (0.3387)^2 \varrho_5} \leq \nu_{HE}^- \\
 & \sqrt{(1.2093)^2 \varrho_1 + (2.203)^2 \varrho_2 + (0.4754)^2 \varrho_3 + (2.1479)^2 \varrho_4 + (0.8479)^2 \varrho_5} \leq \nu_{HE}^- \\
 & \sqrt{(0.172)^2 \varrho_1 + (0.6342)^2 \varrho_2 + (0.4754)^2 \varrho_3 + (1.2284)^2 \varrho_4 + (1.2255)^2 \varrho_5} \leq \nu_{HE}^- \\
 & \text{where } \varrho_i \geq 0, i = 1, \dots, 5, \sum_{i=1}^5 \varrho_i = 1
 \end{aligned} \right.
 \end{aligned}$$

To simplify the above mentioned programming problem, we square the both sides of the constraints of this problem and we suppose that $(\nu_{HE}^-)^2 = \nu_{HE}'$. Doing this substitution, the above problem reduces to the following programming problem.

$$\begin{aligned}
 & \min \quad \sqrt{\nu_{HE}'} \equiv \min \nu_{HE}' \\
 & \text{s.t.} \quad \left\{ \begin{aligned}
 & 0.4882\varrho_1 + 0.1229\varrho_2 + 4.2423\varrho_3 + 0.4513\varrho_4 + 0.7031\varrho_5 \leq \nu_{HE}' \\
 & 1.0872\varrho_1 + 0.1229\varrho_2 + 2.265\varrho_3 + 0.8056\varrho_4 + 0.3755\varrho_5 \leq \nu_{HE}' \\
 & 1.4624\varrho_1 + 0.1229\varrho_2 + 0.0141\varrho_3 + 1.2499\varrho_4 + 0.599\varrho_5 \leq \nu_{HE}' \\
 & 2.036\varrho_1 + 0.4022\varrho_2 + 0.0564\varrho_3 + 0.0909\varrho_4 + 0.1147\varrho_5 \leq \nu_{HE}' \\
 & 1.4624\varrho_1 + 4.8532\varrho_2 + 0.226\varrho_3 + 4.6134\varrho_4 + 0.7219\varrho_5 \leq \nu_{HE}' \\
 & 0.029\varrho_1 + 0.4022\varrho_2 + 0.226\varrho_3 + 1.5089\varrho_4 + 1.5018\varrho_5 \leq \nu_{HE}' \\
 & \text{where } \varrho_i \geq 0, i = 1, \dots, 5, \sum_{i=1}^5 \varrho_i = 1
 \end{aligned} \right.
 \end{aligned}$$

Setting $Z_i = \frac{\varrho_i}{\nu_{HE}'}$ and $V_{HE}' = \frac{1}{\nu_{HE}'}$, the final hyperentropy linear programming problem

becomes

$$\begin{array}{ll} \max & V_{HE}' \\ \text{s.t.} & \left\{ \begin{array}{l} 0.4882Z_1 + 0.1229Z_2 + 4.2423Z_3 + 0.4513Z_4 + 0.7031Z_5 \leq 1 \\ 1.0872Z_1 + 0.1229Z_2 + 2.265Z_3 + 0.8056Z_4 + 0.3755Z_5 \leq 1 \\ 1.4624Z_1 + 0.1229Z_2 + 0.0141Z_3 + 1.2499Z_4 + 0.599Z_5 \leq 1 \\ 2.036Z_1 + 0.4022Z_2 + 0.0564Z_3 + 0.0909Z_4 + 0.1147Z_5 \leq 1 \\ 1.4624Z_1 + 4.8532Z_2 + 0.226Z_3 + 4.6134Z_4 + 0.7219Z_5 \leq 1 \\ 0.029Z_1 + 0.4022Z_2 + 0.226Z_3 + 1.5089Z_4 + 1.5018Z_5 \leq 1 \\ \text{where } Z_i \geq 0, i = 1, \dots, 5 \end{array} \right. \end{array}$$

Solving the above LPP, solution come out as

$$Z_1 = 0.3514, Z_2 = 0, Z_3 = 0.0883, Z_4 = 0, Z_5 = 0.6458, V_{HE}' = 1.0854$$

Substituting back to original variables, the hyperentropy value $v_{HE}^- = 0.96$, and the strategies are $(0.3237, 0, 0.08135, 0, 0.5949)$.

Summarizing and writing collectively, the strategies in terms of PULC are:

$$\varrho_1^* = (0, 0.7717, 0.3237), \varrho_2^* = 0, \varrho_3^* = (0.305, 0.2283, 0.0813), \varrho_4^* = 0, \varrho_5^* = (0.695, 0, 0.5949)$$

and, the PULC value of the game is

$$v^- = (1.36, 2.93, 0.96)$$

Further, the probabilistic unbalanced linguistic value of the game can be calculated and is equal to

$$\tilde{v}^-(p) = \{t_{20}(0.4612), t_{21}(0.5388)\}$$

Similarly, solving the game problem for Player II, i.e. for “Year”, the expectation, entropy and hyperentropy games are as given below:

Expectation Game

$$\begin{array}{ll} \min & v_{EX}^+ \\ \text{s.t.} & \left\{ \begin{array}{l} 0.6298\varphi_1 + 0.9399\varphi_2 + 1.0901\varphi_3 + 1.6112\varphi_4 + 2.6473\varphi_5 + 0.3765\varphi_6 \leq v_{EX}^+ \\ 0\varphi_1 + 0\varphi_2 + 0\varphi_3 + 0.5717\varphi_4 + 1.9864\varphi_5 + 0.5717\varphi_6 \leq v_{EX}^+ \\ 2.3255\varphi_1 + 3.2944\varphi_2 + 0.6284\varphi_3 + 1.2569\varphi_4 + 2.5139\varphi_5 + 2.5139\varphi_6 \leq v_{EX}^+ \\ 0.6056\varphi_1 + 0.8091\varphi_2 + 1.0077\varphi_3 + 0.3405\varphi_4 + 2.4252\varphi_5 + 1.387\varphi_6 \leq v_{EX}^+ \\ 0.9468\varphi_1 + 1.3413\varphi_2 + 1.6943\varphi_3 + 1.7911\varphi_4 + 4.4936\varphi_5 + 2.6826\varphi_6 \leq v_{EX}^+ \\ \text{where } \varphi_j \geq 0, \sum_{j=1}^6 \varphi_j = 1 \end{array} \right. \end{array}$$

Solving by doing required substitutions, we get the optimal solution as : $v_{EX}^+ = 1.36$, and the strategies are $(0.4359, 0, 0.564, 0, 0, 0)$.

Entropy Game

$$\begin{array}{ll} \max & v_{EN}^+ \\ \text{s.t.} & \left\{ \begin{array}{l} \sqrt{(1.7603)^2\varphi_1 + (2.6269)^2\varphi_2 + (3.0467)^2\varphi_3 + (2.2799)^2\varphi_4 + (3.0467)^2\varphi_5 + (0.4333)^2\varphi_6} \geq v_{EN}^+ \\ \sqrt{(1.854)^2\varphi_1 + (1.854)^2\varphi_2 + (1.854)^2\varphi_3 + (1.5978)^2\varphi_4 + (5.5518)^2\varphi_5 + (1.5978)^2\varphi_6} \geq v_{EN}^+ \\ \sqrt{(3.2907)^2\varphi_1 + (3.7914)^2\varphi_2 + (0.6284)^2\varphi_3 + (1.2569)^2\varphi_4 + (2.5139)^2\varphi_5 + (2.5139)^2\varphi_6} \geq v_{EN}^+ \\ \sqrt{(1.6926)^2\varphi_1 + (2.2613)^2\varphi_2 + (2.8165)^2\varphi_3 + (0.482)^2\varphi_4 + (3.4317)^2\varphi_5 + (1.9626)^2\varphi_6} \geq v_{EN}^+ \\ \sqrt{(1.3397)^2\varphi_1 + (1.5436)^2\varphi_2 + (1.9499)^2\varphi_3 + (1.7911)^2\varphi_4 + (4.4936)^2\varphi_5 + (3.0873)^2\varphi_6} \geq v_{EN}^+ \\ \text{where } \varphi_j \geq 0, j = 1, \dots, 6, \sum_{j=1}^6 \varphi_j = 1 \end{array} \right. \end{array}$$

Solving just as for Player I , we obtain result for entropy game as $(0, 0.2839, 0, 0, 0.716, 0)$ and value $v_{EN}^+ = 2.93$.

Hyperentropy Game

$$\begin{aligned} \max \quad & \nu_{HE}^+ \\ \text{s.t.} \quad & \left\{ \begin{aligned} & \sqrt{(0.6987)^2\varphi_1 + (1.0427)^2\varphi_2 + (1.2093)^2\varphi_3 + (1.427)^2\varphi_4 + (1.2093)^2\varphi_5 + (0.172)^2\varphi_6} \geq \nu_{HE}^+ \\ & \sqrt{(0.3506)^2\varphi_1 + (0.3506)^2\varphi_2 + (0.3506)^2\varphi_3 + (0.6342)^2\varphi_4 + (2.203)^2\varphi_5 + (0.6342)^2\varphi_6} \geq \nu_{HE}^+ \\ & \sqrt{(2.0596)^2\varphi_1 + (1.505)^2\varphi_2 + (0.1188)^2\varphi_3 + (0.3015)^2\varphi_4 + (2.1479)^2\varphi_5 + (1.2284)^2\varphi_6} \geq \nu_{HE}^+ \\ & \sqrt{(0.6718)^2\varphi_1 + (0.8976)^2\varphi_2 + (1.118)^2\varphi_3 + (0.3015)^2\varphi_4 + (2.1479)^2\varphi_5 + (1.2284)^2\varphi_6} \geq \nu_{HE}^+ \\ & \sqrt{(0.8385)^2\varphi_1 + (0.6127)^2\varphi_2 + (0.774)^2\varphi_3 + (0.3387)^2\varphi_4 + (0.8497)^2\varphi_5 + (1.2255)^2\varphi_6} \geq \nu_{HE}^+ \\ & \text{where } \varphi_j \geq 0, j = 1, \dots, 6, \sum_{j=1}^6 \varphi_j = 1 \end{aligned} \right. \end{aligned}$$

The solution comes out as: $(0.173, 0, 0, 0, 0.5669, 0.2598)$ and $\nu_{HE}^+ = 0.96$. Hence, collectively the optimal PULC strategies and PULC value of Player *II* are:

$$\varphi_1^* = (0.4359, 0, 0.173), \varphi_2^* = (0, 0.2839, 0), \varphi_3^* = (0.564, 0, 0), \varphi_4^* = 0, \varphi_5^* = (0, 0.716, 0.5669), \varphi_6^* = (0, 0, 0.2598)$$

Value $\nu^+ = (1.36, 2.93, 0.96)$. Hence, $\nu^- = \nu^+ = \nu =$ Value of the game.

Further, the scores of strategies may be evaluated to finalize which strategy should a player use to win the game. For that, we may find different score function values for different values of γ . Without loss of generality, we fix the value of $\gamma = 0.5$ for current calculations. For Player *I*, $\hat{S}(\varrho_1^*) = -0.4184, \hat{S}(\varrho_2^*) = 0, \hat{S}(\varrho_3^*) = 0.3825, \hat{S}(\varrho_4^*) = 0, \hat{S}(\varrho_5^*) = 0.3975$. Thus based on the score values, the IT company Larsen & Toubro would be the best to invest the money in. Based on the above-obtained results, the DM can also plan where to invest in the future. Also, the trend of increment of stock prices may be determined by finding the score values of Player *II*, i.e. “Year”.

Note 5.4.1. The above problem is solved for the case when the DMs record only a single linguistic term as a payoff entry. The same procedure may be followed for the case when multiple linguistic terms consist a payoff or the case when the payoffs are of a set type. In that case, we could break the individual term of the set and then form separate PULCLP for each broken term set.

5.5 Comparative Analysis and Discussion

This section provides a comparison analysis of the proposed methodology with the existing studies that involve game theoretic models using PULCs.

The idea of PLTS was first proposed by [174] which allows the players or DMs to evaluate situations more effectively by using multiple terms rather than just one, by giving each term a probability-based weight. Apart from this, there are other variants of PLTS, such as [12, 238, 261]. But these PLTSs have the limitation that all of them use symmetric and balanced information for decision analysis. Further, this symmetric information inculcated in these PLTSs is also used in matrix game theory in the paper [162]. The theory defined by Mi et al. [162] uses triangular membership function to represent probabilistic linguistic information. This information is defuzzified using the triangular membership function, and the calculated games values are then retranslated to probabilistic data by using linear interpolation method. This process is limited in its form as the defuzzification and then, retranslation process causes major loss of information. To overcome the problem of symmetric probabilistic linguistic information, Han et al. [87] defined a probabilistic unbalanced linguistic term set (PULTS) that makes use of unbalanced linguistic data to represent probabilistic linguistic information. But this method uses the unbalanced linguistic model defined by [97] which is computationally very expensive method. Thus in the paper [154], the authors solve linguistic matrix games that have probabilistic unbalanced linguistic data that comes from a PM-ULTS defined by Malhotra & Gupta [153].

The theory defined in our study is useful for solving matrix games having the same PM-UL information, but with a different point of view. The authors in [154] only considered the fuzziness involved in a linguistic concept. However in this chapter, the linguistic concept is analysed in the way that both randomness and fuzziness are omitted from the qualitative concept. The present chapter uses cloud model defined in Chapter 4 to thoroughly capture the intrinsic fuzziness and vagueness of decision information present in matrix game problems. Apart from that, the game problem considered in this chapter consists of payoffs that are randomly irregularly distributed in a multiplicative manner. Such linguistic data addressess all types of linguistic information. This preserves the literal meaning of unbalanced linguistic matrix games. The suggested approach not only prevents data loss, it also studies different types of uncertainties lying in the linguistic information of matrix game problems. Also, the loss of information occurring in [162] due to the retranslation process is well handled in this chapter as the probabilistic information is first converted into PULCs and the PULC data is then solved us-

ing cloud aggregation operators. Also the retranslation process is done using the bijective operators that overcome the information loss. Furthermore, the methodology defined in this chapter converts the probabilistic unbalanced payoff into a PULC in a way that it forms three independent easy-to-solve games, namely Expectation game, Entropy game and Hyperentropy game which are basically three independent PULLPs. These PULLPs are converted into LPPs using required substitution and are then solved as a classical LPP. The objective function of these PULLPs runs into $O(n)$ time, where n is the cardinality of the PULTS.

The novelty of this methodology lies in the fact that it allows the DMs to study the underlying uncertainty in the information of an uncertain game problem and still solve a decision-making problem just as a classical problem without any loss of information.

5.6 Conclusion

This study proposed and investigated the two-person matrix games having PM-UL information. This allows the theory of games to accept incomplete information as players' responses. The uncertain payoffs are dealt with by PULCs, and then cloud aggregation operators are applied to aggregate the probabilistic cloud information. Then, the game problem is converted into three independent games, i.e., expectation game, entropy game, and hypertrophy game. The LPPs of each game are individually solved to get the cloud value of the game. The significance of transforming the unbalanced linguistic payoffs to PULCs is that the cloud model studies the uncertainty lying in a qualitative concept by thoroughly encapsulating the underlying randomness and fuzziness of the unbalanced payoff. The results of all three LPPs are collected to form a single strategy and single value, that is, further a cloud. The cloud value of the game provides the dispersion of cloud droplets using entropy value. This methodology is applied to solve a real-time decision problem of investment in stock of different IT companies.

This methodology is advantageous as it provides more accurate results than other existing methodologies to solve matrix game problems having probabilistic linguistic information in which only one kind of uncertainty is dealt with. As for future research directions, one can extend this methodology towards interval clouds as this methodology maps qualitative information to a certain precise value. Fur-

ther, one can work towards developing a simpler theory for the case where the payoff information is in the form of multiple probabilistic linguistic terms.

Chapter 6

The unbalanced 2-tuple intuitionistic fuzzy linguistic numbers and aggregation operators

Intuitionistic fuzzy linguistic sets are outlined by a linguistic membership and non-membership degree; because of which, the decision-makers are capable of considering a linguistic hesitancy degree, where they don't simply convey their perception using one linguistic term. This chapter ¹ defines the unbalanced 2-tuple intuitionistic fuzzy linguistic sets in which the intuitionistic fuzzy linguistic numbers are composed of linguistic 2-tuples belonging to an unbalanced multiplicative set. Also, new score and accuracy functions are defined to determine the comparison between two unbalanced 2-tuple intuitionistic fuzzy linguistic numbers (2-TIFLNs). This is done by partitioning the linguistic term set into two parts, namely membership terms and non-membership terms. The logic behind partitioning is that it helps decision-makers independently and individually define the membership and non-membership degree for a linguistic term belonging to either set. There is ambiguity in selecting membership and non-membership degrees in the existing literature as both are selected from the same scale set. Partitioning gives clarity in determining the membership and non-membership degrees from different scale sets. Next, we define some operational laws of unbalanced 2-TIFLNs and use

¹The work presented in this chapter comprises of the paper entitled, “The unbalanced 2-tuple intuitionistic fuzzy linguistic numbers and aggregation operators” (Submitted in “International Journal of Fuzzy Systems”)

them to determine aggregation operators: unbalanced 2-tuple intuitionistic fuzzy linguistic weighted average, ordered weighted average, weighted geometric, and ordered weighted geometric operators. The defined operators are of the characteristic that all of them satisfy the closure property, which was lacking in existing theories of IFLNs. The prominent features of these aggregation operators are also studied theoretically as well as graphically. Finally, some examples are applied to illustrate the performance of the proposed operators. The graphical results show the effectiveness of the developed method of aggregating unbalanced 2-TIFLNs.

6.1 Introduction

The fuzzy set theory introduced by Zadeh [255] plays a significant role in handling uncertainty. This theory has exhibited great zeal and competency in modeling the uncertainty persisting in decision-making to confront the prevailing probabilistic theories. Since the introduction of fuzzy sets, many attempts have been made to generalize the notion of fuzzy sets. In fuzzy set theory, the membership of an element to a set happens to be a value between 0 and 1. However in reality, it might not be the case that the degree of non-membership of an element is always equal to 1 minus the membership degree because of the presence of some hesitation degree. Therefore, Atanssov [8] established the concept of intuitionistic fuzzy sets (IFS) identified by a degree of membership and a degree of non-membership. All the fuzzy sets are IFS but not conversely. Moreover, there are problems dealt with by fuzzy set theory that might also be explored by IFS theory, but the existence of some intricate problems [8, 29, 70] indicate that the IFS theory is appropriately suitable rather than fuzzy set theory. Atanassov [?, 8, 9] proposes a generalization of fuzzy sets, making the degrees of membership and non-membership intervene to describe the association of an element to a set so that the sum of these degrees is always less than or equal to 1. Several aggregation operators emerged as a new research direction since then, for example, [24, 73, 131, 229, 236, 241, 242]. Further, experts usually wish to communicate the results in a common language instead of precise numerical value. This brings about the idea of the linguistic approach [256, 257] based on which several computation models have been defined in the literature. Moreover, as decision-making is one of the most distinguished application areas and has been studied considerably in literature, the linguistic computational models have been applied in several decision-making problems [57, 94, 243, 245, 246].

Analogous to IFS, intuitionistic fuzzy linguistic set (IFLS) is outlined by a linguistic membership degree and a linguistic non-membership degree, respectively based on the notion of linguistic set [256, 257]. By using IFLS, the DMs are capable of considering a linguistic hesitancy degree, where they cannot simply convey their perception using one linguistic term. Zhang [264] defined the concept of IFLS and develops various aggregation operators for IFLS. Since then, several authors have significantly solved many decision-making problems in the realm of

IFLSs [50, 133, 159]. Moreover, several aggregation operators have also been defined in literature for aggregation of IFL variables such as [7, 23, 77, 134, 202]. Wei [234] has defined a 2-tuple intuitionistic fuzzy linguistic set (2-TIFLS) and developed the aggregation operators for the same. However, the linguistic set used by Wei [234] is a balanced and symmetric set in which the members of the set are distributed evenly and equally.

Adding to the contribution in the present research, we try to develop an intuitionistic model in terms of multiplicative unbalanced linguistic terms set. In literature, it has been noticed that membership and non-membership are taken to be in the same evaluation scale set. If we try to understand the scenario of membership and non-membership in the real sense, there is no clarity on how to differentiate between membership and non-membership because both belong to the same domain. Here, we are partitioning the scale set as left and right set to deal with this ambiguity. The left set symbolizes a non-membership set and the right set symbolizes the membership scale set so that it gives precise meaning to a DM regarding the degree of membership and non-membership of the linguistic element contained within the term set. Membership and non-membership have an inverse relationship with each other. Note that it is not necessary to always term left set only as non-membership set. One can also name the right set as a non-membership set, but we assume that an optimistic person would give a high membership value to an element, and a pessimistic person would give a high non-membership value to an element. WLOG, here we proceed, assuming that a DM is an optimistic person. The importance of partitioning the set is that it facilitates the DM to interpret a problem in a real-time domain by individually considering the degree of membership lying in a membership set very clearly and independently. One may quickly obtain either value by defining the range of both degrees. So the partitioning may help the DM independently and individually define the membership and non-membership degree for a linguistic term belonging to either set. Moreover, partitioning clarifies the terms to quickly interpret the degrees, missing in earlier definitions of intuitionistic sets defined in the balanced domain as both membership and non-membership belong to the same single set. So, we define an unbalanced 2-tuple intuitionistic fuzzy linguistic set by partitioning the set $LS = \{t_{a^{-n}}, t_{a^{-(n-1)}}, \dots, t_{a^0}, \dots, t_{a^m}\}$ where $a > 1, n, m \in \mathbb{Z}^+$ into two parts about the central term. That is, we partition LS as union of LS_1 and LS_2 where LS_1 consists

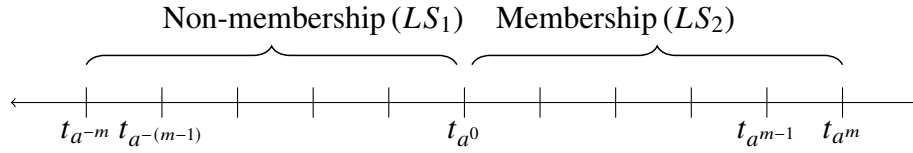


Figure 6.1: Partition of the set LS about the central term t_{a^0}

of terms having non-membership degree and lies on the left of central term and LS_2 consists of terms representing membership degree only and it lies on the right side of central term as shown in Figure 6.1.

The main highlights and contributions of this chapter are as follows:

1. The theory defined in this chapter is beneficial due to two causes: Firstly, a multiplicative linguistic scale set is used to characterize unbalanced linguistic data. Secondly, this approach gives a clear picture of the differentiation of membership and non-membership degrees as these are picked from different scale sets. These scale sets are selected by doing a partition of the original linguistic set.
2. The partitioning process may be viewed as the optimism and pessimism of the DM. The higher values to the right of the central term are denoted as membership set as we assume that a DM is an optimistic person, and an optimistic person would provide high membership to an element. Similarly, we assume that a pessimistic person would give a high non-membership value, so in the case of a pessimistic DM, the right set would denote the non-membership set.
3. This theory defines new score and accuracy functions to determine the comparison between two multiplicative unbalanced 2-TIFLNs. The score and accuracy function values belong to the defined interval $[a^{-n}, a^m]$.
4. Unlike the existing IFLSs defined in the literature, the unbalanced 2-TIFLS defined in this chapter satisfies the closure property. The existing theories are unsuitable for defining such operational laws and aggregation operators, which are closed under their defined operations. On the contrary, the aggregation weighted average, ordered weighted average, weighted geometric, ordered weighted geometric operators defined in this chapter are closed.

To resolve the issues faced in earlier contributions and make advancements in the existing theories, the chapter is set out as follows. Section 6.2 briefly reviews some basic concepts of IFLS introduced by Zhang [264] and 2-TIFLSs defined by Wei [234]. In Section 6.3, we introduce unbalanced 2-TIFLSs and define some desirable operational laws and their properties. In Section 6.4, we develop some novel aggregation operators for aggregating a collection of unbalanced 2-TIFL variables. These operators are namely the arithmetic and geometric operators. A numerical example is also provided in Section 6.5 to illustrate the relevance of defined aggregation operators practically. Section 6.6 concludes the chapter with some remarks.

6.2 Preliminaries

This section reviews certain essential concepts of the IFLS and 2-TIFLSs relevant for our work.

6.2.1 Intuitionistic Fuzzy Linguistic Sets

The concept of IFLS introduced by Zhang [264] is given as follows:

Definition 6.2.1. Let U be a universal set and $L = \{\ell_\varepsilon | \ell_0 < \ell_\varepsilon < \ell_g, \varepsilon \in [0, g]\}$ be a continuous linguistic term set. Then an IFLS in U is given as

$$P = \{(u, \ell_\mu(u), \ell_\nu(u)) | u \in U\},$$

where $\ell_\mu, \ell_\nu \in L$ stand for the linguistic membership degree and linguistic non-membership of the element u to P , respectively. The pair (ℓ_μ, ℓ_ν) is called Intuitionistic Fuzzy Linguistic Number (IFLN).

Definition 6.2.2. Let $\mathbf{p}_1 = (\ell_{\mu_1}, \ell_{\nu_1})$ and $\mathbf{p}_2 = (\ell_{\mu_2}, \ell_{\nu_2})$ be two Intuitionistic Fuzzy Linguistic Numbers (IFLNs). The score function of $\mathbf{p}_1$ is defined as $\mathfrak{s}(\mathbf{p}_1) = \ell_{(g+\mu_1-\nu_1)/2}$ and the accuracy function is defined as $\mathfrak{a}(\mathbf{p}_1) = \ell_{\mu_1+\nu_1}$

Definition 6.2.3. The comparison of $\mathbf{p}_1 = (\ell_{\mu_1}, \ell_{\nu_1})$ and $\mathbf{p}_2 = (\ell_{\mu_2}, \ell_{\nu_2})$ can be done as follows.

- If $\mathfrak{s}(\mathbf{p}_1) > \mathfrak{s}(\mathbf{p}_2)$, then $\mathbf{p}_1 > \mathbf{p}_2$.
- If $\mathfrak{s}(\mathbf{p}_1) = \mathfrak{s}(\mathbf{p}_2)$ and

- * $\alpha(p_1) = \alpha(p_2)$, then $p_1 = p_2$.
- * $\alpha(p_1) > \alpha(p_2)$, then $p_1 > p_2$.

6.2.2 2-tuple Intuitionistic Fuzzy Linguistic Sets

Wei [234] introduced the 2-TIFLSs based on symmetrically distributed linguistic term sets. Since, for the IFS, the degrees of membership and non-membership are non-negative real numbers. However, IFSs cannot depict and deal with qualitative membership degrees and qualitative non-membership degrees. Thus, Wei proposed the notion of 2-TIFLS given below to solve this kind of issue.

Definition 6.2.4. A 2-TIFLS P in X is given as

$$P = \{ \langle (\ell_{\mu(u)}, \alpha), (\ell_{\nu(u)}, \xi) \rangle, u \in U \}$$

where $\ell_{\mu(u)}, \ell_{\nu(u)} \in S = \{\ell_1, \ell_2, \dots, \ell_g\}$ and $\alpha, \xi \in [-0.5, 0.5]$ with the condition that $2 \leq \Delta^{-1}(\ell_{\mu(u)}, \alpha) + \Delta^{-1}(\ell_{\nu(u)}, \xi) \leq g+1, \forall u \in U$. $(\ell_{\mu(u)}, \alpha)$ and $(\ell_{\nu(u)}, \xi)$ represent the degree of membership and degree of non-membership of u to P . The Δ function is the function defined by Herrera & Martinez in [92], and is explained briefly as follows. Suppose $S = \{\ell_1, \ell_2, \dots, \ell_g\}$ be a linguistic term set and $\varsigma, \varpi \in [1, g]$ be two numbers exhibiting the aggregation results of linguistic symbolic. Then the function Δ used to obtain the 2-TIFL information equivalent to ς, ϖ are defined as :

$$\Delta: [1, g] \rightarrow S \times [-0.5, 0.5)$$

$$\Delta(\varsigma) = \begin{cases} \ell_{\mu}, \mu = \text{round}(\varsigma) \\ \alpha = \varsigma - \mu, \alpha \in [-0.5, 0.5) \end{cases}$$

$$\Delta(\varpi) = \begin{cases} \ell_{\nu}, \nu = \text{round}(\varpi) \\ \xi = \varpi - \nu, \xi \in [-0.5, 0.5) \end{cases}$$

$\text{round}(\cdot)$ being the usual round operation.

$\tilde{p} = \langle (\ell_{\mu}, \alpha), (\ell_{\nu}, \xi) \rangle$ is called a 2-tuple intuitionistic fuzzy linguistic number (2-TIFLN) such that $\ell_{\mu}, \ell_{\nu} \in S, \alpha, \xi \in [-0.5, 0.5), 2 \leq \Delta^{-1}(\ell_{\mu}, \alpha) + \Delta^{-1}(\ell_{\nu}, \xi) \leq g+1$.

Definition 6.2.5. Let $\tilde{p} = \langle (\ell_{\mu(u)}, \alpha), (\ell_{\nu(u)}, \xi) \rangle$ be a 2-TIFLN, then its score function is

defined as :

$$s(\tilde{p}) = \Delta \left(\frac{g + \Delta^{-1}(\ell_\mu, \alpha) - \Delta^{-1}(\ell_\nu, \xi)}{2} \right), \Delta^{-1}(s(\tilde{p})) \in [1, g]$$

Definition 6.2.6. Let $\tilde{p} = \{(\ell_\mu(u), \alpha), (\ell_\nu(u), \xi)\}$ be a 2-TIFLN, then its accuracy function is defined as :

$$a(\tilde{p}) = \Delta \left(\frac{\Delta^{-1}(\ell_\mu, \alpha) + \Delta^{-1}(\ell_\nu, \xi)}{2} \right), \Delta^{-1}(a(\tilde{p})) \in [1, n]$$

Definition 6.2.7. Comparison of 2-TIFLNs

Let $\tilde{p}_1 = \langle (\ell_{\mu_1}, \alpha_1), (\alpha_{\nu_1}, \xi_1) \rangle$ and $\tilde{p}_2 = \langle (\ell_{\mu_2}, \alpha_2), (\ell_{\nu_2}, \xi_2) \rangle$ be two 2-TIFLNs, then if $s(\tilde{p}_1) < s(\tilde{p}_2)$, then $\tilde{p}_1 < \tilde{p}_2$; if $s(\tilde{p}_1) = s(\tilde{p}_2)$, then

- If $a(\tilde{p}_1) = a(\tilde{p}_2)$, then $\tilde{p}_1 = \tilde{p}_2$.
- If $a(\tilde{p}_1) < a(\tilde{p}_2)$, then $\tilde{p}_1 < \tilde{p}_2$.

Apart from the above defined concepts, this theory makes use of the unbalanced 2-tuple linguistic computational model introduced by Malhotra & Gupta [153], defined in this thesis in Section 1.5.4 of Chapter 1. In this unbalanced 2-tuple linguistic computational model, we assume $n = m$ in the current chapter for the sake of convenience of computations.

6.3 Unbalanced 2-tuple intuitionistic fuzzy linguistic sets

In this section, we shall propose the theory and principal operations of the unbalanced 2-TIFLSs.

Definition 6.3.1. Let U be a universal set consisting of finite number of terms, and suppose that the set $\mathbb{L}\mathbb{S} = \{t_{a^{-m}}, t_{a^{-(m-1)}}, \dots, t_{a^0}, \dots, t_{a^m}\}$, where $a > 0, m \in \mathbb{Z}^+$ be a multiplicative unbalanced LT set with cardinality $\tau = 2m + 1$. Partition the set $\mathbb{L}\mathbb{S}$ across the middle term t_{a^0} such that we call the left partition as LS_1 and right partition as LS_2 . An unbalanced 2-tuple intuitionistic fuzzy linguistic set (2-TIFLS) $\tilde{S}$ in U is given as

$$\tilde{S} = \{\langle u, (t_{\lambda(u)}, \gamma), (t_{\Lambda(u)}, \Gamma) \rangle | u \in U\}$$

where $(t_{\lambda(u)}, \gamma)$ and $(t_{\Lambda(u)}, \Gamma)$ are unbalanced 2-tuple linguistic variables denoting the linguistic degree of membership and degree of non-membership respectively, of the element $u \in U$ to $\tilde{\mathcal{S}}$ such that $(t_{\lambda(u)}, \gamma) \in LS_2 \times [-0.5, 0.5]$, and $(t_{\Lambda(u)}, \Gamma) \in LS_1 \times [-0.5, 0.5]$ satisfying the condition that $a^{-m} \leq \Omega^{-1}(t_{\lambda(u)}, \gamma) \cdot \Omega^{-1}(t_{\Lambda(u)}, \Gamma) \leq a^m$. Here, t_{λ} and t_{Λ} are linguistic terms from pre-defined linguistic terms set $\mathbb{LS}$, and $\gamma, \Gamma \in [-0.5, 0.5]$. For convenience, we call the pair $\langle (t_{\lambda(u)}, \gamma), (t_{\Lambda(u)}, \Gamma) \rangle$ an unbalanced 2-TIFL number (2-TIFLN), and represent it as

$$\tilde{\mathfrak{p}} = \langle (t_{\lambda}, \gamma), (t_{\Lambda}, \Gamma) \rangle$$

with

$$a^{-m} \leq \Omega^{-1}(t_{\lambda}, \gamma) \cdot \Omega^{-1}(t_{\Lambda}, \Gamma) \leq a^m$$

Definition 6.3.2. Let $\beta, \varkappa$ be the results of aggregation of the indices of a set of labels assessed in a linguistic term set $\mathbb{LS}$, i.e. $\beta \in [a^0, a^m], \varkappa \in [a^{-m}, a^0]$. Then the transformation function Ω used to obtain the unbalanced 2-tuple intuitionistic fuzzy linguistic information equivalent to $\beta, \varkappa$ are defined as

$$\Omega: [a^0, a^m] \rightarrow LS_2 \times [-0.5, 0.5],$$

$$\Omega(\beta) = (t_{\lambda}, \gamma)$$

$$\text{where } \begin{cases} \lambda = a^{i_L} \text{ or } a^{i_R}, & \text{for } i_L, i_R \in \{0, \dots, m\} \\ \gamma = \frac{\beta - \lambda}{d_L + |d_R|}, & \text{for } \gamma \in [-0.5, 0.5] \end{cases}$$

and

$$\Omega: [a^{-m}, a^0] \rightarrow LS_1 \times [-0.5, 0.5],$$

$$\Omega(\varkappa) = (t_{\Lambda}, \Gamma)$$

$$\text{where } \begin{cases} \Lambda = a^{i_L} \text{ or } a^{i_R}, & \text{for } i_L, i_R \in \{-m, \dots, 0\} \\ \Gamma = \frac{\varkappa - \Lambda}{d_L + |d_R|}, & \text{for } \Gamma \in [-0.5, 0.5] \end{cases}$$

Here d_L and $|d_R|$ represent nearest left and right distance of Λ from $\varkappa$ (and λ from β) respectively. Moreover, the values λ and Λ are evaluated by the nearest distance measure from numerical values β and $\varkappa$ respectively, whereas γ and Γ are termed as the values of symbolic translation.

Definition 6.3.3. Let $\mathbb{LS} = \{t_{a^{-m}}, t_{a^{-(m-1)}}, \dots, t_{a^0}, \dots, t_{a^m}\}$ be an unbalanced LT set and $\tilde{\mathfrak{p}} = \langle (t_{\lambda}, \gamma), (t_{\Lambda}, \Gamma) \rangle$ be an unbalanced 2-TIFLN. Then there always a function Ω^{-1} can be defined, such that, from an unbalanced 2-TIFLN $\langle (t_{\lambda}, \gamma), (t_{\Lambda}, \Gamma) \rangle$, it returns its equivalent

numerical values $\beta \in [a^0, a^m]$, $\varkappa \in [a^{-m}, a^0] \subset \mathbb{R}$, which is

$$\Omega^{-1}: LS_2 \times [-0.5, 0.5] \rightarrow [a^0, a^m]$$

$$\Omega^{-1}(t_\lambda, \gamma) = \gamma(d_L + |d_R|) + \lambda = \beta$$

and,

$$\Omega^{-1}: LS_1 \times [-0.5, 0.5] \rightarrow [a^{-m}, a^0]$$

$$\Omega^{-1}(t_\Lambda, \Gamma) = \Gamma(d_L + |d_R|) + \Lambda = \varkappa$$

Moreover, it can be concluded that the conversion of two linguistic terms into an unbalanced 2-TIFLN consists of adding a value 0 as symbolic translation, i.e. $\Omega(\langle(t_\lambda, t_\Lambda)\rangle) = \langle(t_\lambda, 0), (t_\Lambda, 0)\rangle$.

Definition 6.3.4. Score Function

For an unbalanced 2-TIFLN $\tilde{\mathfrak{p}} = \langle(t_\lambda, \gamma), (t_\Lambda, \Gamma)\rangle$, the score function $\mathfrak{s}(\tilde{\mathfrak{p}})$ can be represented as follows:

$$\mathfrak{s}(\tilde{\mathfrak{p}}) = \Omega\left(\sqrt{\frac{\Omega^{-1}(t_\lambda, \gamma)}{\Omega^{-1}(t_\Lambda, \Gamma)}}\right)$$

s.t. $\Omega^{-1}(\mathfrak{s}(\tilde{\mathfrak{p}})) \in [a^{-m}, a^m]$.

Definition 6.3.5. Accuracy Function

For an unbalanced 2-TIFLN $\tilde{\mathfrak{p}} = \langle(t_\lambda, \gamma), (t_\Lambda, \Gamma)\rangle$, the accuracy function $\mathfrak{a}(\tilde{\mathfrak{p}})$ can be represented as follows:

$$\mathfrak{a}(\tilde{\mathfrak{p}}) = \Omega\left(\sqrt{\Omega^{-1}(t_\lambda, \gamma) \cdot \Omega^{-1}(t_\Lambda, \Gamma)}\right)$$

s.t. $\Omega^{-1}(\mathfrak{a}(\tilde{\mathfrak{p}})) \in [a^{-m}, a^m]$

Definition 6.3.6. Comparison of two unbalanced 2-TIFLNs

Let $\tilde{\mathfrak{p}}_1 = \langle(t_{\lambda_1}, \gamma_1), (t_{\Lambda_1}, \Gamma_1)\rangle$ and $\tilde{\mathfrak{p}}_2 = \langle(t_{\lambda_2}, \gamma_2), (t_{\Lambda_2}, \Gamma_2)\rangle$ be two unbalanced 2-TIFLNs. Then, if $\mathfrak{s}(\tilde{\mathfrak{p}}_1) < \mathfrak{s}(\tilde{\mathfrak{p}}_2)$, then we say that $\tilde{\mathfrak{p}}_1$ is smaller than $\tilde{\mathfrak{p}}_2$, and we denote it as $\tilde{\mathfrak{p}}_1 < \tilde{\mathfrak{p}}_2$. In case $\mathfrak{s}(\tilde{\mathfrak{p}}_1) = \mathfrak{s}(\tilde{\mathfrak{p}}_2)$, then

- If $\mathfrak{a}(\tilde{\mathfrak{p}}_1) = \mathfrak{a}(\tilde{\mathfrak{p}}_2)$, then $\tilde{\mathfrak{p}}_1 = \tilde{\mathfrak{p}}_2$;
- If $\mathfrak{a}(\tilde{\mathfrak{p}}_1) < \mathfrak{a}(\tilde{\mathfrak{p}}_2)$, then $\tilde{\mathfrak{p}}_1 < \tilde{\mathfrak{p}}_2$.

For example, let $\mathbb{LS} = \{t_{2^{-2}}, t_{2^{-1}}, t_{2^0}, t_{2^1}, t_{2^2}\}$ be a multiplicative LT set. Choose $\tilde{\mathfrak{p}}_1 = \langle(t_{2^0}, 0.139), (t_{2^{-1}}, 0.391)\rangle$. Clearly $(t_{2^0}, 0.139) \in LS_2 \times [-0.5, 0.5]$ and $(t_{2^{-1}}, 0.391) \in$

$LS_1 \times [-0.5, 0.5]$, and $\Omega^{-1}((t_{20}, 0.139)) \cdot \Omega^{-1}((t_{2-1}, 0.391)) = (1.1391) \cdot (0.6955) = 0.792244 \in [2^{-2}, 2^2]$, thus $\tilde{p}_1$ is a valid 2-TIFLN.

Similarly, let $\tilde{p}_2 = \langle (t_{21}, 0.189), (t_{2-2}, 0.046) \rangle$, then $\tilde{p}_2$ is also a valid 2-TIFLN. Now, $s(\tilde{p}_1) = (t_{20}, 0.2783)$ and, $s(\tilde{p}_2) = (t_{22}, -0.4924)$. Since $s(\tilde{p}_1) < s(\tilde{p}_2)$, it can be concluded that $\tilde{p}_1 < \tilde{p}_2$.

Motivated by the operations of the unbalanced 2-tuple intuitionistic linguistic information, in the following, we shall define some operational laws of unbalanced 2-TIFLNs.

Definition 6.3.7. Let $\tilde{p}_1 = \langle (t_{\lambda_1}, \gamma_1), (t_{\Lambda_1}, \Gamma_1) \rangle$ and $\tilde{p}_2 = \langle (t_{\lambda_2}, \gamma_2), (t_{\Lambda_2}, \Gamma_2) \rangle$ be two unbalanced 2-TIFLNs, then

1. $\tilde{p}_1 \oplus \tilde{p}_2 = \langle \Omega \left[\min \left\{ a^m, \Omega^{-1}(t_{\lambda_1}, \gamma_1) + \Omega^{-1}(t_{\lambda_2}, \gamma_2) - \frac{\Omega^{-1}(t_{\lambda_1}, \gamma_1) \cdot \Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} \right\} \right], \Omega \left[\max \left\{ a^{-m}, \Omega^{-1}(t_{\Lambda_1}, \Gamma_1) \cdot \Omega^{-1}(t_{\Lambda_2}, \Gamma_2) \right\} \right] \rangle$
2. $\tilde{p}_1 \otimes \tilde{p}_2 = \langle \Omega \left[\min \left\{ a^m, \Omega^{-1}(t_{\lambda_1}, \gamma_1) \cdot \Omega^{-1}(t_{\lambda_2}, \gamma_2) \right\} \right], \Omega \left[\max \left\{ a^{-m}, \Omega^{-1}(t_{\Lambda_1}, \Gamma_1) + \Omega^{-1}(t_{\Lambda_2}, \Gamma_2) - \Omega^{-1}(t_{\Lambda_1}, \Gamma_1) \cdot \Omega^{-1}(t_{\Lambda_2}, \Gamma_2) \right\} \right] \rangle$
3. Let $\kappa \in [0, 1]$ be a scalar and $\tilde{p} = \langle (t_\lambda, \gamma), (t_\Lambda, \Gamma) \rangle$, then

$$\kappa \tilde{p} = \langle \Omega \left[\max \left\{ a^0, \tau \left| 1 - \left| 1 - \frac{\Omega^{-1}(t_\lambda, \gamma)}{\tau} \right|^\kappa \right| \right\} \right], \Omega \left(\Omega^{-1}(t_\Lambda, \Gamma) \right)^\kappa \rangle$$

4. For any scalar $\kappa \in [0, 1]$, $\tilde{p}^\kappa = \langle \Omega \left(\Omega^{-1}(t_\lambda, \gamma) \right)^\kappa, \Omega \left[\max \left\{ a^{-m}, \tau \left| 1 - \left| 1 - \frac{\Omega^{-1}(t_\Lambda, \Gamma)}{\tau} \right|^\kappa \right| \right\} \right] \rangle$
5. Complement of any 2-TIFLN $\tilde{p} = \langle (t_\lambda, \gamma), (t_\Lambda, \Gamma) \rangle$ is

$$\begin{aligned} \tilde{p}^c &= Neg(\tilde{p}) \\ &= \langle Neg(t_\lambda, \gamma), Neg(t_\Lambda, \Gamma) \rangle \\ &= \langle (Neg(t_\lambda), -\gamma), (Neg(t_\Lambda), -\Gamma) \rangle \end{aligned}$$

where $Neg(t_\lambda)$ and $Neg(t_\Lambda)$ are same as defined in Definition 1.5.13 in Chapter 1.

Note 6.3.1. The above defined operational laws satisfy the closure property that is proved in Theorem 6.3.1. Also, we have developed a code in MS-Excel and generated random unbalanced 2-TIFLNs to verify the same. The values in the following example are also picked from those generated random values.

Example 6.3.1. Let $\mathbb{LS} = \{t_{2^{-2}}, t_{2^{-1}}, t_{2^0}, t_{2^1}, t_{2^2}\}$. Choose

$$\tilde{\mathbf{p}}_1 = \langle (t_{2^1}, 0.39), (t_{2^0}, 0) \rangle \text{ and } \tilde{\mathbf{p}}_2 = \langle (t_{2^2}, 0), (t_{2^{-2}}, 0.17) \rangle$$

$$\begin{aligned} \tilde{\mathbf{p}}_1 \oplus \tilde{\mathbf{p}}_2 &= \langle \Omega \left[\min \left\{ 2^2, 2.78 + 4 - \frac{(2.78) \cdot (4)}{5} \right\} \right], \Omega \left[\max \left\{ 2^{-2}, (1) \cdot (0.29) \right\} \right] \rangle \\ &= \langle \Omega [\min \{4, 4.556\}], \Omega [\max \{0.25, 0.29\}] \rangle \\ &= \langle (t_{2^2}, 0), (t_{2^{-2}}, 0.16) \rangle \end{aligned}$$

$$\begin{aligned} \tilde{\mathbf{p}}_1 \otimes \tilde{\mathbf{p}}_2 &= \langle \Omega [\min \{4, (2.78) \cdot (4)\}], \Omega [\max \{0.25, 1 + 0.29 - (1) \cdot (0.29)\}] \rangle \\ &= \langle \Omega [\min \{4, 11.12\}], \Omega [\max \{0.25, 1\}] \rangle \\ &= \langle (t_4, 0), (t_1, 0) \rangle \end{aligned}$$

Now, suppose $\kappa = 0.73 \in [0, 1]$, then

$$\begin{aligned} \kappa \tilde{\mathbf{p}}_1 &= \langle \Omega \left[\max \left\{ 1, 5 \times \left| 1 - \left| 1 - \frac{2.78}{5} \right|^{0.73} \right| \right\} \right], \Omega (1)^{0.73} \rangle \\ &= \langle \Omega [\max \{1, 2.2359\}], \Omega (1) \rangle \\ &= \langle (t_{2^1}, 0.118), (t_{2^0}, 0) \rangle \end{aligned}$$

$$\begin{aligned} \tilde{\mathbf{p}}_1^\kappa &= \langle \Omega (2.78)^{0.73}, \Omega \left[\max \left\{ 0.25, 5 \times \left| 1 - \left| 1 - \frac{1}{5} \right|^{0.73} \right| \right\} \right] \rangle \\ &= \langle \Omega (2.1094), \Omega [\max \{0.25, 0.7516\}] \rangle \\ &= \langle (t_2, 0.0547), (t_1, -0.4968) \rangle \end{aligned}$$

Further,

$$\begin{aligned} \tilde{\mathbf{p}}_1^c &= \langle \text{Neg}(t_{2^1}, 0.39), \text{Neg}(t_{2^0}, 0) \rangle \\ &= \langle (\text{Neg}(t_{2^1}), -0.39), (\text{Neg}(t_{2^0}), 0) \rangle \\ &= \langle (t_{2^{-1}}, -0.39), (t_{2^0}, 0) \rangle \end{aligned}$$

Similarly,

$$\begin{aligned}
 \tilde{\mathfrak{p}}_2^c &= \langle \text{Neg}(t_{2^2}, 0), \text{Neg}(t_{2^{-2}}, 0.17) \rangle \\
 &= \langle (\text{Neg}(t_{2^2}), 0), (\text{Neg}(t_{2^{-2}}), -0.17) \rangle \\
 &= \langle (t_{2^{-2}}, 0), (t_{2^2}, -0.17) \rangle
 \end{aligned}$$

Theorem 6.3.1. Let $\mathbb{LS} = \{t_{a^{-m}}, \dots, t_{a^m}\}$ be the predefined linguistic terms set, and suppose $\tilde{\mathfrak{p}}_1 = \langle (t_{\lambda_1}, \gamma_1), (t_{\Lambda_1}, \Gamma_1) \rangle$ and $\tilde{\mathfrak{p}}_2 = \langle (t_{\lambda_2}, \gamma_2), (t_{\Lambda_2}, \Gamma_2) \rangle$ be two unbalanced 2-TIFLNs, then the operational laws defined in Definition 6.3.7, are 2-TIFLNs as well.

Proof. Considering $\tilde{\mathfrak{p}}_1$ and $\tilde{\mathfrak{p}}_2$, $(t_{\lambda_1}, \gamma_1), (t_{\lambda_2}, \gamma_2) \in LS_2 \times [-0.5, 0.5]$ and $(t_{\Lambda_1}, \Gamma_1), (t_{\Lambda_2}, \Gamma_2) \in LS_1 \times [-0.5, 0.5]$. Now,

$$\begin{aligned}
 \tilde{\mathfrak{p}}_1 \oplus \tilde{\mathfrak{p}}_2 &= \langle \Omega \left[\min \left\{ a^m, \Omega^{-1}(t_{\lambda_1}, \gamma_1) + \Omega^{-1}(t_{\lambda_2}, \gamma_2) - \frac{\Omega^{-1}(t_{\lambda_1}, \gamma_1) \cdot \Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} \right\} \right], \\
 &\quad \Omega \left[\max \left\{ a^{-m}, \Omega^{-1}(t_{\Lambda_1}, \Gamma_1) \cdot \Omega^{-1}(t_{\Lambda_2}, \Gamma_2) \right\} \right] \rangle
 \end{aligned}$$

Now,

Case 1 : If $\min \left\{ a^m, \Omega^{-1}(t_{\lambda_1}, \gamma_1) + \Omega^{-1}(t_{\lambda_2}, \gamma_2) - \frac{\Omega^{-1}(t_{\lambda_1}, \gamma_1) \cdot \Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} \right\} = a^m$ and,

$$\max \left\{ a^{-m}, \Omega^{-1}(t_{\Lambda_1}, \Gamma_1) \cdot \Omega^{-1}(t_{\Lambda_2}, \Gamma_2) \right\} = a^{-m}$$

then clearly $a^m \in LS_2 \times [-0.5, 0.5]$ and $a^{-m} \in LS_1 \times [-0.5, 0.5]$ and $a^m \times a^{-m} = 1 \in [a^{-m}, a^m]$. Thus, $\tilde{\mathfrak{p}}_1 \oplus \tilde{\mathfrak{p}}_2$ is a 2-TIFLN.

Case 2 : If $\Omega^{-1}(t_{\lambda_1}, \gamma_1) + \Omega^{-1}(t_{\lambda_2}, \gamma_2) - \frac{\Omega^{-1}(t_{\lambda_1}, \gamma_1) \cdot \Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} < a^m$ and

$$\max \left\{ a^{-m}, \Omega^{-1}(t_{\Lambda_1}, \Gamma_1) \cdot \Omega^{-1}(t_{\Lambda_2}, \Gamma_2) \right\} = a^{-m}$$

then as we know that $\Omega^{-1}(t_{\lambda_1}, \gamma_1), \Omega^{-1}(t_{\lambda_2}, \gamma_2) \in [a^0, a^m]$, so we can say that

$$\mu = \Omega^{-1}(t_{\lambda_1}, \gamma_1) + \Omega^{-1}(t_{\lambda_2}, \gamma_2) - \frac{\Omega^{-1}(t_{\lambda_1}, \gamma_1) \cdot \Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} \in [a^0, a^m] \quad (\because LHS < a^m).$$

$$\therefore \mu \times a^{-m} \in [a^{-m}, a^m]$$

making $\tilde{\mathfrak{p}}_1 \oplus \tilde{\mathfrak{p}}_2$ a 2-TIFLN.

Case 3 : If $\Omega^{-1}(t_{\lambda_1}, \gamma_1) + \Omega^{-1}(t_{\lambda_2}, \gamma_2) - \frac{\Omega^{-1}(t_{\lambda_1}, \gamma_1) \cdot \Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} < a^m$ and

$$\Omega^{-1}(t_{\Lambda_1}, \Gamma_1) \cdot \Omega^{-1}(t_{\Lambda_2}, \Gamma_2) > a^{-m}$$

Then as we know that $\Omega^{-1}(t_{\lambda_1}, \gamma_1), \Omega^{-1}(t_{\lambda_2}, \gamma_2) \in [a^0, a^m]$, so we can say that

$$\mu = \Omega^{-1}(t_{\lambda_1}, \gamma_1) + \Omega^{-1}(t_{\lambda_2}, \gamma_2) - \frac{\Omega^{-1}(t_{\lambda_1}, \gamma_1) \cdot \Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} \in [a^0, a^m]$$

and,

$$\nu = \Omega^{-1}(t_{\Lambda_1}, \Gamma_1) \cdot \Omega^{-1}(t_{\Lambda_2}, \Gamma_2) \in [a^{-m}, a^0]$$

$$\implies \mu \times \nu \in [a^{-m}, a^m]$$

$\therefore \tilde{\mathfrak{p}}_1 \oplus \tilde{\mathfrak{p}}_2$ is a 2-TIFLN.

Case 4 : If $\min \left\{ a^m, \Omega^{-1}(t_{\lambda_1}, \gamma_1) + \Omega^{-1}(t_{\lambda_2}, \gamma_2) - \frac{\Omega^{-1}(t_{\lambda_1}, \gamma_1) \cdot \Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} \right\} = a^m$ and,

$$\Omega^{-1}(t_{\Lambda_1}, \Gamma_1) \cdot \Omega^{-1}(t_{\Lambda_2}, \Gamma_2) > a^{-m}$$

Then as $\mu \times \nu = \Omega^{-1}(t_{\Lambda_1}, \Gamma_1) \cdot \Omega^{-1}(t_{\Lambda_2}, \Gamma_2) \in [a^{-m}, a^0]$

$$\implies a^m \times \nu \in [a^{-m}, a^m]$$

$\therefore \tilde{\mathfrak{p}}_1 \oplus \tilde{\mathfrak{p}}_2$ is a 2-TIFLN.

Hence, $\tilde{\mathfrak{p}}_1 \oplus \tilde{\mathfrak{p}}_2$ makes a valid 2-TIFLN for every case.

Similarly, we can prove that the other operational laws, such as, $\tilde{\mathfrak{p}}_1 \otimes \tilde{\mathfrak{p}}_2$, $\kappa \tilde{\mathfrak{p}}_1$ & $\tilde{\mathfrak{p}}_1^\kappa$ for $\kappa \in [0, 1]$ and $\tilde{\mathfrak{p}}_1^c$ are also 2-TIFLNs. $\square$

Theorem 6.3.2. For any three unbalanced 2-TIFLNs,

$$\tilde{\mathfrak{p}}_1 = \langle (t_{\lambda_1}, \gamma_1), (t_{\Lambda_1}, \Gamma_1) \rangle,$$

$$\tilde{\mathfrak{p}}_2 = \langle (t_{\lambda_2}, \gamma_2), (t_{\Lambda_2}, \Gamma_2) \rangle,$$

$$\tilde{\mathfrak{p}}_3 = \langle (t_{\lambda_3}, \gamma_3), (t_{\Lambda_3}, \Gamma_3) \rangle$$

the operators defined in Definition 6.3.7 have some algebraic properties as below:

$$1. \quad \tilde{\mathfrak{p}}_1 \oplus \tilde{\mathfrak{p}}_2 = \tilde{\mathfrak{p}}_2 \oplus \tilde{\mathfrak{p}}_1.$$

$$2. \tilde{p}_1 \oplus (\tilde{p}_2 \oplus \tilde{p}_3) = (\tilde{p}_1 \oplus \tilde{p}_2) \oplus \tilde{p}_3.$$

$$3. \tilde{p}_1 \otimes \tilde{p}_2 = \tilde{p}_2 \otimes \tilde{p}_1.$$

$$4. \tilde{p}_1 \otimes (\tilde{p}_2 \otimes \tilde{p}_3) = (\tilde{p}_1 \otimes \tilde{p}_2) \otimes \tilde{p}_3.$$

6.4 Unbalanced 2-TIFL Aggregation Operators

This section will first develop arithmetic aggregation operators having information as an unbalanced 2-TIFLN. These operators are called unbalanced 2-tuple intuitionistic fuzzy linguistic weighted average (2-TIFLWA) operator and unbalanced 2-tuple intuitionistic fuzzy linguistic ordered weighted average (2-TIFLOWA) operator. Next, we will define the geometric aggregation operators called unbalanced 2-tuple intuitionistic fuzzy linguistic weighted geometric (2-TIFLWG) and ordered weighted geometric (2-TIFLOWG) operators.

6.4.1 Arithmetic Aggregation Operators

Definition 6.4.1. Weighted Average Operator Let $\tilde{p}_i = \langle (t_{\lambda_i}, \gamma_i), (t_{\Lambda_i}, \Gamma_i) \rangle$, $i = 1, 2, \dots, z$ where, $z \in \mathbb{Z}^+$ and $z \in [-m, m]$ be a collection of some unbalanced 2-TIFLNs, and w_i , $i = 1, 2, \dots, z$ be the associated weights such that $w_i \in [0, 1]$ and $\sum_{i=1}^z w_i = 1$, then the 2-TIFLWA operator $\tilde{p}_i^{WA}$ for unbalanced 2-TIFLNs is defined as :

$$\begin{aligned} \tilde{p}_i^{WA} &= \oplus_{i=1}^z w_i \tilde{p}_i \\ &= w_1 \tilde{p}_1 \oplus w_2 \tilde{p}_2 \oplus \dots \oplus w_z \tilde{p}_z \end{aligned}$$

Theorem 6.4.1. Let $\tilde{p}_i = \langle (t_{\lambda_i}, \gamma_i), (t_{\Lambda_i}, \Gamma_i) \rangle$, $i = 1, 2, \dots, z$ be a collection of some unbalanced 2-TIFLNs, and w_i , $i = 1, 2, \dots, z$ be the associated weights such that $w_i \in [0, 1]$ and $\sum_{i=1}^z w_i = 1$, then the 2-TIFLWA operator $\tilde{p}_i^{WA}$ for unbalanced 2-TIFLNs can be given as:

$$\begin{aligned} \tilde{p}_i^{WA} &= \oplus_{i=1}^z w_i \tilde{p}_i \\ &= \left\langle \Omega \left[\min \left\{ a^m, \tau \left| 1 - \prod_{i=1}^z \left| 1 - \frac{\Omega^{-1}(t_{\lambda_i}, \gamma_i)}{\tau} \right|^{w_i} \right| \right\} \right], \Omega \left[\max \left\{ a^{-m}, \prod_{i=1}^z \left(\Omega^{-1}(t_{\Lambda_i}, \Gamma_i) \right)^{w_i} \right\} \right] \right\rangle \end{aligned}$$

Proof. We use the mathematical inductive method to prove theorem 6.4.1.

First, for $z = 2$, by Definition [6.3.7](#)

$$w_1 \tilde{\mathbf{p}}_1 = \langle \Omega \left[\max \left\{ a^0, \tau \left| 1 - \left| 1 - \frac{\Omega^{-1}(t_{\lambda_1}, \gamma_1)}{\tau} \right|^{w_1} \right| \right\} \right], \Omega \left(\Omega^{-1}(t_{\Lambda_1}, \Gamma_1) \right)^{w_1} \rangle$$

and,

$$w_2 \tilde{\mathbf{p}}_2 = \langle \Omega \left[\max \left\{ a^0, \tau \left| 1 - \left| 1 - \frac{\Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} \right|^{w_2} \right| \right\} \right], \Omega \left(\Omega^{-1}(t_{\Lambda_2}, \Gamma_2) \right)^{w_2} \rangle$$

Now for $i = 1, 2$, according to Definition [6.4.1](#),

$$\tilde{\mathbf{p}}_i^{WA} = w_1 \tilde{\mathbf{p}}_1 \oplus w_2 \tilde{\mathbf{p}}_2$$

Now, if in $w_1 \tilde{\mathbf{p}}_1$, $\max \left\{ a^0, \tau \left| 1 - \left| 1 - \frac{\Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} \right|^{w_2} \right| \right\} = a^0$ then since $w_1 + w_2 = 1$ and $w_1, w_2 \geq 0$, so in $w_2 \tilde{\mathbf{p}}_2$, $\max \left\{ a^0, \tau \left| 1 - \left| 1 - \frac{\Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} \right|^{w_2} \right| \right\} = \tau \left| 1 - \left| 1 - \frac{\Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} \right|^{w_2} \right|$. Further, if in $w_2 \tilde{\mathbf{p}}_2$, $\max \{ a^0, \tau \left| 1 - \left| 1 - \frac{\Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} \right|^{w_2} \right| \} = \tau \left| 1 - \left| 1 - \frac{\Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} \right|^{w_2} \right|$, then clearly

$$\tau \left| 1 - \left| 1 - \frac{\Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} \right|^{w_2} \right| \geq 1$$

So, it can be concluded that for $w_1 \tilde{\mathbf{p}}_1 \oplus w_2 \tilde{\mathbf{p}}_2$, if $w_1 + w_2 = 1$, then membership is always greater than or equal to 1.

So we can write

$$\begin{aligned} w_1 \tilde{\mathbf{p}}_1 &= \langle \Omega \left[\tau \left| 1 - \left| 1 - \frac{\Omega^{-1}(t_{\lambda_1}, \gamma_1)}{\tau} \right|^{w_1} \right| \right], \Omega \left(\Omega^{-1}(t_{\Lambda_1}, \Gamma_1) \right)^{w_1} \rangle \\ w_2 \tilde{\mathbf{p}}_2 &= \langle \Omega \left[\tau \left| 1 - \left| 1 - \frac{\Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} \right|^{w_2} \right| \right], \Omega \left(\Omega^{-1}(t_{\Lambda_2}, \Gamma_2) \right)^{w_2} \rangle \end{aligned} \quad (6.4.1)$$

$$\Rightarrow w_1 \tilde{\mathbf{p}}_1 \oplus w_2 \tilde{\mathbf{p}}_2 = \langle \Omega \left[\min \left\{ a^m, \tau \left| 1 - \left| 1 - \frac{\Omega^{-1}(t_{\lambda_1}, \gamma_1)}{\tau} \right|^{w_1} \right| 1 - \frac{\Omega^{-1}(t_{\lambda_2}, \gamma_2)}{\tau} \right|^{w_2} \right| \right] \right], \Omega \left(\Omega^{-1}(t_{\Lambda_1}, \Gamma_1) \right)^{w_1} \Omega \left(\Omega^{-1}(t_{\Lambda_2}, \Gamma_2) \right)^{w_2} \rangle$$

$$\max \left\{ a^{-m}, \Omega \left(\left(\Omega^{-1}(t_{\Lambda_1}, \Gamma_1) \right)^{w_1} \cdot \left(\Omega^{-1}(t_{\Lambda_2}, \Gamma_2) \right)^{w_2} \right) \right\}$$

So the result is true for $z = 2$.

Secondly, we assume that the result is true for $z - 1$, i.e. for $i = 1, 2, \dots, z - 1$

$$\begin{aligned} \tilde{\mathfrak{p}}_i^{WA} &= \oplus_{i=1}^{z-1} w_i \tilde{\mathfrak{p}}_i \\ &= w_1 \tilde{\mathfrak{p}}_1 \oplus w_2 \tilde{\mathfrak{p}}_2 \oplus \dots \oplus w_{z-1} \tilde{\mathfrak{p}}_{z-1} \\ &= \Omega \left[\min \left\{ a^m, \tau \left| 1 - \prod_{i=1}^{z-1} \left| 1 - \frac{\Omega^{-1}(t_{\Lambda_i}, \gamma_i)^{w_i}}{\tau} \right| \right| \right\} \right], \Omega \left[\max \left\{ a^{-m}, \prod_{i=1}^{z-1} \left(\Omega^{-1}(t_{\Lambda_i}, \Gamma_i) \right)^{w_i} \right\} \right] \end{aligned}$$

Then for z ,

$$\begin{aligned} \tilde{\mathfrak{p}}_i^{WA} &= \oplus_{i=1}^z w_i \tilde{\mathfrak{p}}_i = \oplus_{i=1}^{z-1} w_i \tilde{\mathfrak{p}}_i \oplus w_z \tilde{\mathfrak{p}}_z \\ &= \Omega \left[\min \left\{ a^m, \tau \left| 1 - \prod_{i=1}^{z-1} \left| 1 - \frac{\Omega^{-1}(t_{\Lambda_i}, \gamma_i)^{w_i}}{\tau} \right| \right| \right\} \right], \Omega \left[\max \left\{ a^{-m}, \prod_{i=1}^{z-1} \left(\Omega^{-1}(t_{\Lambda_i}, \Gamma_i) \right)^{w_i} \right\} \right] \\ &\quad \oplus \langle \Omega \left[\tau \left| 1 - \left| 1 - \frac{\Omega^{-1}(t_{\Lambda_z}, \gamma_z)^{w_z}}{\tau} \right| \right| \right], \Omega \left(\Omega^{-1}(t_{\Lambda_z}, \Gamma_z) \right)^{w_z} \rangle \\ &= \langle \Omega \left[\min \left\{ a^m, \tau \left| 1 - \prod_{i=1}^z \left| 1 - \frac{\Omega^{-1}(t_{\Lambda_i}, \gamma_i)^{w_i}}{\tau} \right| \right| \right\} \right], \Omega \left[\max \left\{ a^{-m}, \prod_{i=1}^z \left(\Omega^{-1}(t_{\Lambda_i}, \Gamma_i) \right)^{w_i} \right\} \right] \rangle \end{aligned}$$

Hence the result is true for any z . $\square$

It may further be proved that the 2-TIFLWA operator satisfies the following properties:

- **Closure** : If $\tilde{\mathfrak{p}}_i = \langle (t_{\Lambda_i}, \gamma_i), (t_{\Lambda_i}, \Gamma_i) \rangle$ is a collection of some unbalanced 2-TIFLNs, then $\tilde{\mathfrak{p}}_i^{WA}$ is also an unbalanced 2-TIFLN for all $i = 1, 2, \dots, z$.
- **Boundedness** : The 2-TIFLWA operator is bounded by $\tilde{\mathfrak{p}}^- \leq \tilde{\mathfrak{p}}_i^{WA} \leq \tilde{\mathfrak{p}}^+$, where $\tilde{\mathfrak{p}}^- = \min_i(\tilde{\mathfrak{p}}_i)$ and $\tilde{\mathfrak{p}}^+ = \max_i(\tilde{\mathfrak{p}}_i)$, $i = 1, 2, \dots, z$.
- **Monotonicity** : Let $\tilde{\mathfrak{p}}_i$ and $\tilde{\mathfrak{p}}'_i$, ($i = 1, \dots, z$) be two different collections of 2-TIFLNs and suppose that $\tilde{\mathfrak{p}}_i \leq \tilde{\mathfrak{p}}'_i \forall i$, then $\tilde{\mathfrak{p}}_i^{WA} \leq \tilde{\mathfrak{p}}'_i^{WA}$.

Definition 6.4.2. Ordered Weighted Average Operator Let $\tilde{\mathfrak{p}}_i = \langle (t_{\Lambda_i}, \gamma_i), (t_{\Lambda_i}, \Gamma_i) \rangle$, $i = 1, 2, \dots, z$ be a collection of some unbalanced 2-TIFLNs, and $w_i, i = 1, 2, \dots, z$ be the

associated weights such that $w_i \in [0, 1]$ and $\sum_{i=1}^z w_i = 1$, then the unbalanced 2-TIFLOWA operator is computed as :

$$\tilde{\mathbf{p}}_i^{OWA} = \oplus_{i=1}^z w_i \tilde{\mathbf{p}}_{\sigma(i)}$$

where $\sigma(i), i = 1, \dots, z$ represents the permutation of $(1, \dots, z)$ such that $\tilde{\mathbf{p}}_{\sigma(i-1)} \geq \tilde{\mathbf{p}}_{\sigma(i)}, \forall i$, i.e. $\tilde{\mathbf{p}}_{\sigma(i)}$ is the i^{th} largest value according to the ordering given in Definition 6.3.6.

Theorem 6.4.2. Let $\tilde{\mathbf{p}}_i = \langle (t_{\lambda_i}, \gamma_i), (t_{\Lambda_i}, \Gamma_i) \rangle, i = 1, 2, \dots, z$ be a collection of some unbalanced 2-TIFLNs, and $w_i, i = 1, 2, \dots, z$ be the associated weights such that $w_i \in [0, 1]$ and $\sum_{i=1}^z w_i = 1$, then the 2-TIFLOWA operator $\tilde{\mathbf{p}}_i^{OWA}$ for unbalanced 2-TIFLNs can be given as:

$$\begin{aligned} \tilde{\mathbf{p}}_i^{OWA} &= \oplus_{i=1}^z w_i \tilde{\mathbf{p}}_{\sigma(i)} \\ &= \left\langle \Omega \left[\min \left\{ a^m, \tau \left| 1 - \prod_{i=1}^z \left(1 - \frac{\Omega^{-1}(t_{\lambda_{\sigma(i)}}, \phi_{\sigma(i)})}{\tau} \right)^{w_i} \right| \right\} \right], \right. \\ &\quad \left. \Omega \left[\max \left\{ a^{-m}, \prod_{i=1}^z \left(\Omega^{-1}(t_{\gamma_{\sigma(i)}}, \psi_{\sigma(i)}) \right)^{w_i} \right\} \right] \right\rangle \end{aligned}$$

where $\sigma(i), i = 1, \dots, z$ are as defined above.

Proof. The proof is similar to the proof of Theorem 6.4.1. □

The 2-TIFLOWA satisfies all the properties similar to the 2-TIFLWA operator mentioned above, i.e.

- **Closure :** If $\tilde{\mathbf{p}}_i = \langle (t_{\lambda_i}, \gamma_i), (t_{\Lambda_i}, \Gamma_i) \rangle$ is a collection of unbalanced 2-TIFLNs, then $\tilde{\mathbf{p}}_i^{OWA}$ is also an unbalanced 2-TIFLN for all $i = 1, 2, \dots, z$.
- **Boundedness :** The 2-TIFLOWA operator is bounded by $\tilde{\mathbf{p}}^- \leq \tilde{\mathbf{p}}_i^{OWA} \leq \tilde{\mathbf{p}}^+$, where $\tilde{\mathbf{p}}^- = \min_i(\tilde{\mathbf{p}}_i)$ and $\tilde{\mathbf{p}}^+ = \max_i(\tilde{\mathbf{p}}_i), i = 1, 2, \dots, z$.
- **Monotonicity :** Let $\tilde{\mathbf{p}}_i$ and $\tilde{\mathbf{p}}'_i, (i = 1, \dots, z)$ be two different collections of 2-TIFLNs and suppose that $\tilde{\mathbf{p}}_i \leq \tilde{\mathbf{p}}'_i \forall i$, then $\tilde{\mathbf{p}}_i^{OWA} \leq \tilde{\mathbf{p}}'^{OWA}_i$.

Remark 6.4.1. If $w = (\frac{1}{z}, \frac{1}{z}, \dots, \frac{1}{z})^T$ is the associated weight vector for $\tilde{\mathbf{p}}_i (i = 1, 2, \dots, z)$, then the unbalanced 2-TIFLWA operator and the unbalanced 2-TIFLOWA operator reduce

to unbalanced 2-tuple intuitionistic fuzzy linguistic averaging (2-TIFLA) operator which is defined as follows:

$$\tilde{p}_i^A = \frac{1}{z} \tilde{p}_1 \oplus \frac{1}{z} \tilde{p}_2 \oplus \dots \oplus \frac{1}{z} \tilde{p}_z$$

6.4.2 Geometric Aggregation Operators

Definition 6.4.3. Weighted Geometric Operator

Let $\tilde{p}_i = \langle (t_{\lambda_i}, \gamma_i), (t_{\Lambda_i}, \Gamma_i) \rangle$, $i = 1, 2, \dots, z$ be a collection of some unbalanced 2-TIFLNs, and w_i , $i = 1, 2, \dots, z$ be the associated weights such that $w_i \in [0, 1]$ and $\sum_{i=1}^z w_i = 1$, then the 2-TIFLWG operator $\tilde{p}_i^{WG}$ for unbalanced 2-TIFLNs is defined as :

$$\begin{aligned} \tilde{p}_i^{WG} &= \otimes_{i=1}^z (\tilde{p}_i)^{w_i} \\ &= (\tilde{p}_1)^{w_1} \otimes (\tilde{p}_2)^{w_2} \otimes \dots \otimes (\tilde{p}_z)^{w_z} \end{aligned}$$

Theorem 6.4.3. Let $\tilde{p}_i = \langle (t_{\lambda_i}, \gamma_i), (t_{\Lambda_i}, \Gamma_i) \rangle$, $i = 1, 2, \dots, z$ be a collection of some unbalanced 2-TIFLNs, and w_i , $i = 1, 2, \dots, z$ be the associated weights such that $w_i \in [0, 1]$ and $\sum_{i=1}^z w_i = 1$, then the 2-TIFLWG operator $\tilde{p}_i^{WG}$ for unbalanced 2-TIFLNs can be given as:

$$\begin{aligned} \tilde{p}_i^{WG} &= \otimes_{i=1}^z (\tilde{p}_i)^{w_i} \\ &= \left\langle \Omega \left[\min \left\{ a^m, \prod_{i=1}^z \left((\Omega^{-1}(t_{\lambda_i}, \gamma_i))^{w_i} \right) \right\} \right], \right. \\ &\quad \left. \Omega \left[\max \left\{ a^{-m}, \tau \left| 1 - \prod_{i=1}^z \left| 1 - \frac{\Omega^{-1}(t_{\Lambda_i}, \Gamma_i)}{\tau} \right|^{w_i} \right| \right\} \right] \right\rangle \end{aligned}$$

Proof. The proof is similar to the proof of Theorem [6.4.1](#). □

Further, it can be proved that the 2-TIFLWG operator satisfies the following properties:

- **Closure** : If $\tilde{p}_i = \langle (t_{\lambda_i}, \gamma_i), (t_{\Lambda_i}, \Gamma_i) \rangle$ is a collection of some unbalanced 2-TIFLNs, then $\tilde{p}_i^{WG}$ is also an unbalanced 2-TIFLN for all $i = 1, 2, \dots, z$.
- **Boundedness** : The 2-TIFLWG operator is bounded by $\tilde{p}^- \leq \tilde{p}_i^{WG} \leq \tilde{p}^+$, where $\tilde{p}^- = \min_i(\tilde{p}_i)$ and $\tilde{p}^+ = \max_i(\tilde{p}_i)$, $i = 1, 2, \dots, z$.

- **Monotonicity** : Let $\tilde{p}_i$ and $\tilde{p}'_i$, $(i = 1, \dots, z)$ be two different collections of 2-TIFLNs and suppose that $\tilde{p}_i \leq \tilde{p}'_i \forall i$, then $\tilde{p}_i^{OWG} \leq \tilde{p}'_i^{OWG}$.

Definition 6.4.4. Ordered Weighted Geometric Operator

Let $\tilde{p}_i = \langle (t_{\lambda_i}, \gamma_i), (t_{\Lambda_i}, \Gamma_i) \rangle$, $i = 1, 2, \dots, z$ be a collection of some unbalanced 2-TIFLNs, and w_i , $i = 1, 2, \dots, z$ be the associated weights such that $w_i \in [0, 1]$ and $\sum_{i=1}^z w_i = 1$, then the unbalanced 2-TIFLOWG operator is computed as :

$$\tilde{p}_i^{OWG} = \otimes_{i=1}^z (\tilde{p}_{\sigma(i)})^{w_i}$$

where $\tilde{p}_{\sigma(i)}$ is the i^{th} largest value according to the ordering given in Definition 6.3.6.

Theorem 6.4.4. Let $\tilde{p}_i = \langle (t_{\lambda_i}, \gamma_i), (t_{\Lambda_i}, \Gamma_i) \rangle$, $i = 1, 2, \dots, z$ be a collection of some unbalanced 2-TIFLNs, and w_i , $i = 1, 2, \dots, z$ be the associated weights such that $w_i \in [0, 1]$ and $\sum_{i=1}^z w_i = 1$, then the 2-TIFLOWG operator $\tilde{p}_i^{OWG}$ for unbalanced 2-TIFLNs can be given as:

$$\begin{aligned} \tilde{p}_i^{OWG} &= \otimes_{i=1}^z (\tilde{p}_{\sigma(i)})^{w_i} \\ &= \langle \Omega \left[\min \left\{ a^m, \prod_{i=1}^z ((\Omega^{-1}(t_{\lambda_{\sigma(i)}}, \gamma_{\sigma(i)}))^{w_i}) \right\} \right], \\ &\quad \Omega \left[\max \left\{ a^{-m}, \tau \left| 1 - \prod_{i=1}^z \left| 1 - \frac{\Omega^{-1}(t_{\Lambda_{\sigma(i)}}, \Gamma_{\sigma(i)})}{\tau} \right|^{w_i} \right| \right\} \right] \rangle \end{aligned}$$

Proof. The proof is similar to the proof of Theorem 6.4.1. □

The 2-TIFLOWG operator satisfies all the properties similar to the 2-TIFLWG, i.e.

- **Closure** : If $\tilde{p}_i = \langle (t_{\lambda_i}, \gamma_i), (t_{\Lambda_i}, \Gamma_i) \rangle$, $i = 1, 2, \dots, z$ be a collection of some unbalanced 2-TIFLNs, then $\tilde{p}_i^{OWG}$ is also a 2-TIFLN.
- **Boundedness** : The 2-TIFLOWG operator is bounded by $\tilde{p}^- \leq \tilde{p}_i^{OWG} \leq \tilde{p}^+$, where $\tilde{p}^- = \min_i(\tilde{p}_i)$ and $\tilde{p}^+ = \max_i(\tilde{p}_i)$, $i = 1, 2, \dots, z$.
- **Monotonicity** : Let $\tilde{p}_i$ and $\tilde{p}'_i$, $(i = 1, \dots, z)$ be two different collections of 2-TIFLNs and suppose that $\tilde{p}_i \leq \tilde{p}'_i \forall i$, then $\tilde{p}_i^{OWG} \leq \tilde{p}'_i^{OWG}$.

Remark 6.4.2. If $w = (\frac{1}{z}, \frac{1}{z}, \dots, \frac{1}{z})^T$ is the associated weight vector for $\tilde{p}_i$ ($i = 1, 2, \dots, z$), then the unbalanced 2-TIFLWG operator and the unbalanced 2-TIFLOWG operator reduce to unbalanced 2-tuple intuitionistic fuzzy linguistic geometric (2-TIFLG) operator

which is defined as follows:

$$\tilde{\mathbf{p}}_i^G = \tilde{\mathbf{p}}_1^{1/z} \otimes \tilde{\mathbf{p}}_2^{1/z} \otimes \dots \otimes \tilde{\mathbf{p}}_z^{1/z}$$

6.5 Numerical Illustration

In this section, we consider two numerical examples to exhibit the practical working of the proposed aggregation operators. In Example 6.5.1, we consider a usual problem to explain how these operators apply to unbalanced 2-TIFLNs to aggregate them. Example 6.5.2 shows the illustration of these operators graphically.

Example 6.5.1. Suppose $\mathbb{LS} = \{t_{3^{-2}}, t_{3^{-1}}, t_{3^0}, t_{3^1}, t_{3^2}\}$ and let $\tilde{\mathbf{p}}_1 = \langle (t_{3^0}, 0.195), (t_{3^{-1}}, 0.19) \rangle$, $\tilde{\mathbf{p}}_2 = \langle (t_{3^1}, -0.21), (t_{3^{-2}}, 0.22) \rangle$, $\tilde{\mathbf{p}}_3 = \langle (t_{3^1}, 0.145), (t_{3^0}, -0.125) \rangle$, $\tilde{\mathbf{p}}_4 = \langle (t_{3^2}, -0.12), (t_{3^{-2}}, 0.129) \rangle$, $\tilde{\mathbf{p}}_5 = \langle (t_{3^0}, 0), (t_{3^{-1}}, -0.135) \rangle$ be five unbalanced 2-TIFLNs, and let $w = (0.32, 0.15, 0.04, 0.11, 0.38)^T$ be the weight vector of $\tilde{\mathbf{p}}_i$ ($i = 1, 2, \dots, 5$). The weights are those random numbers that are greater than 0 and sum to 1; then

$$\begin{aligned} \tilde{\mathbf{p}}_i^{WA} &= \left\langle \Omega \left[\min \left\{ 3^2, 5 \times \left| 1 - \left(\left| 1 - \frac{1.39}{5} \right|^{0.32} \right) \times \left(\left| 1 - \frac{2.59}{5} \right|^{0.15} \right) \times \left(\left| 1 - \frac{3.87}{5} \right|^{0.04} \right) \times \left(\left| 1 - \frac{8.31}{5} \right|^{0.11} \right) \times \left(\left| 1 - \frac{1}{5} \right|^{0.38} \right) \right| \right] \right. \\ &\quad \left. \Omega \left[\max \left\{ 3^{-2}, \left((0.46)^{0.32} \times (0.16)^{0.15} \times (0.9165)^{0.04} \times (0.14)^{0.11} \times (0.3033)^{0.38} \right) \right\} \right] \right\rangle \\ &= \left\langle \Omega \left[\min \left\{ 3^2, 1.6597 \right\} \right], \Omega \left[\max \left\{ 3^{-2}, 0.302 \right\} \right] \right\rangle \\ &= \langle (t_{3^0}, 0.329), (t_{3^{-1}}, -0.141) \rangle \end{aligned}$$

Note that $\Omega^{-1}(t_{3^0}, 0.329) \cdot \Omega^{-1}(t_{3^{-1}}, -0.141) = 0.50122 \in [3^{-2}, 3^2]$, i.e. $\tilde{\mathbf{p}}_i^{WA}$ is closed.

Next, we find $\tilde{\mathbf{p}}_i^{OWA}$ as follows.

$$\begin{aligned} \tilde{\mathbf{p}}_i^{OWA} &= \left\langle \Omega \left[\min \left\{ a^m, \tau \left(1 - \prod_{i=1}^z \left(1 - \frac{\Omega^{-1}(t_{\lambda_{\sigma(i)}}, \gamma_{\sigma(i)})}{\tau} \right)^{w_i} \right) \right\} \right], \right. \\ &\quad \left. \Omega \left[\max \left\{ a^{-m}, \prod_{i=1}^z \left(\Omega^{-1}(t_{\lambda_{\sigma(i)}}, \gamma_{\sigma(i)}) \right)^{w_i} \right\} \right] \right\rangle \end{aligned}$$

Now, finding scores of $\tilde{\mathbf{p}}'_i$ ($i = 1, 2, \dots, 5$) using Definition 6.3.4

$$s(\tilde{\mathbf{p}}_1) = \Omega \left(\sqrt{\frac{\Omega^{-1}(t_{3^0}, 0.39)}{\Omega^{-1}(t_{3^{-1}}, 0.38)}} \right) = \Omega(1.7383) = (t_{3^0}, 0.3692)$$

$$\mathfrak{s}(\tilde{\mathfrak{p}}_2) = \Omega \left(\sqrt{\frac{\Omega^{-1}(t_{3^1}, -0.41)}{\Omega^{-1}(t_{3^{-2}}, 0.44)}} \right) = \Omega(4.0233) = (t_{3^1}, 0.1705)$$

$$\mathfrak{s}(\tilde{\mathfrak{p}}_3) = \Omega \left(\sqrt{\frac{\Omega^{-1}(t_{3^1}, -0.29)}{\Omega^{-1}(t_{3^0}, -0.25)}} \right) = \Omega(2.0547) = (t_{3^1}, -0.4772)$$

$$\mathfrak{s}(\tilde{\mathfrak{p}}_4) = \Omega \left(\sqrt{\frac{\Omega^{-1}(t_{3^2}, 0.49)}{\Omega^{-1}(t_{3^{-2}}, 0.28)}} \right) = \Omega(7.7043) = (t_{3^2}, -0.2159)$$

$$\mathfrak{s}(\tilde{\mathfrak{p}}_5) = \Omega \left(\sqrt{\frac{\Omega^{-1}(t_{3^0}, 0)}{\Omega^{-1}(t_{3^{-1}}, -0.27)}} \right) = \Omega(1.8157) = (t_{3^0}, 0.4079)$$

Since

$$\mathfrak{s}(\tilde{\mathfrak{p}}_4) > \mathfrak{s}(\tilde{\mathfrak{p}}_2) > \mathfrak{s}(\tilde{\mathfrak{p}}_3) > \mathfrak{s}(\tilde{\mathfrak{p}}_5) > \mathfrak{s}(\tilde{\mathfrak{p}}_1)$$

then,

$$\tilde{\mathfrak{p}}_{\sigma(1)} = \langle (t_{3^2}, -0.12), (t_{3^{-2}}, 0.129) \rangle, \quad \tilde{\mathfrak{p}}_{\sigma(2)} = \langle (t_{3^1}, -0.21), (t_{3^{-2}}, 0.22) \rangle,$$

$$\tilde{\mathfrak{p}}_{\sigma(3)} = \langle (t_{3^1}, 0.145), (t_{3^0}, -0.125) \rangle, \quad \tilde{\mathfrak{p}}_{\sigma(4)} = \langle (t_{3^0}, 0), (t_{3^{-1}}, -0.135) \rangle,$$

$$\tilde{\mathfrak{p}}_{\sigma(5)} = \langle (t_{3^0}, 0.195), (t_{3^{-1}}, 0.19) \rangle.$$

Then it follows that

$$\begin{aligned} \tilde{\mathfrak{p}}_i^{OWA} &= \langle \Omega \left[\min \left\{ 3^2, 5 \times \left| 1 - \left(\left| 1 - \frac{8.31}{5} \right|^{0.32} \right) \times \left(\left| 1 - \frac{2.59}{5} \right|^{0.15} \right) \times \left(\left| 1 - \frac{3.87}{5} \right|^{0.04} \right) \times \left(\left| 1 - \frac{1}{5} \right|^{0.11} \right) \left(\left| 1 - \frac{1.39}{5} \right|^{0.38} \right) \right| \right] \right. \\ &\quad \left. \Omega \left[\max \left\{ 3^{-2}, (0.14)^{0.32} \times (0.16)^{0.15} \times (0.9167)^{0.04} \times (0.3033)^{0.11} \times (0.46)^{0.38} \right\} \right] \right\rangle \\ &= \langle \Omega \left[\min \left\{ 3^2, 1.8096 \right\} \right], \Omega \left[\max \left\{ 2^{-3}, 0.263 \right\} \right] \rangle \\ &= \langle (t_{3^0}, 0.4048), (t_{3^{-1}}, -0.3165) \rangle \end{aligned}$$

Also, $\Omega^{-1}(t_{3^0}, 0.4048) \cdot \Omega^{-1}(t_{3^{-1}}, -0.3165) = 0.4759 \in [3^{-2}, 3^2]$, which implies that $\tilde{\mathfrak{p}}_i^{OWA}$ is closed.

Next for finding the weighted geometric mean we proceed as follows.

$$\begin{aligned} \tilde{\mathfrak{p}}_i^{WG} &= \langle \Omega \left[\min \left\{ 3^2, (1.39)^{0.32} \times (2.59)^{0.15} \times (3.87)^{0.04} \times (8.31)^{0.11} \times (1)^{0.38} \right\} \right] \right. \\ &\quad \left. \Omega \left[\max \left\{ 3^{-2}, 5 \times \left| 1 - \left(\left| 1 - \frac{0.46}{5} \right|^{0.32} \right) \times \left(\left| 1 - \frac{0.16}{5} \right|^{0.15} \right) \times \left(\left| 1 - \frac{0.9167}{5} \right|^{0.04} \right) \times \left(\left| 1 - \frac{0.14}{5} \right|^{0.11} \right) \times \left(\left| 1 - \frac{0.3033}{5} \right|^{0.38} \right) \right| \right] \right] \right\rangle \\ &= \langle \Omega \left[\min \left\{ 3^2, 1.707 \right\} \right], \Omega \left[\max \left\{ 3^{-2}, 0.342 \right\} \right] \rangle \\ &= \langle (t_{3^0}, 0.3535), (t_{3^{-1}}, 0.013) \rangle \end{aligned}$$

Also, $\Omega^{-1}(t_{3^0}, 0.3535) \cdot \Omega^{-1}(t_{3^{-1}}, 0.013) = 0.5838 \in [3^{-2}, 3^2]$ implying that $\tilde{\mathfrak{p}}_i^{WG}$ is closed.

Proceeding towards finding the Ordered Weighted Geometric mean, we have

$$\begin{aligned}
 \tilde{p}_i^{OWG} &= \langle \Omega \left[\min \left\{ 3^2, (8.31)^{0.32} \times (2.59)^{0.15} \times (3.87)^{0.04} \times (1)^{0.11} \times (1.39)^{0.38} \right\} \right], \\
 &\quad \Omega \left[\max \left\{ 3^{-2}, 5 \times \left| 1 - \left(1 - \frac{0.14}{5} \right)^{0.32} \right| \times \left(1 - \frac{0.16}{5} \right)^{0.15} \right| \times \left(1 - \frac{0.9167}{5} \right)^{0.04} \right| \times \left(1 - \frac{0.3033}{5} \right)^{0.11} \right| \times \left(1 - \frac{0.46}{5} \right)^{0.38} \right| \right] \rangle \\
 &= \langle \Omega \left[\min \left\{ 3^2, 1.8096 \right\} \right], \Omega \left[\max \left\{ 3^{-2}, 0.318 \right\} \right] \rangle \\
 &= \langle (t_{30}, 0.4048), (t_{3-2}, 0) \rangle
 \end{aligned}$$

It is to be noted that $\Omega^{-1}(t_{30}, 0.4048) \cdot \Omega^{-1}(t_{3-2}, 0) = 0.5754 \in [3^{-2}, 3^2]$, hence $\tilde{p}_i^{OWG}$ is closed.

The next example shows the graphs of membership, non-membership, and score functions of some alternatives based on different aggregation operators.

Example 6.5.2. An organization wants to install 15 smart classes equipments (or alternatives $A_i, i = 1, 2, \dots, 15$) in its classes based on 6 options O_i . The organization hires some professionals to aid this decision-making. The alternatives are to be evaluated using the unbalanced 2-TIFLNs whose linguistic terms belong to the set

$$\mathbb{LS} = \{t_{2-3}:\text{worst (to install)}, t_{2-2}:\text{very poor}, t_{2-1}:\text{poor}, t_{20}:\text{average}, t_{21}:\text{good}, t_{22}:\text{very good}, t_{23}:\text{best}\}$$

The weighting vector is $w = (0.16, 0.17, 0.18, 0.15, 0.16, 0.18)$, and construct the following matrix as shown in Table 6.1. Further, to analyse the aggregation results, we utilize all

	O_1	O_2	O_3	O_4	O_5	O_6
A_1	$\langle (t_{20}, 0.089), (t_{2-1}, 0.23) \rangle$	$\langle (t_{20}, 0.196), (t_{2-1}, 0.2) \rangle$	$\langle (t_{22}, -0.164), (t_{2-2}, 0.4) \rangle$	$\langle (t_{23}, 0), (t_{2-1}, -0.411) \rangle$	$\langle (t_{21}, -0.188), (t_{2-1}, 0.244) \rangle$	$\langle (t_{21}, 0.141), (t_{2-2}, 0.143) \rangle$
A_2	$\langle (t_{20}, 0.078), (t_{2-2}, 0.101) \rangle$	$\langle (t_{21}, 0.288), (t_{2-2}, 0.46) \rangle$	$\langle (t_{22}, 0.052), (t_{2-2}, 0.189) \rangle$	$\langle (t_{20}, 0.164), (t_{2-1}, 0.301) \rangle$	$\langle (t_{23}, 0), (t_{2-2}, 0.128) \rangle$	$\langle (t_{20}, 0.196), (t_{2-2}, 0.4) \rangle$
A_3	$\langle (t_{23}, 0), (t_{2-2}, 0.248) \rangle$	$\langle (t_{21}, 0.127), (t_{2-2}, 0.264) \rangle$	$\langle (t_{21}, 0.471), (t_{2-2}, 0.282) \rangle$	$\langle (t_{23}, 0), (t_{2-2}, 0.493) \rangle$	$\langle (t_{23}, -0.41), (t_{2-1}, 0.221) \rangle$	$\langle (t_{21}, 0.402), (t_{20}, 0) \rangle$
A_4	$\langle (t_{22}, 0.179), (t_{2-2}, 0.224) \rangle$	$\langle (t_{20}, 0.388), (t_{20}, 0) \rangle$	$\langle (t_{22}, 0.124), (t_{2-1}, 0.048) \rangle$	$\langle (t_{20}, 0.004), (t_{2-2}, 0.298) \rangle$	$\langle (t_{23}, 0), (t_{2-2}, 0.457) \rangle$	$\langle (t_{21}, 0.332), (t_{2-1}, 0.441) \rangle$
A_5	$\langle (t_{21}, 0.381), (t_{2-2}, 0.4) \rangle$	$\langle (t_{20}, 0.249), (t_{2-2}, 0.071) \rangle$	$\langle (t_{21}, 0.461), (t_{2-1}, 0.492) \rangle$	$\langle (t_{22}, 0.418), (t_{20}, 0) \rangle$	$\langle (t_{22}, -0.416), (t_{20}, -0.31) \rangle$	$\langle (t_{22}, 0.472), (t_{2-2}, 0.011) \rangle$
A_6	$\langle (t_{21}, 0.224), (t_{2-2}, 0.433) \rangle$	$\langle (t_{20}, 0.382), (t_{2-2}, 0.406) \rangle$	$\langle (t_{23}, 0), (t_{2-2}, 0.178) \rangle$	$\langle (t_{20}, 0.388), (t_{2-1}, 0.413) \rangle$	$\langle (t_{20}, 0.392), (t_{20}, 0) \rangle$	$\langle (t_{23}, 0), (t_{20}, -0.441) \rangle$
A_7	$\langle (t_{21}, 0.355), (t_{2-2}, 0.353) \rangle$	$\langle (t_{22}, 0.205), (t_{2-2}, 0.234) \rangle$	$\langle (t_{21}, 0.352), (t_{2-2}, 0.143) \rangle$	$\langle (t_{20}, 0.187), (t_{2-2}, 0.12) \rangle$	$\langle (t_{22}, 0.469), (t_{2-2}, 0.158) \rangle$	$\langle (t_{21}, 0.174), (t_{20}, 0) \rangle$
A_8	$\langle (t_{20}, 0.075), (t_{2-2}, -0.346) \rangle$	$\langle (t_{20}, 0.051), (t_{2-2}, 0.226) \rangle$	$\langle (t_{23}, 0), (t_{20}, 0) \rangle$	$\langle (t_{20}, 0.048), (t_{20}, 0) \rangle$	$\langle (t_{23}, 0), (t_{2-2}, 0.105) \rangle$	$\langle (t_{23}, -0.469), (t_{2-1}, 0.235) \rangle$
A_9	$\langle (t_{23}, 0), (t_{20}, 0) \rangle$	$\langle (t_{20}, 0.42), (t_{2-2}, 0.098) \rangle$	$\langle (t_{22}, 0.221), (t_{2-2}, 0.199) \rangle$	$\langle (t_{22}, 0.186), (t_{2-1}, 0.133) \rangle$	$\langle (t_{23}, -0.048), (t_{20}, 0) \rangle$	$\langle (t_{23}, 0), (t_{2-2}, 0.319) \rangle$
A_{10}	$\langle (t_{23}, 0), (t_{20}, 0) \rangle$	$\langle (t_{21}, 0.059), (t_{20}, -0.245) \rangle$	$\langle (t_{20}, 0.386), (t_{2-2}, 0.339) \rangle$	$\langle (t_{21}, 0.114), (t_{2-1}, 0.064) \rangle$	$\langle (t_{23}, 0), (t_{2-2}, 0.393) \rangle$	$\langle (t_{21}, 0.183), (t_{2-2}, 0.125) \rangle$
A_{11}	$\langle (t_{21}, 0.006), (t_{2-2}, 0.237) \rangle$	$\langle (t_{22}, 0.295), (t_{2-2}, 0.485) \rangle$	$\langle (t_{20}, 0.201), (t_{2-1}, 0.298) \rangle$	$\langle (t_{23}, -0.156), (t_{20}, -0.135) \rangle$	$\langle (t_{22}, 0.459), (t_{2-2}, 0.257) \rangle$	$\langle (t_{23}, 0), (t_{20}, 0) \rangle$
A_{12}	$\langle (t_{22}, 0.39), (t_{20}, 0) \rangle$	$\langle (t_{21}, 0.004), (t_{2-2}, 0.428) \rangle$	$\langle (t_{22}, 0.363), (t_{2-2}, 0.095) \rangle$	$\langle (t_{22}, 0.348), (t_{2-2}, 0.001) \rangle$	$\langle (t_{22}, 0.422), (t_{2-1}, 0.371) \rangle$	$\langle (t_{23}, 0), (t_{2-2}, 0.398) \rangle$
A_{13}	$\langle (t_{21}, 0.098), (t_{2-1}, 0.049) \rangle$	$\langle (t_{23}, 0), (t_{2-2}, 0.428) \rangle$	$\langle (t_{21}, 0.332), (t_{2-2}, 0.342) \rangle$	$\langle (t_{22}, 0.198), (t_{2-2}, 0.393) \rangle$	$\langle (t_{22}, 0.179), (t_{2-1}, 0.495) \rangle$	$\langle (t_{22}, 0.036), (t_{2-2}, 0.092) \rangle$
A_{14}	$\langle (t_{21}, 0.16), (t_{20}, 0) \rangle$	$\langle (t_{21}, 0.152), (t_{2-1}, 0.143) \rangle$	$\langle (t_{21}, 0.304), (t_{2-1}, 0.388) \rangle$	$\langle (t_{23}, 0.01), (t_{2-2}, 0.284) \rangle$	$\langle (t_{23}, -0.13), (t_{2-1}, 0.286) \rangle$	$\langle (t_{21}, 0.222), (t_{2-2}, 0.336) \rangle$
A_{15}	$\langle (t_{20}, 0.131), (t_{2-2}, 0.32) \rangle$	$\langle (t_{23}, 0), (t_{2-2}, 0.036) \rangle$	$\langle (t_{20}, 0.272), (t_{2-2}, 0.034) \rangle$	$\langle (t_{21}, 0.5), (t_{2-2}, 0.022) \rangle$	$\langle (t_{20}, 0.356), (t_{2-1}, -0.27) \rangle$	$\langle (t_{23}, 0), (t_{2-1}, 0.046) \rangle$

Table 6.1: The unbalanced 2-TIFL decision matrix

the earlier defined aggregation operators and the results are as given in Table 6.2. Also,

	2-TIFLWA	2-TIFLOWA	2-TIFLWG	2-TIFLOWG
A ₁	$\langle(t_2^2, -0.183), (t_2^{-1}, -0.188)\rangle$	$\langle(t_2^2, -0.176), (t_2^{-1}, -0.188)\rangle$	$\langle(t_2, 0.437), (t_2^{-1}, -0.096)\rangle$	$\langle(t_2, 0.43), (t_2^{-1}, -0.096)\rangle$
A ₂	$\langle(t_2^2, 0.042), (t_2^{-2}, 0.104)\rangle$	$\langle(t_2^2, 0.041), (t_2^{-2}, 0.104)\rangle$	$\langle(t_2^2, -0.256), (t_2^{-2}, 0.224)\rangle$	$\langle(t_2^2, -0.261), (t_2^{-2}, 0.232)\rangle$
A ₃	$\langle(t_2^2, 0.245), (t_2^{-2}, 0.256)\rangle$	$\langle(t_2^2, 0.26), (t_2^{-2}, 0.268)\rangle$	$\langle(t_2^2, 0.074), (t_2^{-1}, -0.32)\rangle$	$\langle(t_2^2, 0.1025), (t_2^{-1}, -0.316)\rangle$
A ₄	$\langle(t_2^2, -0.038), (t_2^{-2}, 0.428)\rangle$	$\langle(t_2^2, -0.032), (t_2^{-2}, 0.412)\rangle$	$\langle(t_2, 0.48), (t_2^{-1}, -0.088)\rangle$	$\langle(t_2, 0.48), (t_2^{-1}, -0.1)\rangle$
A ₅	$\langle(t_2^2, 0.043), (t_2^{-1}, -0.468)\rangle$	$\langle(t_2^2, 0.011), (t_2^{-1}, -0.432)\rangle$	$\langle(t_2^2, -0.408), (t_2^{-1}, 0.064)\rangle$	$\langle(t_2^2, -0.43), (t_2^{-1}, 0.076)\rangle$
A ₆	$\langle(t_2^2, 0.02), (t_2^{-1}, -0.36)\rangle$	$\langle(t_2^2, -0.007), (t_2^{-1}, -0.292)\rangle$	$\langle(t_2, 0.427), (t_2^{-1}, 0.06)\rangle$	$\langle(t_2, 0.386), (t_2^{-1}, 0.098)\rangle$
A ₇	$\langle(t_2^2, -0.14), (t_2^{-2}, 0.184)\rangle$	$\langle(t_2^2, -0.134), (t_2^{-2}, 0.18)\rangle$	$\langle(t_2, 0.457), (t_2^{-1}, -0.48)\rangle$	$\langle(t_2, 0.447), (t_2^{-1}, -0.484)\rangle$
A ₈	$\langle(t_2^2, 0.176), (t_2^{-1}, -0.288)\rangle$	$\langle(t_2^2, 0.165), (t_2^{-1}, -0.232)\rangle$	$\langle(t_2, 0.444), (t_2^{-1}, 0.12)\rangle$	$\langle(t_2, 0.418), (t_2^{-1}, 0.158)\rangle$
A ₉	$\langle(t_2^2, 0.332), (t_2^{-1}, -0.06)\rangle$	$\langle(t_2^2, 0.317), (t_2^{-1}, -0.052)\rangle$	$\langle(t_2^2, 0.255), (t_2^{-1}, 0.146)\rangle$	$\langle(t_2^2, 0.221), (t_2^{-1}, 0.146)\rangle$
A ₁₀	$\langle(t_2^2, 0.006), (t_2^{-1}, -0.26)\rangle$	$\langle(t_2^2, -0.013), (t_2^{-1}, -0.252)\rangle$	$\langle(t_2^2, -0.455), (t_2^{-1}, 0.068)\rangle$	$\langle(t_2^2, -0.476), (t_2^{-1}, 0.07)\rangle$
A ₁₁	$\langle(t_2^2, 0.315), (t_2^{-1}, -0.48)\rangle$	$\langle(t_2^2, 0.305), (t_2^{-2}, 0.464)\rangle$	$\langle(t_2^2, -0.009), (t_2^{-1}, 0.058)\rangle$	$\langle(t_2^2, -0.065), (t_2^{-1}, 0.022)\rangle$
A ₁₂	$\langle(t_2^2, 0.327), (t_2^{-2}, 0.06)\rangle$	$\langle(t_2^2, 0.334), (t_2^{-2}, 0.088)\rangle$	$\langle(t_2^2, 0.244), (t_2^{-1}, -0.46)\rangle$	$\langle(t_2^2, 0.259), (t_2^{-1}, -0.396)\rangle$
A ₁₃	$\langle(t_2^2, 0.101), (t_2^{-2}, 0.004)\rangle$	$\langle(t_2^2, 0.1001), (t_2^{-2}, 0.028)\rangle$	$\langle(t_2^2, 0.007), (t_2^{-2}, 0.276)\rangle$	$\langle(t_2^2, 0.007), (t_2^{-2}, 0.304)\rangle$
A ₁₄	$\langle(t_2^2, 0.002), (t_2^{-1}, -0.06)\rangle$	$\langle(t_2^2, 0.041), (t_2^{-1}, -0.016)\rangle$	$\langle(t_2^2, -0.435), (t_2^{-1}, 0.136)\rangle$	$\langle(t_2^2, -0.385), (t_2^{-1}, 0.156)\rangle$
A ₁₅	$\langle(t_2^2, 0.012), (t_2^{-2}, 0.18)\rangle$	$\langle(t_2^2, -0.011), (t_2^{-2}, 0.192)\rangle$	$\langle(t_2, 0.365), (t_2^{-2}, 0.296)\rangle$	$\langle(t_2, 0.352), (t_2^{-2}, 0.304)\rangle$

Table 6.2: The aggregated results using defined operators

	2-TIFLWA	2-TIFLOWA	2-TIFLWG	2-TIFLOWG
A ₁	$(t_2^1, 0.4162)$	$(t_2^1, 0.4189)$	$(t_2^1, 0.22838)$	$(t_2^1, 0.22538)$
A ₂	$(t_2^2, -0.0565)$	$(t_2^2, -0.05821)$	$(t_2^2, -0.311661)$	$(t_2^2, -0.3198)$
A ₃	$(t_2^2, -0.00878)$	$(t_2^2, -0.00594)$	$(t_2^2, -0.4007)$	$(t_2^2, -0.38174)$
A ₄	$(t_2^2, -0.342237)$	$(t_2^2, -0.33037)$	$(t_2, 0.24402)$	$(t_2, 0.24794)$
A ₅	$(t_2^2, -0.36045)$	$(t_2^2, -0.394068)$	$(t_2^1, 0.2234)$	$(t_2^1, 0.20812)$
A ₆	$(t_2^2, -0.4223)$	$(t_2^2, -0.47234)$	$(t_2^1, 0.16026)$	$(t_2^1, 0.12351)$
A ₇	$(t_2^2, -0.22732)$	$(t_2^2, -0.221505)$	$(t_2^1, 0.38435)$	$(t_2^1, 0.38165)$
A ₈	$(t_2^2, -0.34255)$	$(t_2^2, -0.37676)$	$(t_2^1, 0.13546)$	$(t_2^1, 0.10658)$
A ₉	$(t_2^2, -0.3429)$	$(t_2^2, -0.35533)$	$(t_2^1, 0.48008)$	$(t_2^1, 0.45945)$
A ₁₀	$(t_2^2, -0.4794)$	$(t_2^2, -0.49203)$	$(t_2^1, 0.20275)$	$(t_2^1, 0.19344)$
A ₁₁	$(t_2^2, -0.139663)$	$(t_2^2, -0.11152)$	$(t_2^1, 0.37163)$	$(t_2^1, 0.37581)$
A ₁₂	$(t_2^2, 0.119025)$	$(t_2^2, 0.1072)$	$(t_2^2, -0.20263)$	$(t_2^2, -0.22809)$
A ₁₃	$(t_2^2, 0.04697)$	$(t_2^2, 0.03449)$	$(t_2^2, -0.2215)$	$(t_2^2, -0.24265)$
A ₁₄	$(t_2^1, 0.4375)$	$(t_2^1, 0.44851)$	$(t_2^1, 0.17372)$	$(t_2^1, 0.18215)$
A ₁₅	$(t_2^2, -0.147631)$	$(t_2^2, -0.17323)$	$(t_2^1, 0.4511)$	$(t_2^1, 0.44027)$

Table 6.3: The aggregated results using Score functions

the score functions of the aggregated results are as given in Table 6.3. Graphically, the membership, non-membership and the score functions of the aggregated results are depicted in Figure 6.2, 6.3 and 6.4, respectively. It is evident from the graphs that all the aggregation operators provide the same ranking of alternatives and do not affect the order of the ranking. Moreover, as the alternative A₁₂ has highest score function value in graph 6.4, it would be the best alternative to install. This can be verified from the score functions values given in Table 6.3.

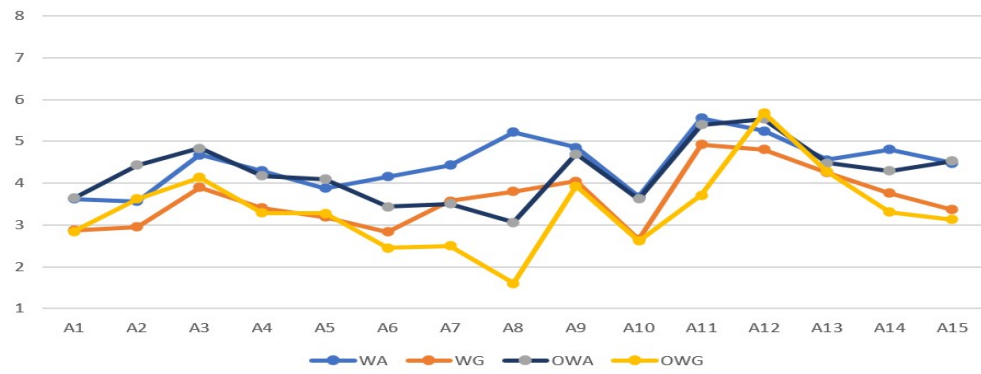


Figure 6.2: Membership Graph of Aggregated Values

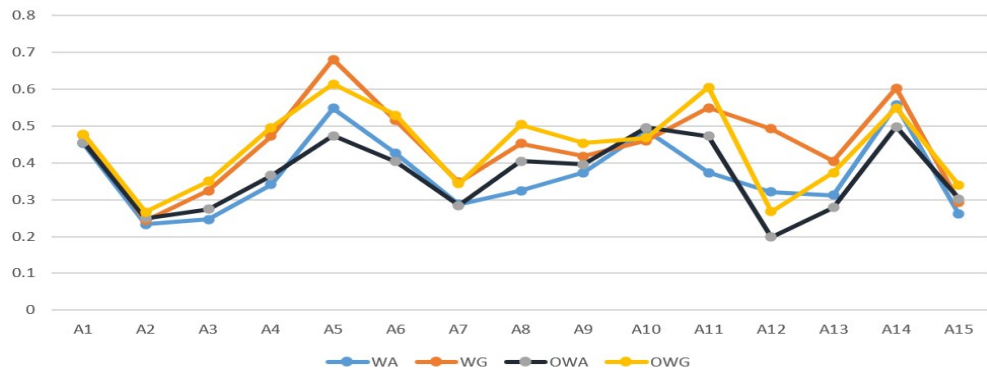


Figure 6.3: Non-membership Graph of Aggregated Values

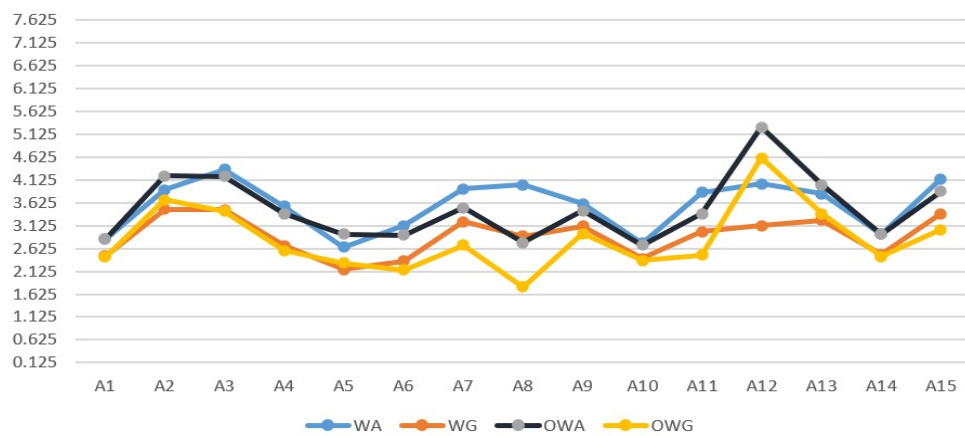


Figure 6.4: Score Function Graph of Aggregated Values

6.6 Conclusions

This chapter develops a 2-tuple intuitionistic linguistic model in which the linguistic terms belong to an unbalanced multiplicative LT set. The idea of the membership and non-membership functions belonging to the same scale set is overruled, and we have taken both membership and non-membership of an element from different partitions of the linguistic set. Based on this fact, we have defined the unbalanced 2-tuple intuitionistic fuzzy linguistic sets (2-TIFLS) and exhibited the comparison of unbalanced 2-tuple intuitionistic fuzzy linguistic numbers (2-TIFLNs) using the newly defined score and accuracy functions. Also, we defined the operational laws for unbalanced 2-TIFLNs whose linguistic terms belong to a multiplicative unbalanced set. Further, we utilize the arithmetic and geometric aggregation operations to develop the unbalanced 2-tuple intuitionistic fuzzy linguistic weighted average (2-TIFLWA) operator, unbalanced 2-tuple intuitionistic fuzzy linguistic ordered weighted average (2-TIFLOWA) operator, unbalanced 2-tuple intuitionistic fuzzy weighted geometric (2-TIFLWG) operator and unbalanced 2-tuple intuitionistic fuzzy linguistic ordered weighted geometric (2-TIFLOWG) operator. The aggregated values of membership, non-membership, and the score functions of several unbalanced 2-TIFLNs have been shown graphically.

Unlike the earlier defined approaches, the novel approach defined by us is advantageous because it considers linguistic terms that belong to an unbalanced multiplicative LT set. Moreover, the membership and non-membership functions picked by partitioning the LT set clarify choosing these functions independently and individually. Further, the unbalanced 2-TIFLS model defined by us satisfies the closure property that was missing in existing theories. In the future, we look forward to define some additional aggregation operators for unbalanced 2-TIFLNs.

Summary and Future Scope of the work

Many practical situations certainly deal with uncertainty and vagueness and further increase the complexity of the problem. Using probabilistic models to deal with such uncertainty has resulted in unsatisfactory results. There are different computational models based on linguistic information that accomplish the process of CW. Further, game theory is essentially the study of strategic communication of information through language in a rigorous and stylized way based on the linguistic term set. It is anticipated that within the traditional setup of games, the payoffs are well-known with certainty. Nevertheless, getting to the practical world, the assumption of certainty is not meaningful on several occasions. The postulations made for the exact payoffs can be considered the stringent ideology in the real-world scenario involving uncertain and ambiguous information. Imprecision and uncertainty have been incorporated into game theory by using various frameworks like fuzzy, probability, etc. Hence it is necessary to study methodologies that can handle the linguistic uncertainty prevailing in game problems. The primary purpose of the work done in this thesis is to handle the vagueness and impreciseness prevailing in matrix games in the form of fuzzy linguistic variables. An ample literature survey is performed, and based on this review, different methodologies have been defined to deal with matrix games having linguistic information.

Chapter 2 addresses the two-person non-zero sum matrix games comprising linguistic information, for which a theory of bimatrix matrix games with payoffs as linguistic 2-tuples is established. The use of linguistic 2-tuple variables to represent game situations has shown to be a powerful approach. A methodology to solve such games is provided in this thesis, and this methodology is sufficient to evaluate the Nash equilibrium and the maximum expected payoffs corresponding to each player simultaneously in a single program. However, this work is based on payoffs that are regularly distributed. But in real-life scenarios, this is not always

the case that the data remains symmetrically distributed since the experts may prefer linguistic labels distributed non-uniformly and non-symmetrically.

Hence to describe two-person constant-sum matrix games with payoffs of asymmetrically distributed sets, two methodologies are designed in Chapter 3 to address matrix games with proportional as well as linguistic distribution assessment payoffs. The proportional linguistic payoffs consist of two linguistic terms. However, a generalization of the proportional linguistic terms set is provided as a distribution assessment linguistic set. To the best of our knowledge, these methodologies are the first to address game problems having irregularly distributed information of payoffs.

Then, in Chapter 4, a concept of a cloud model is generated to represent linguistic information more accurately. A novel concept of the cloud model based on a multiplicative unbalanced linguistic term set is proposed. The innovation of the model overcomes the inherent limitation of the existing cloud and is competent in producing new solutions according to the cloud model theory. Further, based on the defined concept, a generalized multiplicative unbalanced linguistic scale function is introduced to create a mapping between linguistic concepts and numerical intervals. The proposed generalized function creates a map between the linguistic data and numerical interval values. Based on this rational conversion method, interval cloud variables are transformed, which overcomes the loss of information existing in the point-based cloud models where linguistic variables are transformed into a specific numerical value. The proposed approach can be perceived as a convenient technique for MCDM problems.

Chapter 5 presents the concept of a probabilistic multiplicative unbalanced linguistic game in which the players record their expression in terms of unbalanced linguistic terms that have probabilities attached to them. In many game theoretic environments, the precise, crisp values of payoffs are not always easy to gather as real-life decision-making problems are not inevitably precise and symmetrically distributed in nature. Hence there is a need to explore matrix games having unbalanced probabilistic linguistic information. To thoroughly capture the uncertainty involved in the payoffs of such games, they are converted to probabilistic unbalanced linguistic clouds that describe the uncertainties of qualitative payoffs by implementing the uncertain transformation between payoffs and quantitative values. Hence this model overcomes the inherent limitation of existing linguistic

games where only the fuzziness of concepts is evaluated by providing an unbalanced cloud strategy and unbalanced cloud value of the game. This procedure overcomes the loss of information occurring in the process of transformation of payoffs. The real-life applications of the defined model are shown to prove its consistency and applicability in the practical world.

Lastly, Chapter 6 presents unbalanced 2-tuple intuitionistic fuzzy sets in which the intuitionistic fuzzy linguistic numbers are composed of linguistic 2-tuples belonging to an unbalanced multiplicative set. Intuitionistic fuzzy linguistic sets are characterized by a linguistic membership and non-membership degree; because of this, the decision-makers are capable of considering a linguistic hesitancy degree, where they don't simply convey their perception using one linguistic term. Hence, defining unbalanced 2-tuple intuitionistic fuzzy linguistic sets in which the intuitionistic fuzzy linguistic numbers are composed of linguistic 2-tuples makes a significant contribution to the literature of intuitionistic fuzzy set theory. This theory manipulates imprecise and uncertain information in decision-making.

Since the inception of linguistic computational models, it is foreseen that a lot of research has been dedicated to these LCMs in game theory. Our research is also fully dedicated to the development of such games that have linguistic uncertainty prevailing in them. Although this research is advantageous and novel in its own way, there is always a scope for future work. In the near future, we look forward to extending the methodology developed for 2-TL bimatrix games to such non-zero sum games that consist of asymmetric linguistic information. Moreover, further study can also be focussed on constructing dynamics in repeated bimatrix games. A dynamic game is played over many time phases. A bimatrix game requires solving at every possible system state in every phase. As a result, many bimatrix games must be computed over various time phases. This originates the demand for an efficient computational method that can be used to determine Nash equilibrium points of dynamic bimatrix games. Further, we envisage extending the methodology to solve matrix games with unbalanced linguistic information to solve n -person games to evaluate the results in cooperative game theory. Moreover, the cloud model developed to represent linguistic information addresses the linguistic information by adding a dummy variable to the linguistic set in case the LT set is not regularly distributed around the central term. Some MCDM problems may not have an accurate solution after adding this dummy variable. Therefore,

further extension of this model is required to get over the corresponding limitation. Furthermore, the concept of a probabilistic multiplicative unbalanced cloud model using linguistic cloud currently addresses such games in which information consists of a single linguistic term. Further work can be done toward developing a simpler theory for the case where the payoff information is in the form of multiple probabilistic linguistic terms. Also, some more aggregation operators may be defined for unbalanced 2-TIFLNs, and more properties of currently defined operators may be explored. These problems are worth being explored from the beginning.

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List of Publications

1. **Parul Chauhan**, Anjana Gupta; *Probabilistic Multiplicative Unbalanced Linguistic Game using Linguistic Cloud model*, The Journal of Supercomputing, Springer (2024). DOI: <https://doi.org/10.1007/s11227-024-06240-4>. (**SCIE, Impact Factor: 3.3**).
2. **Parul Chauhan**, Anjana Gupta; *A methodology for solving bimatrix games under 2-tuple linguistic environment*, International Journal of Systems Science: Operations & Logistics, Taylor & Francis, **10 (1)**, 2199368 (2023) (**SCIE, Impact Factor: 4.2**).
3. **Parul Chauhan**, Anjana Gupta, Tanya Malhotra; *A Novel Cloud Model based on a Multiplicative Unbalanced Linguistic Term Set*, The Journal of Supercomputing, Springer, **79**, 16378–16408 (2023). (**SCIE, Impact Factor : 3.3**).
4. **Parul Chauhan**, Anjana Gupta; *Matrix Games with Linguistic Distribution Assessment Payoffs*, In: Saraswat, M., Sharma, H., Balachandran, K., Kim, J.H., Bansal, J.C. (eds) Congress on Intelligent Systems. Lecture Notes on Data Engineering and Communications Technologies, **vol 111**. Springer, Singapore (2022) (**Conference Proceedings Citation Index (CPCI)**).
5. **Parul Chauhan**, Anjana Gupta; *Matrix games with proportional linguistic payoffs*, Soft Computing, Springer, **25**, 15067-15081 (2021) (**SCIE, Impact Factor: 4.1**).
6. **Parul Chauhan**, Anjana Gupta; *The unbalanced 2-tuple intuitionistic fuzzy linguistic numbers and aggregation operators* (**Communicated** in “International Journal of Fuzzy Systems, Springer”).

Papers Presented in International Conferences

- Presented a research paper entitled “**2-person Nonzero Sum Unbalanced Linguistic Games**” in *International Symposium on Applied Optimization and Game Theoretic Models for Decision Making* held at Indian Statistical Institute, Delhi centre, during February 01-03, 2023.
 - Presented a research paper entitled “**Matrix Games with Linguistic Distribution Assessment Payoffs**” in *Congress on Intelligent Systems (CIS 2021)* held online by Soft Computing Research Society & CHRIST (Deemed to be University), Bengaluru during September 04-05, 2021.
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Probabilistic multiplicative unbalanced linguistic game using linguistic cloud model

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Abstract

In many game theoretic environments, the precise crisp values of payoffs are not always easy to gather as real-life decision-making problems are not inevitably precise and symmetrically distributed in nature. Hence, there is a need to explore matrix games having unbalanced probabilistic linguistic information. To fill this research gap, this study proposes a concept of probabilistic multiplicative unbalanced linguistic (PM-UL) game in which the players express their responses in terms of PM-UL term. This type of PM-UL game is a useful technique for multiple criteria analysis. Further, to thoroughly capture the uncertainty involved in payoffs of such games, they are converted to probabilistic unbalanced linguistic clouds (PULCs) that describe the uncertainties of qualitative payoffs by implementing the uncertain transformation between PM-UL payoffs and quantitative values. The PULCs are represented by three numerical characteristics, i.e. expectation, entropy and hyper-entropy. Hence, this model overcomes the inherent limitation of existing linguistic games where only the fuzziness of concepts is evaluated, by providing a PULC strategy and PULC value of the game. This procedure overcomes the loss of information occurring in the process of transformation of payoff values into quantitative ones. The defined methodology is implemented on a real-life working problem to demonstrate the effect and applicability in decision-making circumstances.

Keywords Probabilistic unbalanced linguistic term set · Normal cloud model · Two-person zero sum games · Probabilistic linguistic information

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1 Introduction

Game theory has already proved its tremendous potential for conflict resolution problems in decision-making problems. Decision making is a social activity that is characterized by inherent uncertainty and ambiguous information communicated in day-to-day language. When a game theoretic approach is implemented to solve any decision-making problem, it is necessary to analyse which concept should be implemented to solve that particular problem. The results and methods of a problem are not always realistic and do not always match a real-life phenomenon. The ambiguity present in a concept make it difficult to assess a decision-making problem in a particular way. Hence, it is necessary to consider the uncertainty pertaining in a decision maker's (DM's) judgement to solve a problem underlying in a fuzzy environment. The development of fuzzy set theory [49] made it possible to deal with imprecise information explicitly. Several academics have looked into matrix games with payoffs that are fuzzy numbers [7, 8, 10, 14, 22, 25, 32] in order to reflect the uncertainty in the context of payoffs and goals. Also, the fuzzy programming approach plays a vital role in managing the group decision-making problems [35–38].

Although fuzzy games have provided phenomenal results in literature, these are still incapable of addressing games having qualitative information as the best expression in strategic forms occurs in the form of natural language only. Thus linguistic variables [50] play an important role in real-life decision making. Arfi [1, 2] introduced game theory based on fuzzy linguistic logic, which is a more straightforward and adaptable way to define payoffs in terms of linguistic labels. The utility of linguistic concepts implies a broad outlook in the current specialized literature on Computing with Words (CWW)-related methodologies. Also, a number of linguistic computational models based on balanced [17, 18] as well as unbalanced linguistic data [6, 9, 15, 28, 44, 52] are developed in existing literature to avoid information loss occurring in the process of CWW. For example, Herrera et al. [19] defined a computational model based on hierarchical structure to manage unbalanced linguistic information. Zhang et al. [54] revised the Herrera's model to define a simplified computational model to fuse multi-granular unbalanced linguistic terms. In fact, some authors have focused on developing linguistic computation models to managing multigranular unbalanced hesitant fuzzy linguistic information [53] in multi-attribute large-scale group decision making. Further, it is also necessary to model DMs' personalized individual semantics (PISs) in linguistic group decision making [55]. Zhang and Li's [51] work focuses on developing some PIS-based consistency control and consensus reaching models for this purpose. Furthermore, the DMs have extended the use of balanced as well as unbalanced linguistic information to deal with matrix games having such linguistic information [12, 13, 33, 34].

However, to deal with the ambiguity involved in the qualitative concepts, it is sometimes not possible to express the information with the help of a single particular term, or there are probabilities of occurrence of a qualitative term attached with them. For example, if $\mathbb{L}\mathbb{T} = \{\text{Good, Very Good}\}$ is a linguistic term set, then it might be possible that the player expresses his linguistic information as “50%

chances of being good", then we might say that the linguistic term "Good" has a probability attached as "Good (0.5)". For expressing such type of information, there is a need to use probabilistic linguistic term sets (PLTS) introduced in [31]. The PLTS enables the players or DMs to evaluate using numerous linguistic terms rather than a single term, more efficiently by giving each term a weight through probability. Moreover, a PLTS may be transformed into linguistic distribution so that each and every term of the original linguistic set is present in the final evaluation by the DMs [45]. A lot of research has been carried out based on PLTSs [3, 20, 46–48] as these types of sets allow the DMs to use more than one linguistic term with associated probabilities. Mi et al. [30] put forward this extensive and comprehensive way of expressing information in matrix games taking PLTS as input of matrix game. However, this PLTS consists of symmetrically distributed linguistic terms only. As it is well known that the real-life decision-making problems are not just based on symmetric data sets. Rather, a practical decision-making problem considers terms that are irregularly distributed. In order to deal with unbalanced linguistic information, Han et al. [16] established probabilistic unbalanced linguistic term set (PULTS), in which the authors proposed a computational model based on copula and co-copula. However, an additive scale set is still used to represent the data, therefore the term "unbalanced" does not retain its literal meaning. This is the first research gap in the literature that is addressed in the present study.

To handle unbalanced linguistic probabilistic information, Malhotra and Gupta [27] introduced a probabilistic multiplicative ULTS (PM-ULTS) that takes into account the irregular distribution of probabilistic linguistic terms and is sufficient enough to deal with all types of ULTSs. Further they have solved matrix games that have payoffs in terms of PM-ULTS. By the concept "PM-UL payoffs", it is meant that the players of the game express their responses in the form of linguistic terms that are distributed asymmetrically and are multiplicative in their distribution with some attached probabilities. Moreover, the set consisting of such types of linguistic terms is referred to as probabilistic multiplicative unbalanced linguistic term set. However, despite such study of PM-UL game, the methodology defined by [27] also has a limitation that it handles only one kind of uncertainty, i.e. fuzziness. The two most significant uncertainties in natural language, however, are randomness and fuzziness that have received a lot of attention in the field of artificial intelligence. It is typically exceedingly challenging to ascertain precise information or define the bounds of extensions if we only take fuzziness into account. For instance, there is no specific value that can be used to describe the concept's distinct border when we define the linguistic term "old". This is the second research gap addressed in the current study. In general, it is always possible for a linguistic variable to take on distinct meanings for different people, and even for the same person, to have different interpretations in different circumstances due to the abstractness and generality of notions. As a result, some notions frequently include randomness as well. Many linguistic terms have both randomness and fuzziness at the same time, despite the fact that they are of very different kinds. For instance, the linguistic terms "about", "maybe", "probably", etc., although represent random thoughts, also exhibit the fuzziness due to the vague boundary of the extension. In the light of this, it seems more practical for a formal

representation model of concepts to simultaneously account for both randomness and the fuzziness. Li et al. [21] presented a cloud model that can synthetically express the concept of randomness and fuzziness of decision information at the same time to formally characterize the underlying relationship between randomness and fuzziness and effectively integrate both uncertainties. The cloud models use EX (expectation), EN (entropy), and HE (hyperentropy) to illustrate a linguistic concept. Since the development of cloud models, numerous academics have dedicated their efforts to advance cloud-specific studies [23, 24, 26, 29, 39–42]. Although existing cloud models are believed to be able to preserve the original information, they are still unable to accurately represent labels that are asymmetrically distributed. To solve this problem, Wang et al. [43] proposed an Asymmetric Trapezoidal Clouds (ATCs) model as a solution to this problem. The EN and HE of the ATCs differ on the left and right sides in this model, and the authors expressed the expectation as an interval rather than a specific number. To address multiplicative unbalanced linguistic information, Chauhan et al. [11] have developed an accurate and innovative cloud decision support model. Here, in the present paper, we solve two-person matrix games having payoffs in the form of PM-ULTS by dealing with both randomness and fuzziness of the linguistic concepts. Further, the operational laws used to aggregate the payoffs in the paper [27] are based on probabilistic linguistic terms, while in present paper, well-defined cloud aggregation operators are used that are already well established in literature and do not follow any complex representation. This is the third research gap to be addressed in the current paper.

To fill all the discussed research gaps, the following contributions are offered in this paper:

- The matrix games having payoffs in the form of PM-ULTS are considered and solved using the defined methodology. These PM-ULTSs are able to handle all types of regular as well as irregular distribution of the linguistic concepts. Moreover, the defined methodology is able to deal with the most important uncertainties, i.e. randomness and fuzziness underlying in a qualitative information.
- An unbalanced linguistic cloud model is used to transform the probabilistic multiplicative data into its numerical form without making any significant loss of information. The probabilistic multiplicative unbalanced linguistic variables that are used as payoffs of a game problem are transformed into probabilistic unbalanced linguistic clouds (PULCs), i.e. the clouds generated from the probabilistic unbalanced linguistic terms, to effectively capture the uncertainty of the probabilistic linguistic payoff.
- The dispersion of cloud drops is measured by standard deviation S.D. defined in the paper that suggests the relative stability distribution of the linguistic term. Also, the score function is defined to compare different PULCs.
- To address the practical issues, the defined approach is then applied to address matrix games having probabilistic multiplicative unbalanced linguistic payoffs. The values of the players are defined in terms of PULCs. Also, the proposed method provides more accurate results by providing the values and strategies of the game in terms of expectation, entropy and hyperentropy. The expectation is the most representative and typical sample of the qualitative concept,

while entropy represents the randomness and fuzziness extent of the cloud drop. Hyperentropy reflects the uncertainty degree of entropy.

The paper is organized in the following manner. Section 2 reviews the basic concepts related to PM-ULTS, unbalanced linguistic cloud model and two-person matrix games that are required in the present paper. Section 3 provides the transformation of probabilistic multiplicative unbalanced linguistic variables to PULCs and operational laws. In Sect. 4, the application of PULC model in two-person constant sum matrix games is illustrated by first defining a PM-ULC game (PM-ULCG) along with the methodology for solving PM-ULCG followed by a numerical problem. Some analysis of results obtained and comparison with other defined methodologies, along with discussion, is provided in Sect. 5. Section 6 concludes the paper along with future scope.

2 Preliminaries

This section reviews some basic concepts that are required in this paper. The PM-ULTS defined in Sect. 2.1 consists of unbalanced linguistic terms having probabilities associated with them. Section 2.2 discusses the unbalanced linguistic cloud model based on which the payoffs are aggregated in this paper. In Sect. 2.3, the fundamentals of two-person matrix games are discussed.

2.1 Probabilistic multiplicative unbalanced linguistic term set (PM-ULTS)

Definition 1 [27] Let $\mathbb{LS} = \{t_{a^i} | i = -n, \dots, 0, \dots, m\}$ be a ULTS, where n and m may or may not be equal, $a > 1$. Then, a probabilistic multiplicative ULTS is defined as

$$\mathbb{LS}(p) = \{t_{a^i}^k(p^k) | k = 1, 2, \dots, \#\mathbb{LS}(p)\}$$

where $t_{a^i}^k \in \mathbb{LS}$, $p^k \geq 0$, $\sum_{k=1}^{\#\mathbb{LS}(p)} p^k$ may or may not be equal to 1 and $\#\mathbb{LS}(p)$ is the cardinality of all the given distinct multiplicative unbalanced linguistic terms in $\mathbb{LS}(p)$. The term $t_{a^i}^k(p^k)$ is interpreted as unbalanced linguistic term t_{a^i} associated with the probability p^k .

Note If $\sum_{k=1}^{\#\mathbb{LS}(p)} p^k < 1$ or $\sum_{k=1}^{\#\mathbb{LS}(p)} p^k > 1$, then the probability information within the set is incomplete and we normalize the incomplete PM-ULTS to transform the set into complete PM-ULTS using the following process.

1. Normalize the probability. If $\sum_{k=1}^{\#\mathbb{LS}(p)} p^k \neq 1$, then the given PM-ULTS is defined as $\widetilde{\mathbb{LS}}(p) = \{t_{a^i}^k(\bar{p}^k) | \sum_{k=1}^{\#\mathbb{LS}(p)} \bar{p}^k = 1\}$, where $\bar{p} = \frac{p^k}{\sum_{k=1}^{\#\mathbb{LS}(p)} p^k}$, $t_{a^i} \in \mathbb{LS}$, $p^k \geq 0$, $k = 1, \dots, \#\mathbb{LS}(p)$;

2. Normalize the granularity. If $\mathbb{L}\mathbb{S}_1(p)$ and $\mathbb{L}\mathbb{S}_2(p)$ are two PM-ULTS. If $\#\mathbb{L}\mathbb{S}_1(p) < \#\mathbb{L}\mathbb{S}_2(p)$, add $\#\mathbb{L}\mathbb{S}_2(p) - \#\mathbb{L}\mathbb{S}_1(p)$ multiplicative unbalanced linguistic terms to $\mathbb{L}\mathbb{S}_1(p)$

Definition 2 [27] Let $\mathbb{L}\mathbb{S}(p) = \{t_{a^i}^k(\bar{p}^k) | k = 1, \dots, \#\mathbb{L}\mathbb{S}(p)\}$ be a normalized PM-ULTS. The negation of $\mathbb{L}\mathbb{S}(p)$ is defined as $Neg(\mathbb{L}\mathbb{S}(p)) = \{neg(t_{a^i}^k)(\bar{p}^k) | k = 1, \dots, \#\mathbb{L}\mathbb{S}(p)\}$.

Definition 3 [27] Let $\mathbb{L}\mathbb{S}_1(p_1) = \{t_{a^i}^k(p^k) | k = 1, 2, \dots, \#\mathbb{L}\mathbb{S}_1(p_1)\}$ and $\mathbb{L}\mathbb{S}_2(p_2) = \{t_{a^j}^k(p^k) | k = 1, 2, \dots, \#\mathbb{L}\mathbb{S}_2(p_2)\}$, $i, j = -n, \dots, m$ be any two ordered PM-ULTS. Then, the operational laws are given as follows:

1. $\mathbb{L}\mathbb{S}_1(p_1) \oplus \mathbb{L}\mathbb{S}_2(p_2) = \{t_{a^{i \oplus j}}^k(\bar{p}_1^k \oplus \bar{p}_2^k) | k = 1, 2, \dots, \#\mathbb{L}\mathbb{S}_1(p_1)\}$
2. $\mathbb{L}\mathbb{S}_1(p_1) \otimes \mathbb{L}\mathbb{S}_2(p_2) = \{t_{a^{i \otimes j}}^k(\bar{p}_1^k \otimes \bar{p}_2^k) | k = 1, 2, \dots, \#\mathbb{L}\mathbb{S}_1(p_1)\}$
3. Let $\lambda \in (0, 1)$ be any arbitrary scalar, $\lambda \odot \mathbb{L}\mathbb{S}(p) = \{t_{\lambda \odot a^i}^k(\bar{p}^k) | k = 1, 2, \dots, \#\mathbb{L}\mathbb{S}(p)\}$
4. $(\mathbb{L}\mathbb{S}(p))^\lambda = \{t_{a^{\lambda i}}^k(\bar{p}^k) | k = 1, 2, \dots, \#\mathbb{L}\mathbb{S}(p)\}$

2.2 Unbalanced linguistic cloud model

Definition 4 [11] Let $\mathbb{L}\mathbb{S} = \{t_{a^i} | i = -n, \dots, 0, \dots, m\}$, $a > 1$ (n and m may or may not be equal) be any finite ordered multiplicative unbalanced LT set. The linguistic scale function (LSF) converting a linguistic term t_{a^i} to a certain number δ_{a^i} , ($i = -n, \dots, 0, \dots, m$) can be expressed as follows:

$$\tilde{g} : t_{a^i} \rightarrow \delta_{a^i} (i = -n, \dots, 0, \dots, m)$$

where

$$\delta_{a^i} = \begin{cases} \left(\frac{b^n - b^{-i}}{2b^n - 2} \right) \frac{1}{\log a}, & -n \leq i \leq 0 \\ \left(\frac{b^n + b^i - 2}{2b^n - 2} \right) \frac{1}{\log a}, & 0 < i \leq n \end{cases} \quad (1)$$

Here, $\delta_{a^i} \in [0, 1/\log a] \subseteq [0, 1]$ is a numeric value. The value of the parameter b can be obtained through experiments such as $b \in [1.36, 1.4]$ [4]. In this paper, we take $b = 1.4$ without any loss of generality.

The next definition introduces the process of generating cloud droplets based on the rational transformation method of unbalanced linguistic variables into clouds.

Definition 5 [11] Let $t_{a^i} \in \mathbb{L}\mathbb{S}$ be any arbitrary unbalanced linguistic term corresponding to the cloud drop $(EX_{a^i}, EN_{a^i}, HE_{a^i})$. Then, the working cloud generation procedure is given as follows:

1. Calculate δ_{a^i} using Eq. 1.
2. Calculate EX_{a^i} .

$$EX_{a^i} = X_{\min} + \delta_{a^i}(X_{\max} - X_{\min})$$

3. Calculate EN_{a^i} .

$$3EN'_{a^i} = \max(X_{\max} - EX_{a^i}, EX_{a^i} - X_{\min})$$

Because of $EN' \sim N(EN, HE^2)$, EN_{a^i} may be considered the mean value of EN'_{a^i} corresponding to the clouds in consideration and their adjacent clouds.

$$EN_{a^{-i}} = EN_{a^i} = \frac{EN'_{a^{i-1}} + EN'_{a^i} + EN'_{a^{i+1}}}{3 \log a} \text{ for } |i| = 0, 1, \dots, n-1, \quad (2)$$

and for $|i| = n$,

$$EN_{a^{-n}} = EN_{a^n} = \frac{EN'_{a^n} + EN'_{a^{n-1}}}{2 \log a} \quad (3)$$

These calculations may be rewritten as the following formulas:

$$EN'_{a^i} = \begin{cases} \frac{(1-\delta_{a^i})(X_{\max}-X_{\min})}{3 \log a}, & -n \leq i \leq 0 \\ \frac{\delta_{a^i}(X_{\max}-X_{\min})}{3 \log a}, & 0 < i \leq n \end{cases} \quad (4)$$

$$EN_{a^{-i}} = EN_{a^i} = \begin{cases} \frac{(\delta_{a^{|i|-1|}} + \delta_{a^{|i|}} + \delta_{a^{|i|+1|}})(X_{\max}-X_{\min})}{9 \log a}, & 0 < |i| < n-1 \\ \frac{(\delta_{a^{|i|}} + \delta_{a^{|i|-1|}})(X_{\max}-X_{\min})}{6 \log a}, & |i| = n \\ \frac{(\delta_{a^i} + 2\delta_{a^{i+1}})(X_{\max}-X_{\min})}{9 \log a}, & i = 0 \end{cases} \quad (5)$$

4. Calculate HE_{a^i} . Due to $EN'_{a^i} \sim N(EN_{a^i}, HE_{a^i}^2)$, the hyperentropy satisfies the 3σ principle. In particular,

$$HE_{a^{-i}} = HE_{a^i} = \frac{EN'^+ - EN_{a^i}}{3 \log a} \quad (6)$$

where, $EN'^+ = \sup_k \{EN'_k\}$.

Note It should be noted that the point-based cloud-generating procedure described above can be used in scenario $n \neq m$ if dummy variables are added to the multiplicative unbalanced linguistic term set in a way that $n = m$.

2.3 Matrix games

Definition 6 [5] A crisp two-player zero-sum matrix game is represented as a triplet (S^r, S^s, A) , where S^r is the set of ‘ r ’ possible strategies of Player I , S^s is the set of ‘ s ’ possible strategies of Player II , $[A]_{r \times s}$ is the payoff matrix of order of Player I corresponding to Player II , and $neg(A)$ refers to the matrix consisting of payoffs of Player II .

Definition 7 [5] A matrix game having $A = (a_{ij})_{r \times s}$ as the matrix of payoffs has the lower value

$$v^- = \max_{i=1, \dots, r} \min_{j=1, \dots, s} a_{ij}$$

and the upper value

$$v^+ = \min_{j=1, \dots, s} \max_{i=1, \dots, r} a_{ij}$$

The upper value v^+ is the guaranteed maximum that Player II may lose, while the lower value v^- is the guaranteed minimum that Player I is guaranteed to get. If $v^+ = v^-$, we say that the game has a value of v .

Definition 8 [5] If no pure strategies exist, a mixed strategy is a vector where $\Phi = (\phi_1, \phi_2, \dots, \phi_r)$ for Player I and $\Psi = (\psi_1, \psi_2, \dots, \psi_s)$ for Player II , where

$$\phi_i \geq 0, \sum_{i=1}^r \phi_i = 1 \quad \& \quad \psi_j \geq 0, \sum_{j=1}^s \psi_j = 1.$$

Player I ’s probability of utilizing row ‘ i ’ and Player II ’s probability of using column ‘ j ’ are each represented by ϕ_i and ψ_j . We represent the collection of mixed strategies having ‘ k ’ components as

$$S^k = \left\{ (z_1, z_2, \dots, z_k) \mid z_r \geq 0, r = 1, 2, \dots, k, \sum_{r=1}^k z_r = 1 \right\}.$$

Given a choice of mixed strategy $\Phi = (\phi_1, \phi_2, \dots, \phi_r)$ for Player I and $\Psi = (\psi_1, \psi_2, \dots, \psi_s)$ for Player II , selected autonomously, the expected payoff of Player I is

$$\begin{aligned} E(\Phi, \Psi) &= \sum_{i=1}^r \sum_{j=1}^s \phi_i a_{ij} \psi_j \\ &= \Phi A \Psi^T \end{aligned}$$

3 Transformation of PM-UL variables to PULCs and operational laws

In this section, we present how to transform a PM-UL variable to its corresponding PULC, and a numerical example is shown. Further, the operational laws and score function of PULCs are defined.

3.1 Method for transformation of probabilistic unbalanced linguistic variables to probabilistic unbalanced linguistic clouds (PULCs)

Definition 9 Let $\mathbb{LS} = \{t_{a^i} | i = -n, \dots, 0, \dots, m\}$ be a ULTS where n and m may or may not be equal and $a > 1$. Suppose p_i be any probability associated with the unbalanced linguistic term t_{a^i} such that $\mathbb{LS}(p) = \{t_{a^i}(p_i) | i = -n, \dots, m\}$ denotes the linguistic set representing linguistic terms having attached probabilities.

Consider a valid universe $X = [X_{\min}, X_{\max}]$. Now to represent the linguistic terms of the set $\mathbb{LS}(p)$, the procedure is as given below.

- First transform the linguistic terms into their corresponding cloud using the method given in Definition 4 and 5, i.e.
- * Find the value of the LSF δ_{a^i} , ($i = -n, \dots, 0, \dots, m$)
- * Calculate EX_{a^i} as $EX_{a^i} = X_{\min} + \delta_{a^i}(X_{\max} - X_{\min})$.
- * Compute EN_{a^i} .
- * Calculate HE_{a^i} .

Note. The above-mentioned procedure is used in the scenario $n = m$. In case $n \neq m$, then we add a dummy linguistic term with probability $\}0'$ in $\mathbb{LS}(p)$ so that $n = m$, where the linguistic term is added on the side of $\min(m, n)$.

- Now consider the attached probabilities with the linguistic terms by multiplying these probabilities with the above-obtained cloud. Hence, a cloud corresponding to a linguistic term $t_{a^i}(p_i)$ is given as $\mathbb{Y}_i(p_i EX_{a^i}, p_i EN_{a^i}, p_i HE_{a^i})$.

$$\therefore t_{a^i}(p_i) \longrightarrow (p_i EX_{a^i}, p_i EN_{a^i}, p_i HE_{a^i})$$

Example 1 Given the universe $[X_{\min}, X_{\max}] = [0, 7]$, let $\mathbb{LS} = \{t_{2^{-2}} : \text{Terrible}, t_{2^{-1}} : \text{Poor}, t_{2^0} : \text{Middle}, t_{2^1} : \text{Satisfactory}, t_{2^2} : \text{Good}, t_{2^3} : \text{Excellent}\}$. As $n \neq m$, we add a dummy linguistic term on the left hand side of $\mathbb{LS}$ with probability $\}0'$. Consider $\mathbb{LS}(p) = \{t_{2^{-3}}(0), t_{2^{-2}}(0), t_{2^{-1}}(0.1), t_{2^0}(0.4), t_{2^1}(0), t_{2^2}(0.2), t_{2^3}(0.3)\}$. By following the steps mentioned above, the cloud is generated as follows. Setting $b = 1.4$.

1. First we calculate δ_{a^i} for each t_{a^i} . $\delta_{2^{-3}} = 0$ $\delta_{2^{-2}} = 0.7467$ $\delta_{2^{-1}} = 1.2801$
 $\delta_{2^0} = 1.6611$ $\delta_{2^1} = 2.0421$ $\delta_{2^2} = 2.5755$ $\delta_{2^3} = 3.3222$
2. $EX_{2^{-3}} = 0$ $EX_{2^{-2}} = 5.22718$ $EX_{2^{-1}} = 8.9607$ $EX_{2^0} = 11.6277$ $EX_{2^1} = 14.2947$
 $EX_{2^2} = 18.0285$ $EX_{2^3} = 23.2557$
3. $EN'_{2^{-3}} = 7.7519$ $EN'_{2^{-2}} = 1.9632$ $EN'_{2^{-1}} = -2.1715$ $EN'_{2^0} = -5.125$
 $EN'_{2^1} = 15.8308$ $EN'_{2^2} = 19.9652$ $EN'_{2^3} = 25.7538$. $\therefore \max EN'_{a^i} = 25.7538$ Next,
 $EN_{2^{-3}} = EN_{2^3} = 22.8595$ $EN_{2^{-2}} = EN_{2^2} = 20.5164$ $EN_{2^{-1}} = EN_{2^1} = 16.2241$
 $EN_{2^0} = 14.8457$
4. $HE_{2^{-3}} = HE_{2^3} = 20.9258$ $HE_{2^{-2}} = HE_{2^2} = 23.9762$ $HE_{2^{-1}} = HE_{2^1} = 26.7236$
 $HE_{2^0} = 21.944$

Hence, the probabilistic multiplicative unbalanced clouds corresponding to each linguistic term of the set $\mathbb{L}\mathbb{S}(p)$ are given as: $t_{2^{-3}}(0) \rightarrow (0, 0, 0)$ $t_{2^{-2}}(0) \rightarrow (0, 0, 0)$
 $t_{2^{-1}}(0.1) \rightarrow (0.89607, 1.6224, 2.6723)$ $t_{2^0}(0.4) \rightarrow (4.651, 5.9382, 8.777)$ $t_{2^1}(0) \rightarrow$
 $t_{2^2}(0) \rightarrow (0, 0, 0)$ $t_{2^2}(0.2) \rightarrow (3.6057, 4.1032, 4.7952)$ $t_{2^3}(0.3) \rightarrow (6.9767,$
 $6.8578, 6.2777)$ These probabilistic multiplicative unbalanced linguistic clouds are represented in Fig. 1.

In the graph, the dispersing extent of cloud drops is depicted by entropy and the hyperentropy EN and HE, respectively. It is also the measurement of fuzziness, representing the scope of the universe that can be accepted by the concept.

3.2 Operations and score functions of probabilistic unbalanced linguistic clouds

In this section, we define some operational laws for probabilistic unbalanced linguistic clouds based on the operators defined for normal clouds.

Definition 10 Given the PULCs $\mathbb{Y}_i = (p_i EX_{a^i}, p_i EN_{a^i}, p_i HE_{a^i})$, the basic operations are developed as follows:

1. $\sum_{i=-n}^m (p_i EX_{a^i}, p_i EN_{a^i}, p_i HE_{a^i}) = \left(\sum_{i=-n}^m p_i EX_{a^i}, \sqrt{\sum_{i=-n}^m (p_i EN_{a^i})^2}, \sqrt{\sum_{i=-n}^m (p_i HE_{a^i})^2} \right)$
2. $\lambda(p_i EX_{a^i}, p_i EN_{a^i}, p_i HE_{a^i}) = (\lambda p_i EX_{a^i}, \lambda p_i EN_{a^i}, \lambda p_i HE_{a^i})$, where λ is any real scalar.
3. Let $\mathbb{Y}_i(p_i EX_{a^i}, p_i EN_{a^i}, p_i HE_{a^i})$, $i = -n, \dots, m$ be a collection of PULCs with associated weights $\omega_i \in [0, 1]$ and $\sum_{i=-n}^m \omega_i = 1$. Then, their aggregated value with use of the weighted average (WA) operator is still a cloud and is denoted as follows:

$$WA(Y_{-n}, \dots, Y_m) = \left(\sum_{i=-n}^m \omega_i p_i EX_{a^i}, \sqrt{\sum_{i=-n}^m \omega_i (p_i EN_{a^i})^2}, \sqrt{\sum_{i=-n}^m \omega_i (p_i HE_{a^i})^2} \right) \quad (7)$$

The score function is a crucial tool for obtaining the ranking results during the decision-making process. For the normal clouds, a number of score functions have previously been defined. Here, we define the score function for PULCs. Before

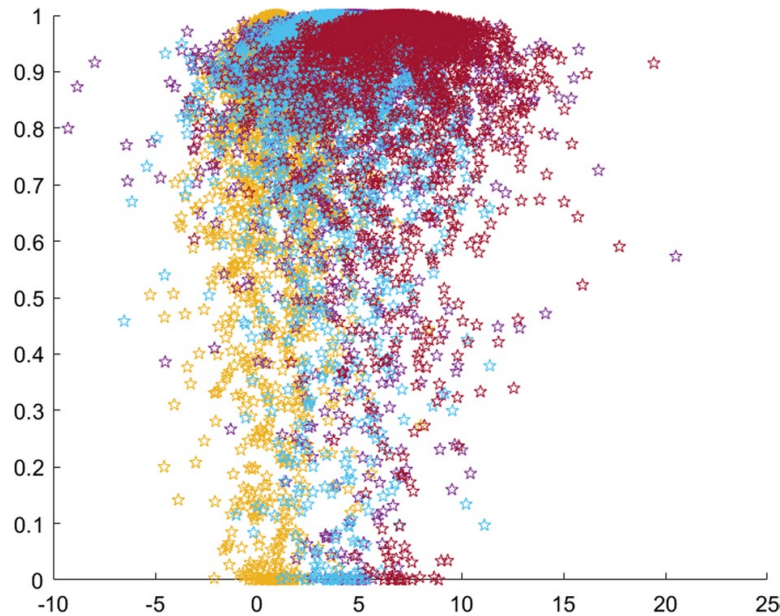


Fig. 1 Representation of probabilistic multiplicative unbalanced linguistic clouds with respect to linguistic terms having attached probabilities based on 1000 droplets

presenting the new function, we first introduce an important concept, the variance of a normal cloud, which is developed by Li et al. [30].

Definition 11 Let $(p_i EX_{ai}, p_i EN_{ai}, p_i HE_{ai})$ be a PULC with $n + m + 1$ generated cloud drops, and the variance of these cloud drops can be computed as

$$V = (p_i EN_{ai})^2 + (p_i HE_{ai})^2 \quad (8)$$

Hence, the standard deviation follows $S.D. = \sqrt{V}$.

The standard deviation ($S.D.$), which can be used to express the ambiguity and fluctuation of a quantitative notion, is used to measure the dispersion of cloud drops. A low standard deviation indicates that the distribution of the linguistic term is reasonably steady. A high standard deviation suggests stability even when the linguistic concept has a high degree of ambiguity and volatility. In this case, the relationship between EX and $S.D$ can be seen as the expected utility and utility dispersion of the utility theory. Therefore, the choice for an option (alternative) can be expressed as a linear trade-off between expected utility and utility dispersion. In response to this approach, we construct a score formula below to compare PULCs.

Definition 12 Given a PULC $(p_i EX_{ai}, p_i EN_{ai}, p_i HE_{ai})$, then a score function of PULC is defined as

$$\hat{S} = p_i EX_{ai} - \gamma S.D. \quad (9)$$

where $S.D.$ is the standard deviation of a PULC, $\gamma \in [0, 1]$ refers to a coefficient to capture the attitude towards the standard deviation. The value of γ can be chosen based on actual decision-making circumstances. $\gamma = 0$ signifies that the DMs do not

care about the uncertainty and fluctuation of the assessment information and simply take into account the average level, which is the expectation EX . The greater the value of γ , the greater the weight given to $S.D$.

The score function $\hat{S}$ is monotonically increasing w.r.t. EX while monotonically decreasing w.r.t. EN and HE . On the basis of score function, two PULCs may be compared in the following manner.

Definition 13 Let $\mathbb{Y}_1$ and $\mathbb{Y}_2$ be two PULCs, and $\hat{S}_1$ and $\hat{S}_2$ denote their score functions, respectively, then

- If $\hat{S}_1 > \hat{S}_2$, then we prefer $\mathbb{Y}_1$ over $\mathbb{Y}_2$ and denote it as $\mathbb{Y}_1 > \mathbb{Y}_2$.
- $\hat{S}_1 = \hat{S}_2$ represents that both the PULCs are equally preferred and are denoted as $\mathbb{Y}_1 = \mathbb{Y}_2$.
- $\max\{\mathbb{Y}_i | i = -n, \dots, m\} = \mathbb{Y}_{\max}$ such that $\hat{S}(\mathbb{Y}_{\max}) \geq \hat{S}(\mathbb{Y}_i) \forall i$.
- $\min\{\mathbb{Y}_i | i = -n, \dots, m\} = \mathbb{Y}_{\min}$ such that $\hat{S}(\mathbb{Y}_{\min}) \leq \hat{S}(\mathbb{Y}_i) \forall i$.

Example 2 Consider the problem given in Example 1, the set consisting of linguistic terms with attached probabilities is $\mathbb{LS}(p) = \{t_{2-3}(0), t_{2-2}(0), t_{2-1}(0.1), t_{20}(0.4), t_{21}(0), t_{22}(0.2), t_{23}(0.3)\}$. The PULCs obtained from these probabilistic unbalanced linguistic terms are

$$\mathbb{Y}_1(0, 0, 0) \quad \mathbb{Y}_2(0, 0, 0) \quad \mathbb{Y}_3(0.89607, 1.6224, 2.6723) \quad \mathbb{Y}_4(4.651, 5.9382, 8.777)$$

$$\mathbb{Y}_5(0, 0, 0) \quad \mathbb{Y}_6(3.6057, 4.1032, 4.7952) \quad \mathbb{Y}_7(6.9767, 6.8578, 6.2777)$$

Using the definition of score function, we find scores of these PULCs for some random values of γ and provide their rankings in Table 1.

Now, from the aspect of information representation form, the PM-UL information is a flexible tool for CWW. Also, the unbalanced linguistic cloud model has provided wonderful results to deal with uncertainty of a linguistic concept. But in the framework of game theory, it is still not investigated how to solve a game problem whose payoffs are of the form PM-UL by capturing the randomness and fuzziness involved in the payoffs. In this regard, Sect. 4 presents the definition of a PM-UL game (PM-ULG) and a methodology is given to solve such type of game by converting its payoffs to PULCs and treating it as a PM-UL cloud game (PM-ULCG).

Table 1 PULC rankings based on scores for different values of γ

Value of γ	$\hat{S}(\mathbb{V}_1)$	$\hat{S}(\mathbb{V}_2)$	$\hat{S}(\mathbb{V}_3)$	$\hat{S}(\mathbb{V}_4)$	$\hat{S}(\mathbb{V}_5)$	$\hat{S}(\mathbb{V}_6)$	$\hat{S}(\mathbb{V}_7)$	Score ranking	PULC ranking
0	0	0	0.896	4.651	0	3.605	6.277	$\hat{S}(\mathbb{V}_7) > \hat{S}(\mathbb{V}_4) > \hat{S}(\mathbb{V}_6) > \hat{S}(\mathbb{V}_3) > \hat{S}(\mathbb{V}_1) = \hat{S}(\mathbb{V}_2) = \hat{S}(\mathbb{V}_5)$	$\mathbb{V}_7 > \mathbb{V}_4 > \mathbb{V}_6 > \mathbb{V}_3 > \mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}_5$
0.1	0	0	0.583	3.591	0	2.974	6.046	$\hat{S}(\mathbb{V}_7) > \hat{S}(\mathbb{V}_4) > \hat{S}(\mathbb{V}_6) > \hat{S}(\mathbb{V}_3) > \hat{S}(\mathbb{V}_1) = \hat{S}(\mathbb{V}_2) = \hat{S}(\mathbb{V}_5)$	$\mathbb{V}_7 > \mathbb{V}_4 > \mathbb{V}_6 > \mathbb{V}_3 > \mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}_5$
0.2	0	0	0.27	2.531	0	2.343	5.117	$\hat{S}(\mathbb{V}_7) > \hat{S}(\mathbb{V}_4) > \hat{S}(\mathbb{V}_6) > \hat{S}(\mathbb{V}_3) > \hat{S}(\mathbb{V}_1) = \hat{S}(\mathbb{V}_2) = \hat{S}(\mathbb{V}_5)$	$\mathbb{V}_7 > \mathbb{V}_4 > \mathbb{V}_6 > \mathbb{V}_3 > \mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}_5$
0.3	0	0	-0.041	1.471	0	1.712	4.187	$\hat{S}(\mathbb{V}_7) > \hat{S}(\mathbb{V}_4) > \hat{S}(\mathbb{V}_6) > \hat{S}(\mathbb{V}_3) > \hat{S}(\mathbb{V}_1) = \hat{S}(\mathbb{V}_2) = \hat{S}(\mathbb{V}_5)$	$\mathbb{V}_7 > \mathbb{V}_4 > \mathbb{V}_6 > \mathbb{V}_3 > \mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}_5$
0.4	0	0	-0.354	0.412	0	1.081	3.257	$\hat{S}(\mathbb{V}_7) > \hat{S}(\mathbb{V}_4) > \hat{S}(\mathbb{V}_6) > \hat{S}(\mathbb{V}_3) > \hat{S}(\mathbb{V}_1) = \hat{S}(\mathbb{V}_2) = \hat{S}(\mathbb{V}_5)$	$\mathbb{V}_7 > \mathbb{V}_4 > \mathbb{V}_6 > \mathbb{V}_3 > \mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}_5$
0.5	0	0	-0.667	-0.647	0	0.45	2.328	$\hat{S}(\mathbb{V}_7) > \hat{S}(\mathbb{V}_4) > \hat{S}(\mathbb{V}_6) > \hat{S}(\mathbb{V}_3) > \hat{S}(\mathbb{V}_1) = \hat{S}(\mathbb{V}_2) = \hat{S}(\mathbb{V}_5)$	$\mathbb{V}_7 > \mathbb{V}_4 > \mathbb{V}_6 > \mathbb{V}_3 > \mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}_5$
0.6	0	0	-0.979	-1.707	0	-0.18	1.398	$\hat{S}(\mathbb{V}_7) > \hat{S}(\mathbb{V}_4) > \hat{S}(\mathbb{V}_6) > \hat{S}(\mathbb{V}_3) > \hat{S}(\mathbb{V}_1) = \hat{S}(\mathbb{V}_2) = \hat{S}(\mathbb{V}_5)$	$\mathbb{V}_7 > \mathbb{V}_4 > \mathbb{V}_6 > \mathbb{V}_3 > \mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}_5$
0.7	0	0	-1.292	-2.766	0	-0.812	0.468	$\hat{S}(\mathbb{V}_7) > \hat{S}(\mathbb{V}_4) > \hat{S}(\mathbb{V}_6) > \hat{S}(\mathbb{V}_3) > \hat{S}(\mathbb{V}_1) = \hat{S}(\mathbb{V}_2) = \hat{S}(\mathbb{V}_5)$	$\mathbb{V}_7 > \mathbb{V}_4 > \mathbb{V}_6 > \mathbb{V}_3 > \mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}_5$
0.8	0	0	-1.604	-3.826	0	-1.443	-0.461	$\hat{S}(\mathbb{V}_7) > \hat{S}(\mathbb{V}_4) > \hat{S}(\mathbb{V}_6) > \hat{S}(\mathbb{V}_3) > \hat{S}(\mathbb{V}_1) = \hat{S}(\mathbb{V}_2) = \hat{S}(\mathbb{V}_5)$	$\mathbb{V}_7 > \mathbb{V}_4 > \mathbb{V}_6 > \mathbb{V}_3 > \mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}_5$
0.9	0	0	-1.917	-4.886	0	-2.074	-1.39	$\hat{S}(\mathbb{V}_7) > \hat{S}(\mathbb{V}_4) > \hat{S}(\mathbb{V}_6) > \hat{S}(\mathbb{V}_3) > \hat{S}(\mathbb{V}_1) = \hat{S}(\mathbb{V}_2) = \hat{S}(\mathbb{V}_5)$	$\mathbb{V}_7 > \mathbb{V}_4 > \mathbb{V}_6 > \mathbb{V}_3 > \mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}_5$
1	0	0	-2.239	-5.946	0	-2.705	-2.32	$\hat{S}(\mathbb{V}_7) > \hat{S}(\mathbb{V}_4) > \hat{S}(\mathbb{V}_6) > \hat{S}(\mathbb{V}_3) > \hat{S}(\mathbb{V}_1) = \hat{S}(\mathbb{V}_2) = \hat{S}(\mathbb{V}_5)$	$\mathbb{V}_7 > \mathbb{V}_4 > \mathbb{V}_6 > \mathbb{V}_3 > \mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}_5$

4 Two-person probabilistic multiplicative unbalanced linguistic cloud games (PM-ULCG)

In uncertain situations, the probabilistic unbalanced linguistic matrix game with payoffs as clouds is introduced in this section to model uncertain cloud payoff values. The payoff matrix $\tilde{A}$ with probabilistic unbalanced linguistic information can be denoted as:

Player <i>I</i>	Player <i>II</i>			
	Strategy 1	Strategy 2	...	Strategy <i>s</i>
Strategy 1	$\tilde{a}_{11}$	$\tilde{a}_{12}$	...	$\tilde{a}_{1s}$
Strategy 2	$\tilde{a}_{21}$	$\tilde{a}_{12}$	...	$\tilde{a}_{2s}$
⋮	⋮	⋮	⋮	⋮
Strategy <i>r</i>	$\tilde{a}_{r1}$	$\tilde{a}_{r2}$	...	$\tilde{a}_{rs}$

where each $\tilde{a}_{ij} \in LS(p)$ is a probabilistic unbalanced linguistic term.

4.1 PM-ULCG: definition and meaning

A two-person constant sum probabilistic multiplicative unbalanced linguistic cloud game is defined as $(S^r, S^s, \mathbb{LS}(p), \tilde{A})$ where S^r and S^s are the strategy sets of Player *I* and Player *II*, respectively. $\mathbb{LS}(p)$ is the set of PM-UL terms having associated probabilities. Now to deal with the uncertainties like randomness and fuzziness underlying the PM-UL terms $\tilde{a}_{ij}$, we first convert the payoffs in the form of PULC. Hence, finally, $\tilde{A}$ is the payoff matrix of Player *I* corresponding to Player *II* given as

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1s} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{r1} & \tilde{a}_{r2} & \cdots & \tilde{a}_{rs} \end{pmatrix}$$

where each $\tilde{a}_{ij} = (p_t EX_{a^t}, p_t EN_{a^t}, p_t HE_{a^t})$ is a PULC. On the contrary, $neg(\tilde{A})$ is the payoff matrix of Player *II*, where $neg(\tilde{A}) = neg(\tilde{a}_{ij})_{r \times s}$, where $neg(\tilde{a}_{ij})$ is as defined in Definition 10.

Conventionally, Player *I* is the row player and wants to maximize his payoff, while column player *II* wants to minimize row player's payoff so as to maximize his own payoff. The rows are pure strategies for Player *I* and columns for Player *II*.

$$v^- = \max_{i=1, \dots, r} \min_{j=1, \dots, s} \tilde{a}_{ij}$$

called the lower value of the game, and is the minimum amount that Player *I* is guaranteed to receive. Here $\tilde{a}_{ij}$ is a PM-UL term. On the other hand,

$$\omega^- = \max_{i=1,\dots,r} \min_{j=1,\dots,s} \text{neg}(\tilde{a}_{ij})$$

is the minimum amount that Player *II* is guaranteed to receive. Obviously,

$$v^+ = \text{neg}(\omega^-) = \min_{j=1,\dots,s} \max_{i=1,\dots,r} \tilde{a}_{ij}$$

is the upper value of the game and is the guaranteed maximum amount that Player *II* can lose. The game is said to have a value $v = (EX_{a^i}, EN_{a^i}, HE_{a^i})$ if $v^- = v^+ = v$, also called the saddle point in pure strategies. However, in most two-person zero sum games a saddle point in pure strategies will not exist because that would interpret that the players must always play in a similar manner. But that is not always possible as in such case, the opposing player will be able to figure out the player's strategy in advance.

Definition 14 A mixed cloud strategy is a vector $\rho = (\rho_1, \rho_2, \dots, \rho_r)$ for Player *I* and $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_s)$ for Player *II*, where

$$\rho_i \geq 0, \rho_i = (EX_{a^i}, EN_{a^i}, HE_{a^i}) \text{ so that } \sum_{i=1}^r \rho_i = 1 \implies \sum_{i=1}^r EX_{a^i} = 1,$$

$$\sqrt{\sum_{i=1}^r EN_{a^i}^2} = 1, \text{ and } \sqrt{\sum_{i=1}^r HE_{a^i}^2} = 1$$

and,

$$\varphi_j \geq 0, \varphi_j = (EX_{a^j}, EN_{a^j}, HE_{a^j}) \text{ so that } \sum_{j=1}^s \varphi_j = 1 \implies \sum_{j=1}^s EX_{a^j} = 1,$$

$$\sqrt{\sum_{j=1}^s EN_{a^j}^2} = 1, \text{ and } \sqrt{\sum_{j=1}^s HE_{a^j}^2} = 1$$

The set of mixed strategies having k components is denoted as

$$S^k = \left\{ (z_1, z_2, \dots, z_k) | z_i \geq 0, \sum_{i=1}^k EX_{a^i} = 1, \sqrt{\sum_{i=1}^k EN_{a^i}^2} = 1, \text{ and } \sqrt{\sum_{i=1}^k HE_{a^i}^2} = 1 \right\}$$

such that $\rho \in S^r$ and $\varphi \in S^s$.

Since it is mentioned that a PM-ULCG has a saddle point in pure strategies if $v^- = v^+$. However, in practical real-life scenarios, this is not always the case that saddle point exists. Hence in that case, the following inequality holds in general.

Theorem 1 Suppose v^- and v^+ be the lower and upper values of a PM-ULCG, respectively, and let both the values exist. Then $v^- \leq v^+$.

Proof For any column j , we know that for any fixed row i , by Definition 13

$$\min_{j=1,\dots,s} \tilde{a}_{ij} \leq \tilde{a}_{ij}$$

Taking max over $i = 1, \dots, r$ both the sides,

$$v^- = \max_{i=1,\dots,r} \min_{j=1,\dots,s} \tilde{a}_{ij} \leq \max_{i=1,\dots,r} \tilde{a}_{ij}$$

This is true for any column $j = 1, \dots, s$. The LHS is just a number independent of i and j and is smaller than RHS for any j . This means that $v^- \leq \min_{j=1,\dots,s} \max_{i=1,\dots,r} \tilde{a}_{ij} = v^+$. This proves the theorem. $\square$

In crisp games, the expected payoff of the player is presented in the form of mathematical expectation; but the uncertainty pertaining in the PM-ULCG does not allow the expected payoffs to be defined simply as in crisp games. Thus, it is necessary to define PM-UL cloud expected payoffs.

Definition 15 Let $\rho \in S^r$ and $\varphi \in S^s$ be the mixed strategies associated with Player I and Player II, respectively. Then, the PM-UL cloud expected payoff of Player I is defined as:

$$E(\rho, \varphi) = \bigoplus_{i=1}^r \left(\rho_i \left(\bigoplus_{j=1}^s \tilde{a}_{ij} \varphi_j \right) \right)$$

where $\tilde{a}_{ij} = (p_t EX_{at}, p_t EN_{at}, p_t HE_{at})$, $t = -n, \dots, m$ is a PULC.

$$\Rightarrow E(\rho, \varphi) = \left(\sum_{i=1}^r \sum_{j=1}^s \rho_i \varphi_j p_t EX_{at}, \sqrt{\sum_{i=1}^r \sum_{j=1}^s \rho_i \varphi_j (p_t EN_{at})^2}, \sqrt{\sum_{i=1}^r \sum_{j=1}^s \rho_i \varphi_j (p_t HE_{at})^2} \right) \quad (10)$$

and this expected payoff is further a PULC. Similarly, the PM-UL cloud expected payoff of Player II is given by $Neg(E(\rho, \varphi))$.

4.2 PM-ULCG: methodology and solution

This section discusses the probabilistic unbalanced linguistic cloud linear programming (PULCLP) method for solving matrix games problems having payoffs in the form of PM-UL terms.

Suppose $\mathbb{LS}(p) = \{t_{a^i}(p_i) | i = -n, \dots, m\}$ where n and m are positive integers and $a > 1$ be the linguistic set having PM-UL terms with probabilities

associated with them; $\tilde{A} = [\tilde{a}_{rs}]$ be the PULC matrix so that each $\tilde{a}_{ij}$ represents a PULC written as $\tilde{a}_{ij} = (p_t EX_{a^t}, p_t EN_{a^t}, p_t HE_{a^t})$. Let v^- refers to the lower value of the game corresponding to Player *I* and let $\rho \in S^r$ be the mixed strategies of Player *I*. Now, the PM-UL cloud expected payoff of Player *I* for s^{th} strategy of Player *II* is the weighted average of $\tilde{a}_{ij}$, i.e. $\bigoplus_{j=1}^s (\rho_j \tilde{a}_{ij})$ where $\tilde{a}_{ij} = (p_t EX_{a^t}, p_t EN_{a^t}, p_t HE_{a^t})$, $t = -n, \dots, m$ is a PULC such that $i = 1, 2, \dots, r$. Analogously, v^+ is the maximum expected payoff of Player *II*, $\varphi \in S^s$ is its mixed strategy, and its expected payoff against Player *I*'s r^{th} strategy is given by $\bigoplus_{i=1}^r (\varphi_i \tilde{a}_{ij})$ where $j = 1, 2, \dots, s$.

For finding the expected payoffs and values of both the players in terms of PULCs, we first convert each PM-UL payoff to a PULC.

Hence, the PULCLP for Player *I* can be given as

$$\begin{aligned} & \max v^- \\ & \text{s.t.} \begin{cases} \bigoplus_{i=1}^r \tilde{a}_{ij} \rho_i \geq v^- \quad j = 1, 2, \dots, s \\ \text{where} \\ \rho_i \geq 0, \sum_{i=1}^r \rho_i = 1. \end{cases} \end{aligned} \quad (\text{P1})$$

and for Player *II*, the problem is

$$\begin{aligned} & \max \omega^- \\ & \text{s.t.} \begin{cases} \bigoplus_{j=1}^s \text{neg}(\tilde{a}_{ij}) \varphi_j \geq \omega^- \quad i = 1, 2, \dots, r \\ \text{where} \\ \varphi_j \geq 0, \sum_{j=1}^s \varphi_j = 1. \end{cases} \end{aligned} \quad (\text{P2})$$

In above problems, each $\tilde{a}_{ij}$ is a PULC. Further, simplifying P2 and using the properties of negation operator, the problem for Player *II* becomes

$$\begin{aligned} & \min v^+ \\ & \text{s.t.} \begin{cases} \bigoplus_{j=1}^s \tilde{a}_{ij} \varphi_j \leq v^+ \quad i = 1, 2, \dots, r \\ \text{where} \\ \varphi_j \geq 0, \sum_{j=1}^s \varphi_j = 1. \end{cases} \end{aligned} \quad (\text{P3})$$

A unified mechanism for solving a probabilistic unbalanced linguistic game for Player *I* is given below in the form of a computation procedure.

Step 1: First normalize the payoff matrix for the row player (i.e. normalize the payoff entries row-wise) so that the sum of probabilities in each row is 1.

Step 2: Convert each payoff to a PULC using the procedure given in Sect. 3.1 to deal with uncertainty in a better way.

Step 3: Since each payoff is a PULC now, suppose the expected value of Player *I* be $v^- = (v_{EX}^-, v_{EN}^-, v_{HE}^-)$ such that the PULCLP becomes

$$\begin{aligned} \max \quad & v^- \implies (\max v_{EX}^-, \min v_{EN}^-, \min v_{HE}^-) \\ \text{s.t.} \quad & \begin{cases} \bigoplus_{i=1}^r \tilde{a}_{ij} \varrho_i \geq v^- \quad j = 1, 2, \dots, s \\ \text{where} \\ \varrho_i \geq 0, \sum_{i=1}^r \varrho_i = 1. \end{cases} \end{aligned} \quad (\text{P4})$$

Here each $\tilde{a}_{ij} = (p_t EX_{a^t}, p_t EN_{a^t}, p_t HE_{a^t})$.

Step 4: From here, the PM-ULCG splits into three individual games, called the expectation game, entropy game, and hyperentropy game given as: Expectation Game

$$\begin{aligned} \max \quad & v_{EX}^- \\ \text{s.t.} \quad & \begin{cases} \sum_{i=1}^r (p_t EX_{a^t})_{ij} \varrho_i \geq v_{EX}^- \quad j = 1, 2, \dots, s \\ \text{where} \\ \varrho_i \geq 0, \sum_{i=1}^r \varrho_i = 1. \end{cases} \end{aligned} \quad (\text{EX:1})$$

Entropy Game

$$\begin{aligned} \min \quad & v_{EN}^- \\ \text{s.t.} \quad & \begin{cases} \sqrt{\sum_{i=1}^r [(p_t EN_{a^t})_{ij}]^2} \varrho_i \leq v_{EN}^- \quad j = 1, 2, \dots, s \\ \text{where} \\ \varrho_i \geq 0, \sum_{i=1}^r \varrho_i = 1. \end{cases} \end{aligned} \quad (\text{EN:1})$$

Hyperentropy Game

$$\begin{aligned} \min \quad & v_{HE}^- \\ \text{s.t.} \quad & \begin{cases} \sqrt{\sum_{i=1}^r [(p_t HE_{a^t})_{ij}]^2} \varrho_i \leq v_{HE}^- \quad j = 1, 2, \dots, s \\ \text{where} \\ \varrho_i \geq 0, \sum_{i=1}^r \varrho_i = 1. \end{cases} \end{aligned} \quad (\text{HE:1})$$

Step 5: Proceeding further, Expectation Game Setting $X_i = \frac{\varrho_i}{v_{EX}^-}$ and letting $V_{EX} = \frac{1}{v_{EX}^-}$, the transformed linear programming problem becomes

$$\begin{aligned} \min \quad & V_{EX} = \sum_{i=1}^r X_i \\ \text{s.t.} \quad & \begin{cases} \sum_{i=1}^r (p_t EX_{a^t})_{ij} X_i \geq 1 \quad j = 1, 2, \dots, s \\ \text{where } X_i \geq 0 \end{cases} \end{aligned} \quad (\text{EX Game PI})$$

Entropy Game Squaring both sides of the constraints and letting $(v_{EN}^-)^2 = v_{EN}'$. Further setting $Y_i = \frac{\varrho_i}{v_{EN}'}$, and letting $V_{EN}' = \frac{1}{v_{EN}'} v_{EN}^- = \sqrt{v_{EN}'} = \frac{1}{\sqrt{V_{EN}'}}$. $\therefore \min v_{EN}^- \equiv \max \sqrt{V_{EN}'} V_{EN}'$ being a positive quantity, we maximize V_{EN}' instead of $\sqrt{V_{EN}'}$ and square the value later on. Thus, the final entropy linear programming problem becomes

$$\begin{aligned} \max \quad & V_{EN}' = \sum_{i=1}^r Y_i \\ \text{s.t.} \quad & \begin{cases} \sum_{i=1}^r [(p_t EN_{a^t})_{ij}]^2 Y_i \leq 1, \quad j = 1, 2, \dots, s \\ \text{where } Y_i \geq 0 \end{cases} \end{aligned} \quad (\text{EN Game PI})$$

Hyperentropy Game Similar to entropy game, squaring both sides and letting $(v_{HE})^2 = v_{HE}'$. Further setting $Z_i = \frac{\varrho_i}{v_{HE}'}$, and suppose $V_{HE}' = \frac{1}{v_{HE}'} v_{HE}^- = \sqrt{v_{HE}'} = \frac{1}{\sqrt{V_{HE}'}} \therefore \min v_{HE}^- \equiv \max \sqrt{V_{HE}'} V_{HE}'$ being a positive quantity, we maximize V_{HE}' instead of $\sqrt{V_{HE}'}$ and square the value later on. Thus, the final hyperentropy linear programming problem becomes

$$\begin{aligned} \max \quad & V_{HE}' \\ \text{s.t.} \quad & \begin{cases} \sum_{i=1}^r [(p_t HE_{a^t})_{ij}]^2 Z_i \leq 1, \quad j = 1, 2, \dots, s \\ \text{where } Z_i \geq 0 \end{cases} \end{aligned} \quad (\text{HE Game PI})$$

Step 6: After the formulation of final LPPs each for expectation, entropy and hyperentropy, we solve them to get the PULC strategies and PULC value of each game. Combining all three, we get Lower value of the game: $v^- = (v_{EX}^-, v_{EN}^-, v_{HE}^-)$ and mixed strategies ϱ_i for each game such that $\sum_{i=1}^r \varrho_{i_{EX}} = 1, \sqrt{\sum_{i=1}^r \varrho_{i_{EN}}^2} = 1, \sqrt{\sum_{i=1}^r \varrho_{i_{HE}}^2} = 1$

Step 7: Substituting back to the original variables, we get the final results. Also, the final strategy to be played by the player may be decided by using score of each strategy. The higher the score, the better (optimum) is the strategy.

Figure 2 demonstrates the process of evaluation of a PM-ULCG problem for the row player.

The exact same procedure is to be followed for Player II, where the payoffs are $neg(\tilde{a}_{ij})$. The expected value of Player II would be $v^+ = (v_{EX}^+, v_{EN}^+, v_{HE}^+)$ such that as per P3, the PULCLP becomes

$$\begin{aligned} \min \quad & v^+ \implies (\min v_{EX}^+, \max v_{EN}^+, \max v_{HE}^+) \\ \text{s.t.} \quad & \begin{cases} \bigoplus_{j=1}^s \tilde{a}_{ij} \varphi_j \leq v^+ \quad i = 1, 2, \dots, r \\ \text{where} \\ \varrho_j \geq 0, \sum_{j=1}^s \varphi_j = 1. \end{cases} \end{aligned} \quad (\text{P5})$$

and thus, the PULCLP splits into three problems given as: Expectation Game

$$\begin{aligned} \min \quad & v_{EX}^+ \\ \text{s.t.} \quad & \begin{cases} \sum_{j=1}^s (p_t EX_{a^t})_{ij} \varrho_j \leq v_{EX}^+ \quad i = 1, 2, \dots, r \\ \text{where} \\ \varrho_j \geq 0, \sum_{j=1}^s \varphi_j = 1. \end{cases} \end{aligned} \quad (\text{EX Game PII})$$

Entropy Game

$$\begin{aligned} \max \quad & v_{EN}^+ \\ \text{s.t.} \quad & \begin{cases} \sqrt{\sum_{j=1}^s [(p_t EN_{a^t})_{ij}]^2} \varrho_j \geq v_{EN}^+ \quad i = 1, 2, \dots, r \\ \text{where} \\ \varrho_j \geq 0, \sum_{j=1}^s \varphi_j = 1. \end{cases} \end{aligned} \quad (\text{EN Game PII})$$

Hyperentropy Game

$$\begin{aligned} \max \quad & v_{HE}^+ \\ \text{s.t.} \quad & \begin{cases} \sqrt{\sum_{j=1}^s [(p_t HE_{a^t})_{ij}]^2} \varrho_j \geq v_{HE}^+ \quad i = 1, 2, \dots, r \\ \text{where} \\ \varrho_j \geq 0, \sum_{j=1}^s \varphi_j = 1. \end{cases} \end{aligned} \quad (\text{HE Game PII})$$

Then, solving just as in Step 5 mentioned above, we combine the strategies and values obtained from all three games so that we get upper value of the game, i.e. $v^+ = (v_{EX}^+, v_{EN}^+, v_{HE}^+)$.

We illustrate the defined methodology using an example.

4.3 Numerical illustration

Example 3 An investor wishes to invest his money in the stocks of some IT companies. He does so by analysing the average stock closing values for the past 6 years 2017–2022. The possible alternatives under consideration are the five IT companies: Infosys, Wipro, TCS, HCL Technologies, and Larsen & Toubro Infotech. The following linguistic variables represent the stock values:

$$LS = \{t_{2-2} : \text{Very Low (VL)}, t_{2-1} : \text{Low (L)}, t_{20} : \text{Average (Avg)}, t_{21} : \text{High (H)}, t_{22} : \text{Very High (VH)}\}$$

Table 2 provides the values of stocks of the IT companies for the financial years 2017–2022. These data are taken from <https://www.moneycontrol.com/moneycontrol>.

Based on the values listed in Table 2, the variations that occurred in the stock values of different IT companies are evaluated in terms of PM-ULTS (just an interpretation). The interpretation here is done by supposing that the values less than 300 assume the linguistic term “VL”, 300–900 assume “L”, 900–1500 as “Avg”, 1500–2100 as “H” and greater than 2100 assume the linguistic term “VH”. Also, the probabilistic values are interpreted based on the deviation of numeric values from the original linguistic terms. Thus, the probabilistic unbalanced linguistic stock data for the years 2017–2022 for the above-mentioned IT companies are provided in the following “IT company versus year” matrix:

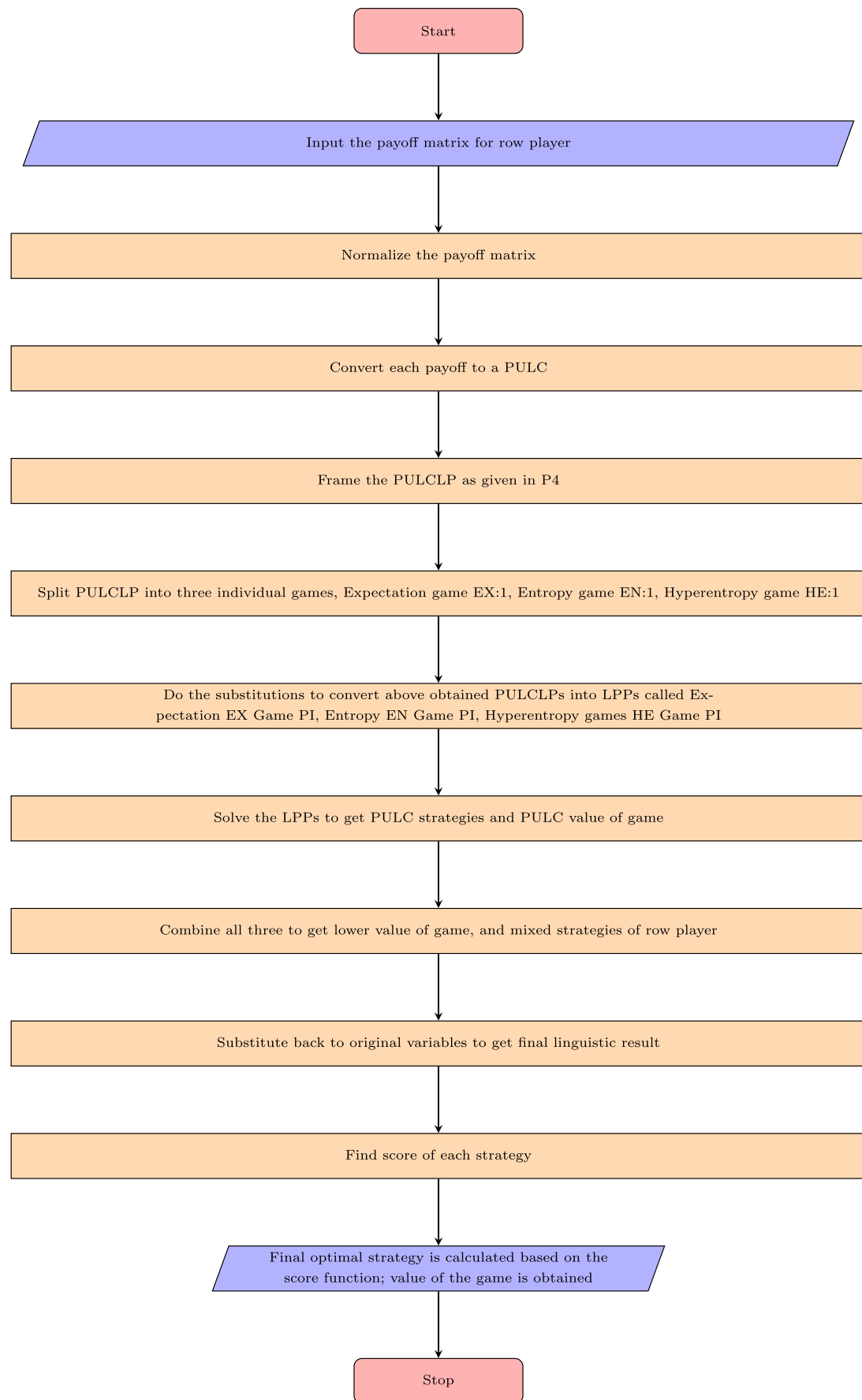


Fig. 2 Flowchart for evaluation of PM-ULCG problem for row player

$$\tilde{A} = \begin{array}{c} \text{Infosys} \\ \text{Wipro} \\ \text{TCS} \\ \text{HCL Technologies} \\ \text{Larsen \& Toubro Infotech} \end{array} \begin{bmatrix} 2017 & 2018 & 2019 & 2020 & 2021 & 2022 \\ t_{2-1}(0.4) & t_{2-1}(0.6) & t_{2-1}(0.7) & t_{2^0}(0.6) & t_{2^1}(0.7) & t_{2^1}(0.1) \\ t_{2-2}(0.2) & t_{2-2}(0.2) & t_{2-2}(0.2) & t_{2-1}(0.2) & t_{2-1}(0.7) & t_{2-1}(0.2) \\ t_{2^0}(0.7) & t_{2^1}(0.7) & t_{2^2}(0.1) & t_{2^2}(0.2) & t_{2^2}(0.4) & t_{2^2}(0.4) \\ t_{2-1}(0.3) & t_{2-1}(0.4) & t_{2-1}(0.5) & t_{2^0}(0.1) & t_{2^0}(0.7) & t_{2^0}(0.4) \\ t_{2^0}(0.4) & t_{2^1}(0.4) & t_{2^1}(0.5) & t_{2^2}(0.4) & t_{2^2}(1) & t_{2^1}(0.8) \end{bmatrix}$$

The payoff $\tilde{a}_{ij}$ of Player I represents the response to strategy i of Player I for the case when Player II selects strategy j in order to win the game. Also, a probabilistic linguistic payoff signifies that any strategy played by a player can be evaluated on an unbalanced linguistic scale.

Next, we apply our methodology to deal with PM-UL information to analyse which company has had a higher growth in share market from financial year 2017 to 2022. On the basis of this analysis, one can interpret which company to invest in, for future purpose. Now first we normalize the payoff matrix. So, the normalized payoff matrix may be given as:

$$\tilde{A} = \begin{array}{c} \text{Infosys} \\ \text{Wipro} \\ \text{TCS} \\ \text{HCL Technologies} \\ \text{Larsen \& Toubro Infotech} \end{array} \begin{bmatrix} 2017 & 2018 & 2019 & 2020 & 2021 & 2022 \\ t_{2-1}(0.13) & t_{2-1}(0.194) & t_{2-1}(0.225) & t_{2^0}(0.194) & t_{2^1}(0.225) & t_{2^1}(0.032) \\ t_{2-2}(0.118) & t_{2-2}(0.118) & t_{2-2}(0.118) & t_{2-1}(0.118) & t_{2-1}(0.41) & t_{2-1}(0.118) \\ t_{2^0}(0.28) & t_{2^1}(0.28) & t_{2^2}(0.04) & t_{2^2}(0.08) & t_{2^2}(0.16) & t_{2^2}(0.16) \\ t_{2-1}(0.125) & t_{2-1}(0.167) & t_{2-1}(0.208) & t_{2^0}(0.041) & t_{2^0}(0.292) & t_{2^0}(0.167) \\ t_{2^0}(0.114) & t_{2^1}(0.114) & t_{2^1}(0.144) & t_{2^2}(0.114) & t_{2^2}(0.286) & t_{2^1}(0.228) \end{bmatrix}$$

The ambiguity and volatility of stock values are represented by the dispersion of cloud droplets. So we convert the probabilistic unbalanced linguistic terms to PULCs so as to choose a specific alternative. Hence, we use the procedure given in Sect. 3.1 to convert the payoffs to PULCs, and thus, the PULC payoff matrix is given as:

$$\tilde{A}_{PULC} = \begin{array}{c} \text{Infosys} \\ \text{Wipro} \\ \text{TCS} \\ \text{HCL Technologies} \\ \text{Larsen \& Toubro Infotech} \end{array} \begin{bmatrix} 2017 & 2018 & 2019 & 2020 & 2021 & 2022 \\ (0.6298, 1.76033, 0.6987) & (0.9399, 2.6269, 1.0427) & (1.0901, 3.0467, 1.2093) & (1.6112, 2.2799, 1.42706) & (2.6473, 3.0467, 1.2093) & (0.3765, 0.4333, 0.172) \\ (0, 1.854, 0.3506) & (0, 1.854, 0.3506) & (0, 1.854, 0.3506) & (0.5717, 1.5978, 0.6342) & (1.9864, 5.5518, 2.203) & (0.5717, 1.5978, 0.6342) \\ (2.3255, 3.2907, 2.0596) & (3.2944, 3.7914, 1.505) & (0.6284, 0.6284, 0.1188) & (1.2569, 1.2569, 0.2376) & (2.5139, 2.5139, 0.4753) & (2.5139, 2.5139, 0.4753) \\ (0.6056, 1.6926, 0.6718) & (0.8091, 2.2613, 0.8976) & (1.0077, 2.8165, 1.118) & (0.3405, 0.4818, 0.3015) & (2.4252, 3.4317, 2.1479) & (1.387, 1.9626, 1.2284) \\ (0.9468, 1.3397, 0.8385) & (1.3413, 1.5436, 0.6127) & (1.6943, 1.9499, 0.774) & (1.7911, 1.7911, 0.3387) & (4.4936, 4.4936, 0.8497) & (2.6826, 3.0873, 1.2255) \end{bmatrix}$$

Solving this game problem for Player I , i.e. for “IT companies”, we get three separate games as expectation game, entropy game and hyperentropy game given as:
Expectation Game

Table 2 Closing stock prices of IT companies for different financial years

	2017	2018	2019	2020	2021	2022
Infosys	521.03	658.95	731.15	1255.8	1887.75	1508.2
Wipro	235.69	248.14	245.8	386.25	715.35	392.75
TCS	1350.6	1893.05	2161.7	2862.75	3738.35	3256.7
HCL Technologies	445.25	482.18	568.1	946.15	1319.1	1039.3
Larsen & Toubro Infotech	1118.85	1731.2	1750.5	3659.5	7332	2085.8

$$\begin{aligned} \max \quad & v_{EX}^- \\ \text{s.t.} \quad & \begin{cases} 0.6298\varrho_1 + 0\varrho_2 + 2.3255\varrho_3 + 0.6056\varrho_4 + 0.9468\varrho_5 \geq v_{EX}^- \\ 0.9399\varrho_1 + 0\varrho_2 + 3.29448\varrho_3 + 0.8091\varrho_4 + 1.3413\varrho_5 \geq v_{EX}^- \\ 1.0901\varrho_1 + 0\varrho_2 + 0.6284\varrho_3 + 1.0077\varrho_4 + 1.6943\varrho_5 \geq v_{EX}^- \\ 1.6112\varrho_1 + 0.5717\varrho_2 + 1.2569\varrho_3 + 0.3405\varrho_4 + 1.7911\varrho_5 \geq v_{EX}^- \\ 2.6473\varrho_1 + 1.9864\varrho_2 + 2.5139\varrho_3 + 2.4252\varrho_4 + 4.4936\varrho_5 \geq v_{EX}^- \\ 0.3765\varrho_1 + 0.5717\varrho_2 + 2.5139\varrho_3 + 1.387\varrho_4 + 2.6826\varrho_5 \geq v_{EX}^- \\ \text{where } \varrho_i \geq 0, \sum_{i=1}^5 \varrho_i = 1 \end{cases} \end{aligned}$$

Setting $X_i = \frac{\varrho_i}{v_{EX}^-}$, the problem becomes

$$\begin{aligned} \min \quad & V_{EX} = X_1 + X_2 + X_3 + X_4 + X_5 \\ \text{s.t.} \quad & \begin{cases} 0.6298X_1 + 0X_2 + 2.3255X_3 + 0.6056X_4 + 0.9468X_5 \geq 1 \\ 0.9399X_1 + 0X_2 + 3.29448X_3 + 0.8091X_4 + 1.3413X_5 \geq 1 \\ 1.0901X_1 + 0X_2 + 0.6284X_3 + 1.0077X_4 + 1.6943X_5 \geq 1 \\ 1.6112X_1 + 0.5717X_2 + 1.2569X_3 + 0.3405X_4 + 1.7911X_5 \geq 1 \\ 2.6473X_1 + 1.9864X_2 + 2.5139X_3 + 2.4252X_4 + 4.4936X_5 \geq 1 \\ 0.3765X_1 + 0.5717X_2 + 2.5139X_3 + 1.387X_4 + 2.6826X_5 \geq 1 \\ \text{where } X_i \geq 0, i = 1, \dots, 5 \end{cases} \end{aligned}$$

Solving the above LPP, we get the solution as: $X_1 = 0$, $X_2 = 0$, $X_3 = 0.223$, $X_4 = 0$, $X_5 = 0.508$, $V_{EX} = 0.731$. Substituting back to the original variables, we get the optimal solution of the expectation game problem as: $v_{EX}^- = \frac{1}{V_{EX}} = 1.36$, $\varrho_1 = 0$, $\varrho_2 = 0$, $\varrho_3 = 0.305$, $\varrho_4 = 0$, $\varrho_5 = 0.695$. Hence, the strategies are $(0, 0, 0.305, 0, 0.695)$ and expectation value is 1.36. **Entropy Game**

$$\begin{aligned} \min \quad & v_{EN}^- \\ \text{s.t.} \quad & \begin{cases} \sqrt{(1.7603)^2\varrho_1 + (1.854)^2\varrho_2 + (3.2907)^2\varrho_3 + (1.6926)^2\varrho_4 + (1.3397)^2\varrho_5} \leq v_{EN}^- \\ \sqrt{(2.6269)^2\varrho_1 + (1.854)^2\varrho_2 + (3.7914)^2\varrho_3 + (2.2613)^2\varrho_4 + (1.5436)^2\varrho_5} \leq v_{EN}^- \\ \sqrt{(3.0467)^2\varrho_1 + (1.854)^2\varrho_2 + (0.6285)^2\varrho_3 + (2.8165)^2\varrho_4 + (1.95)^2\varrho_5} \leq v_{EN}^- \\ \sqrt{(2.2799)^2\varrho_1 + (1.5978)^2\varrho_2 + (1.2569)^2\varrho_3 + (0.482)^2\varrho_4 + (1.791)^2\varrho_5} \leq v_{EN}^- \\ \sqrt{(3.0467)^2\varrho_1 + (5.5518)^2\varrho_2 + (2.5139)^2\varrho_3 + (3.4317)^2\varrho_4 + (4.4936)^2\varrho_5} \leq v_{EN}^- \\ \sqrt{(0.4333)^2\varrho_1 + (1.5978)^2\varrho_2 + (2.5139)^2\varrho_3 + (1.9626)^2\varrho_4 + (3.0873)^2\varrho_5} \leq v_{EN}^- \\ \text{where } \varrho_i \geq 0, i = 1, \dots, 5, \sum_{i=1}^5 \varrho_i = 1 \end{cases} \end{aligned}$$

Squaring both sides, and letting $(v_{EN}^-)^2 = v_{EN}'$, we get

$$\begin{aligned} \min \quad & \sqrt{v_{EN}'} \equiv \min v_{EN}' \\ \text{s.t.} \quad & \begin{cases} 3.0987\varrho_1 + 3.4374\varrho_2 + 10.8287\varrho_3 + 2.8649\varrho_4 + 1.7948\varrho_5 \leq v_{EN}' \\ 6.9\varrho_1 + 3.437\varrho_2 + 14.375\varrho_3 + 5.113\varrho_4 + 2.382\varrho_5 \leq v_{EN}' \\ 9.282\varrho_1 + 3.437\varrho_2 + 0.395\varrho_3 + 7.933\varrho_4 + 3.802\varrho_5 \leq v_{EN}' \\ 5.198\varrho_1 + 2.553\varrho_2 + 1.579\varrho_3 + 0.232\varrho_4 + 3.208\varrho_5 \leq v_{EN}' \\ 9.282\varrho_1 + 30.822\varrho_2 + 6.319\varrho_3 + 11.653\varrho_4 + 20.192\varrho_5 \leq v_{EN}' \\ 0.187\varrho_1 + 2.553\varrho_2 + 6.319\varrho_3 + 3.851\varrho_4 + 9.531\varrho_5 \leq v_{EN}' \\ \text{where } \varrho_i \geq 0, i = 1, \dots, 5 \sum_{i=1}^5 \varrho_i = 1 \end{cases} \end{aligned}$$

Setting $Y_i = \frac{\varrho_i}{v_{EN}'}$ and $V_{EN}' = \frac{1}{v_{EN}'}$, hence the final entropy LPP becomes

$$\begin{aligned} \max \quad & V_{EN}' = Y_1 + Y_2 + Y_3 + Y_4 + Y_5 \\ \text{s.t.} \quad & \begin{cases} 3.0987Y_1 + 3.4374Y_2 + 10.8287Y_3 + 2.8649Y_4 + 1.7948Y_5 \leq 1 \\ 6.9Y_1 + 3.437Y_2 + 14.375Y_3 + 5.113Y_4 + 2.382Y_5 \leq 1 \\ 9.282Y_1 + 3.437Y_2 + 0.395Y_3 + 7.933Y_4 + 3.802Y_5 \leq 1 \\ 5.198Y_1 + 2.553Y_2 + 1.579Y_3 + 0.232Y_4 + 3.208Y_5 \leq 1 \\ 9.282Y_1 + 30.822Y_2 + 6.319Y_3 + 11.653Y_4 + 20.192Y_5 \leq 1 \\ 0.187Y_1 + 2.553Y_2 + 6.319Y_3 + 3.851Y_4 + 9.531Y_5 \leq 1 \\ \text{where } Y_i \geq 0, i = 1, \dots, 5 \end{cases} \end{aligned}$$

Solving the above LPP, the solution comes out to be $Y_1 = 0.08968$, $Y_2 = 0$, $Y_3 = 0.0265$, $Y_4 = 0$, $Y_5 = 0$. Further, $V_{EN}' = 0.1162$. $\therefore v_{EN}^- = 2.93$ and the strategies are $(0.7717, 0, 0.2282, 0, 0)$. On a similar note, **Hyperentropy Game**

$$\begin{aligned} \min \quad & v_{HE}^- \\ \text{s.t.} \quad & \begin{cases} \sqrt{(0.6987)^2\varrho_1 + (0.3506)^2\varrho_2 + (2.0596)^2\varrho_3 + (0.6718)^2\varrho_4 + (0.8385)^2\varrho_5} \leq v_{HE}^- \\ \sqrt{(1.0427)^2\varrho_1 + (0.3506)^2\varrho_2 + (1.505)^2\varrho_3 + (0.8976)^2\varrho_4 + (0.6127)^2\varrho_5} \leq v_{HE}^- \\ \sqrt{(1.2093)^2\varrho_1 + (0.3506)^2\varrho_2 + (0.1188)^2\varrho_3 + (1.118)^2\varrho_4 + (0.774)^2\varrho_5} \leq v_{HE}^- \\ \sqrt{(1.427)^2\varrho_1 + (0.6342)^2\varrho_2 + (0.2376)^2\varrho_3 + (0.3015)^2\varrho_4 + (0.3387)^2\varrho_5} \leq v_{HE}^- \\ \sqrt{(1.2093)^2\varrho_1 + (2.203)^2\varrho_2 + (0.4754)^2\varrho_3 + (2.1479)^2\varrho_4 + (0.8479)^2\varrho_5} \leq v_{HE}^- \\ \sqrt{(0.172)^2\varrho_1 + (0.6342)^2\varrho_2 + (0.4754)^2\varrho_3 + (1.2284)^2\varrho_4 + (1.2255)^2\varrho_5} \leq v_{HE}^- \\ \text{where } \varrho_i \geq 0, i = 1, \dots, 5 \sum_{i=1}^5 \varrho_i = 1 \end{cases} \end{aligned}$$

Squaring both sides and letting $(v_{HE}^-)^2 = v_{HE}'$

$$\begin{aligned} \min \quad & \sqrt{v_{HE}'} \equiv \min v_{HE}' \\ \text{s.t.} \quad & \begin{cases} 0.4882\rho_1 + 0.1229\rho_2 + 4.2423\rho_3 + 0.4513\rho_4 + 0.7031\rho_5 \leq v_{HE}' \\ 1.0872\rho_1 + 0.1229\rho_2 + 2.265\rho_3 + 0.8056\rho_4 + 0.3755\rho_5 \leq v_{HE}' \\ 1.4624\rho_1 + 0.1229\rho_2 + 0.0141\rho_3 + 1.2499\rho_4 + 0.599\rho_5 \leq v_{HE}' \\ 2.036\rho_1 + 0.4022\rho_2 + 0.0564\rho_3 + 0.0909\rho_4 + 0.1147\rho_5 \leq v_{HE}' \\ 1.4624\rho_1 + 4.8532\rho_2 + 0.226\rho_3 + 4.6134\rho_4 + 0.7219\rho_5 \leq v_{HE}' \\ 0.029\rho_1 + 0.4022\rho_2 + 0.226\rho_3 + 1.5089\rho_4 + 1.5018\rho_5 \leq v_{HE}' \\ \text{where } \rho_i \geq 0, i = 1, \dots, 5 \sum_{i=1}^5 \rho_i = 1 \end{cases} \end{aligned}$$

Setting $Z_i = \frac{\rho_i}{v_{HE}'}$ and $V_{HE}' = \frac{1}{v_{HE}'}$, the final hyperentropy LPP becomes

$$\begin{aligned} \max \quad & V_{HE}' \\ \text{s.t.} \quad & \begin{cases} 0.4882Z_1 + 0.1229Z_2 + 4.2423Z_3 + 0.4513Z_4 + 0.7031Z_5 \leq 1 \\ 1.0872Z_1 + 0.1229Z_2 + 2.265Z_3 + 0.8056Z_4 + 0.3755Z_5 \leq 1 \\ 1.4624Z_1 + 0.1229Z_2 + 0.0141Z_3 + 1.2499Z_4 + 0.599Z_5 \leq 1 \\ 2.036Z_1 + 0.4022Z_2 + 0.0564Z_3 + 0.0909Z_4 + 0.1147Z_5 \leq 1 \\ 1.4624Z_1 + 4.8532Z_2 + 0.226Z_3 + 4.6134Z_4 + 0.7219Z_5 \leq 1 \\ 0.029Z_1 + 0.4022Z_2 + 0.226Z_3 + 1.5089Z_4 + 1.5018Z_5 \leq 1 \\ \text{where } Z_i \geq 0, i = 1, \dots, 5 \end{cases} \end{aligned}$$

Solving the above LPP, solution come out as $Z_1 = 0.3514, Z_2 = 0, Z_3 = 0.0883, Z_4 = 0, Z_5 = 0.6458, V_{HE}' = 1.0854$. Substituting back to original variables, the hyperentropy value $v_{HE}^- = 0.96$, and the strategies are $(0.3237, 0, 0.08135, 0, 0.5949)$. Summarizing and writing collectively, the strategies in terms of PULC are: $\rho_1^* = (0, 0.7717, 0.3237)$, $\rho_2^* = 0, \rho_3^* = (0.305, 0.2283, 0.0813)$, $\rho_4^* = 0, \rho_5^* = (0.695, 0, 0.5949)$ and the PULC value of the game is $v^- = (1.36, 2.93, 0.96)$. Further, the probabilistic unbalanced linguistic value of the game can be calculated and is equal to $\tilde{v}^-(p) = \{t_{20}(0.4612), t_{21}(0.5388)\}$.

Similarly, solving the game problem for Player II, i.e. for “Year”, the expectation, entropy and hyperentropy games are as given below: Expectation Game

$$\begin{aligned} \min \quad & v_{EX}^+ \\ \text{s.t.} \quad & \begin{cases} 0.6298\varphi_1 + 0.9399\varphi_2 + 1.0901\varphi_3 + 1.6112\varphi_4 + 2.6473\varphi_5 + 0.3765\varphi_6 \leq v_{EX}^+ \\ 0\varphi_1 + 0\varphi_2 + 0\varphi_3 + 0.5717\varphi_4 + 1.9864\varphi_5 + 0.5717\varphi_6 \leq v_{EX}^+ \\ 2.3255\varphi_1 + 3.2944\varphi_2 + 0.6284\varphi_3 + 1.2569\varphi_4 + 2.5139\varphi_5 + 2.5139\varphi_6 \leq v_{EX}^+ \\ 0.6056\varphi_1 + 0.8091\varphi_2 + 1.0077\varphi_3 + 0.3405\varphi_4 + 2.4252\varphi_5 + 1.387\varphi_6 \leq v_{EX}^+ \\ 0.9468\varphi_1 + 1.3413\varphi_2 + 1.6943\varphi_3 + 1.7911\varphi_4 + 4.4936\varphi_5 + 2.6826\varphi_6 \leq v_{EX}^+ \\ \text{where } \varphi_j \geq 0, \sum_{j=1}^6 \varphi_j = 1 \end{cases} \end{aligned}$$

Solving by doing required substitutions, we get the optimal solution as: $v_{EX}^+ = 1.36$, and the strategies are $(0.4359, 0, 0.564, 0, 0, 0)$. Entropy Game

$$\begin{aligned}
& \max \quad v_{EN}^+ \\
& \text{s.t.} \quad \left\{ \begin{array}{l}
\sqrt{(1.7603)^2\varphi_1 + (2.6269)^2\varphi_2 + (3.0467)^2\varphi_3 + (2.2799)^2\varphi_4 + (3.0467)^2\varphi_5 + (0.4333)^2\varphi_6} \geq v_{EN}^+ \\
\sqrt{(1.854)^2\varphi_1 + (1.854)^2\varphi_2 + (1.854)^2\varphi_3 + (1.5978)^2\varphi_4 + (5.5518)^2\varphi_5 + (1.5978)^2\varphi_6} \geq v_{EN}^+ \\
\sqrt{(3.2907)^2\varphi_1 + (3.7914)^2\varphi_2 + (0.6284)^2\varphi_3 + (1.2569)^2\varphi_4 + (2.5139)^2\varphi_5 + (2.5139)^2\varphi_6} \geq v_{EN}^+ \\
\sqrt{(1.6926)^2\varphi_1 + (2.2613)^2\varphi_2 + (2.8165)^2\varphi_3 + (0.482)^2\varphi_4 + (3.4317)^2\varphi_5 + (1.9626)^2\varphi_6} \geq v_{EN}^+ \\
\sqrt{(1.3397)^2\varphi_1 + (1.5436)^2\varphi_2 + (1.9499)^2\varphi_3 + (1.7911)^2\varphi_4 + (4.4936)^2\varphi_5 + (3.0873)^2\varphi_6} \geq v_{EN}^+ \\
\text{where } \varphi_j \geq 0, j = 1, \dots, 6 \sum_{j=1}^6 \varphi_j = 1
\end{array} \right.
\end{aligned}$$

Solving just as for Player *I*, we obtain result for entropy game as (0, 0.2839, 0, 0, 0.716, 0) and value $v_{EN}^+ = 2.93$. Similarly for hyperentropy, Hyper-entropy Game

$$\begin{aligned}
& \max \quad v_{HE}^+ \\
& \text{s.t.} \quad \left\{ \begin{array}{l}
\sqrt{(0.6987)^2\varphi_1 + (1.0427)^2\varphi_2 + (1.2093)^2\varphi_3 + (1.427)^2\varphi_4 + (1.2093)^2\varphi_5 + (0.172)^2\varphi_6} \geq v_{HE}^+ \\
\sqrt{(0.3506)^2\varphi_1 + (0.3506)^2\varphi_2 + (0.3506)^2\varphi_3 + (0.6342)^2\varphi_4 + (2.203)^2\varphi_5 + (0.6342)^2\varphi_6} \geq v_{HE}^+ \\
\sqrt{(2.0596)^2\varphi_1 + (1.505)^2\varphi_2 + (0.1188)^2\varphi_3 + (0.3015)^2\varphi_4 + (2.1479)^2\varphi_5 + (1.2284)^2\varphi_6} \geq v_{HE}^+ \\
\sqrt{(0.6718)^2\varphi_1 + (0.8976)^2\varphi_2 + (1.118)^2\varphi_3 + (0.3015)^2\varphi_4 + (2.1479)^2\varphi_5 + (1.2284)^2\varphi_6} \geq v_{HE}^+ \\
\sqrt{(0.8385)^2\varphi_1 + (0.6127)^2\varphi_2 + (0.774)^2\varphi_3 + (0.3387)^2\varphi_4 + (0.8497)^2\varphi_5 + (1.2255)^2\varphi_6} \geq v_{HE}^+ \\
\text{where } \varphi_j \geq 0, j = 1, \dots, 6 \sum_{j=1}^6 \varphi_j = 1
\end{array} \right.
\end{aligned}$$

The solution comes out as: (0.173, 0, 0, 0, 0.5669, 0.2598) and $v_{HE}^+ = 0.96$. Hence, collectively the optimal PULC strategies and PULC value of Player *II* are: $\varphi_1^* = (0.4359, 0, 0.173)$, $\varphi_2^* = (0, 0.2839, 0)$, $\varphi_3^* = (0.564, 0, 0)$, $\varphi_4^* = 0$, $\varphi_5^* = (0, 0.716, 0.5669)$, $\varphi_6^* = (0, 0, 0.2598)$ Value $v^+ = (1.36, 2.93, 0.96)$. Hence, $v^- = v^+ = v =$ Value of the game. Further, the scores of strategies may be evaluated to finalize which strategy should a player use to win the game. For that, we may find different score function values for different values of γ . Without loss of generality, we fix the value of $\gamma = 0.5$ for current calculations.

For Player *I*, $\hat{S}(\rho_1^*) = -0.4184$, $\hat{S}(\rho_2^*) = 0$, $\hat{S}(\rho_3^*) = 0.3825$, $\hat{S}(\rho_4^*) = 0$, $\hat{S}(\rho_5^*) = 0.3975$. Thus, based on the score values, the IT company Larsen & Toubro would be the best to invest the money in. Thus, based on the above-obtained results, the DM can also plan where to invest in the future. Also, the trend of increment of stock prices may be determined by finding the score values of Player *II*, i.e. "Year".

Note. The above problem is solved for the case when the DMs record only a single linguistic term as a payoff entry. The same procedure may be followed for the case when multiple linguistic terms consist a payoff or the case when the payoffs are of a set type. In that case, we could break the individual term of the set and then form separate PULCLP for each broken term set.

5 Analysis and discussion

This section includes the analysis of numerical results obtained in previous section along with the comparison of proposed approach with the formerly defined literature.

5.1 Numerical discussion

For the numerical example 3, the matrix $\tilde{A}$ corresponds to the first player, i.e. “IT Company” and based on the calculations, the DM is supposed to analyse which IT company is the best to invest in. For this purpose, the original problem is divided into three individual games, namely expectation, entropy and hyperentropy game. Solving the three games, it can be concluded that the optimal strategies are $\rho_1^* = (0, 0.7717, 0.3237)$, $\rho_2^* = 0$, $\rho_3^* = (0.305, 0.2283, 0.0813)$, $\rho_4^* = 0$, $\rho_5^* = (0.695, 0, 0.5949)$ and the PULC value of the game is $v^- = (1.36, 2.93, 0.96)$. The score of the strategies may be evaluated as $\hat{S}(\rho_1^*) = -0.4184$, $\hat{S}(\rho_2^*) = 0$, $\hat{S}(\rho_3^*) = 0.3825$, $\hat{S}(\rho_4^*) = 0.3975$. Thus, based on the score values, the IT company Larsen & Toubro would be the best to invest the money in.

Similar task can be done for the second player, i.e. “Year” that can be used to investigate which year was the best year for investment in a particular IT company. For that purpose, similar expectation, entropy and hyperentropy game problems are constituted and solved.

5.2 Comparative analysis and discussion

This section provides a comparison analysis of the proposed methodology with the existing studies that involve game theoretic models using PULCs.

The idea of probabilistic linguistic term set (PLTS) was first proposed by Pang et al. [31], which allows the players or DMs to evaluate situations more effectively by using multiple terms rather than just one, by giving each term a probability-based weight. Apart from this, there are other variants of PLTS, such as [3, 46, 51]. In the paper [3], the authors have defined a concept “Interval-valued probabilistic linguistic term sets” that can describe two attributes of an object: possible linguistic term sets and their associated interval numbers. The former provides an assessment, such as “perfect” or “low”. The latter describes the hesitancy for the given assessment and denotes the interval-valued probabilistic distribution or membership degrees with respect to the former. Xie et al. [46] extended the analytic hierarchy process to the situations with the probabilistic linguistic information. Zhai et al. [51] focus on improving the accuracy of the multi-granular linguistic recommender system by supporting customers to find out the most eligible items according their own preferences. For this purpose, they propose a probabilistic linguistic vector-term sets to promote the application of multi-granular linguistic information. However, despite having several practical applications, these PLTSs have the limitation that all of them use symmetric and balanced information for decision analysis. Further, this

symmetric information inculcated in these PLTSs is also used in matrix game theory in the paper [30]. The theory defined by Mi et al. [30] uses triangular membership function to represent probabilistic linguistic information. This information is defuzzified using the triangular membership function, and the calculated games values are then retranslated to probabilistic data by using linear interpolation method. This process is limited in its form as the defuzzification, and then, retranslation process causes major loss of information. Also, Mi et al. [30] study the cloud model in game problems where the responses of players belong to a symmetrically distributed linguistic set. To overcome the problem of symmetric probabilistic linguistic information, Han et al. [16] defined a probabilistic unbalanced linguistic term set (PULTS) that makes use of unbalanced linguistic data to represent probabilistic linguistic information. But this method uses the unbalanced linguistic model defined by [19] which is computationally very expensive method. Thus, in the paper [27], the authors solve linguistic matrix games that have probabilistic unbalanced linguistic data that comes from a probabilistic multiplicative unbalanced linguistic set defined by Malhotra & Gupta [28]. Table 3 summarizes the above-mentioned comparison analysis in tabular form.

The theory defined in our paper is useful for solving matrix games having the same probabilistic multiplicative linguistic information, but with a different point of view. The authors in [27] only considered the fuzziness involved in a linguistic concept. However, in our paper, the linguistic concept is analysed in the way that both randomness and fuzziness are omitted from the qualitative concept. The present paper uses cloud model [11] to thoroughly capture the intrinsic fuzziness and vagueness of decision information present in matrix game problems. Apart from that, the game problem considered in this paper consists of payoffs that are randomly irregularly distributed in a multiplicative manner. Such linguistic data address all types of linguistic information. This preserves the literal meaning of unbalanced linguistic matrix games. The suggested approach not only prevents data loss, it also studies different types of uncertainties lying in the linguistic information of matrix game problems. Also, the loss of information occurring in [30] due to the retranslation process is well handled in this paper as the probabilistic information is first converted into PULCs and the PULC data are then solved using cloud aggregation operators. Also, the retranslation process in this paper is done using the bijective operators that overcome the information loss. Furthermore, the methodology defined in this paper converts the probabilistic unbalanced payoff into a PULC in a way that it forms three independent easy-to-solve games, namely Expectation game, Entropy game and Hyperentropy game, which are basically three independent PULLPs. These PULLPs are converted into LPPs using required substitution and are then solved as a classical LPP. The objective function of these PULLPs runs into $\mathcal{O}(n)$ time, where n is the cardinality of the PULTS.

The novelty of this paper lies in the fact that it allows the DMs to study the underlying uncertainty in the information of an uncertain game problem and still solve a decision-making problem just as a classical problem without any loss of information. Unlike existing matrix game problems, this study captures the intrinsic randomness and fuzziness present in a qualitative concept of a matrix game which is based on a PM-ULTS.

Table 3 Comparison of the defined methodology with some existing research

Paper	Nature of information	Distribution of linguistic terms	Application in game	Nature of uncertainty of payoffs
[31]	Probabilistic	Symmetric	No	N/A
[3]	Probabilistic	Symmetric	No	N/A
[46]	Probabilistic	Symmetric and Asymmetric	No	N/A
[30]	Probabilistic	Symmetric	Yes	Fuzziness
[16]	Probabilistic	Asymmetric	No	N/A
[27]	Probabilistic	Asymmetric	Yes	Fuzziness
This paper	Probabilistic	Asymmetric	Yes	Randomness & Fuzziness

6 Conclusion

This study proposed and investigated the two-person matrix games having probabilistic multiplicative unbalanced linguistic information. This allows the theory of games to accept incomplete information as players' response as an input. The uncertain payoffs are dealt with by PULCs, and then, cloud aggregation operators are applied to aggregate the probabilistic cloud information. Then, the game problem is converted into three independent games, i.e. expectation game, entropy game and hyperentropy game. The LPPs of each game are individually solved to get cloud value of the game. The significance of transforming the unbalanced linguistic payoffs to PULCs is that the cloud model studies the uncertainty lying in a qualitative concept by thoroughly encapsulating the underlying randomness and fuzziness of the unbalanced payoff. The results of all three LPPs are collected to form a single strategy and single value, that further form a cloud value. The cloud value of the game provides the dispersion of cloud droplets using entropy value. Further, the hyperentropy studies the uncertainty of entropy. Thus, this methodology is advantageous as it provides more accurate results than other existing methodologies to solve matrix game problems having probabilistic linguistic information, in which only one kind of uncertainty is dealt. The more uncertainty is dealt, the more we proceed towards the preciseness of a certain result. Afterwards, this methodology is applied to solve a real-time decision problem of investment in stock of different IT companies.

As for future research directions, one can extend this methodology towards interval clouds as this methodology maps a qualitative information to a certain precise value. On the other hand, interval provides a more generalized range, thus reducing the loss of information. Further, one can work towards developing a more simpler theory for the case where the payoff information is in the form of multiple probabilistic linguistic terms.

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Declarations

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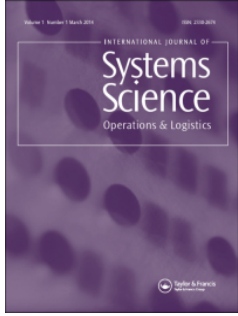
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A methodology for solving bimatrix games under 2-tuple linguistic environment

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ABSTRACT

In a game problem, when the situation is formulated as the gain of one equals the loss of other, the game is called zero-sum. But actually, one player's win is not always the other's loss. Such a game is referred to as non-zero sum or bimatrix game. The work provided in this research is primarily elaborated on linguistic bimatrix games under 2-tuple linguistic environment. This class of bimatrix games consists of payoffs having entries as linguistic 2-tuples, that are based on a finite subscript-symmetric additive linguistic set defined in the domain $[-g, g]$, where g is a positive integer. In this direction, a novel linguistic equilibrium concept is defined to solve such games, and a methodology based on a nonlinear linguistic programme is proposed to compute mixed Nash equilibrium. The proposed methodology is sufficient to evaluate the Nash equilibrium and the maximum expected payoffs corresponding to each player simultaneously in a single programme. The necessary and sufficient conditions for the existence of Nash equilibrium and a computation procedure are also provided. Finally, some hypothetical working examples are given to show the practical applicability of the defined methodology. The proposed methodology is also applied to solve two-player constant sum linguistic games.

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1. Introduction

The tremendous growth and advancement in games have attracted researchers to work in game theory, in which one is the winning player while the other is a loser. This signifies that one player always maximises his profit and the other minimises his loss. Nevertheless, as a matter of fact, both players want to maximise their profit. In that case, the theory of bimatrix game is relevant to control the contradictory situation. Recently, much attention has been focused on bimatrix games, where there are two players, and each player has a finite number of available strategies. In a bimatrix game, the players' interests are not required to be precisely opposed to each other. Both players have a finite number of available strategies, and each player attempts to maximise his payoff.

The traditional game theory assumes that every probable situation to select the payoffs is precisely determinable by the players. However, in real-life scenarios, such restrictive assumptions limit game theory scope as it is difficult to assess the information precisely because of the impreciseness of the game problems. This impreciseness in games was handled with the help of probabilistic methods and the theory of Bayesian games. But this theory was only confined to one kind of imprecision.

To the solution, several impreciseness types are handled well in fuzzy set theory as introduced by Zadeh (1965). However, it was examined that for the study of systems that are too complex or ill-defined for implementing traditional quantitative methods, the representation of the result in numerical form becomes improper. In dealing with accuracy in the face of overpowering intricacy, it is essential to learn the importance of linguistic variables as introduced by Zadeh (1975). Since it is challenging to clarify the precise relationship between payoffs and different strategic profiles, especially when each player's strategy space is numerous; linguistic variables have provided incredible matrix game results.

Further, to overcome the loss of information incurred in the process of Computing with Words, Herrera & Martinez introduced a 2-tuple linguistic (2-TL) model (Herrera & Martínez, 2000) for computing linguistic approaches, and it is implemented to solve two-person constant-sum matrix games having payoffs as 2-TL variables. However, there are no concepts of 'linguistic bimatrix games' yet, to be accepted broadly. Hence, it is a crucial problem to define the concepts of equilibrium solutions in bimatrix games with linguistic payoffs and investigate the characteristics of solutions of such games.

1.1. Research gap

This study aims to introduce a novel concept of ‘Linguistic Bimatrix Games having 2-tuple linguistic payoffs’. This is done by emphasising the following research gaps:

- The bimatrix games considered till now are either based on numerical information or fuzzy data. Such type of data causes information loss while applying fusion processes while aggregating the final data.
- The defuzzification methods used to solve fuzzy bimatrix games are restricted to a particular solution procedure. Hence, a more generalised methodology requires to be developed to directly draw the value of a bimatrix game.
- In the existing methodologies, researchers construct two different auxiliary equations for both players separately and then solve them individually to find their expected payoff. This enhances the time complexity of the computation methods.

1.2. Motivation and contribution

In this paper, we take into account the equilibrium solutions of the bimatrix games with 2-TL payoffs. For doing so, we first define the linguistic bimatrix games having 2-TL payoffs, and then we introduce a methodology for solving such games. The motivation for doing this is to incorporate ambiguity of human judgments in bimatrix game problems having two independent players that always try to maximise their payoffs, irrespective of what other player tries to undertake. The primary reason for this is to handle the linguistic uncertainty in non-zero sum games. For example, two competing companies launch a new product with almost the same features in the market. Each company aims to sell maximum quantities by selecting its market strategies like advertisements, free coupons, etc. Now to approximate the demand for this product, we have to count on the subjective opinions of experts with great managing experiences. The demand may be expressed in terms of language like ‘around 50 units’. Thus to deal with such linguistic ambiguity in bimatrix games, we introduce this novel concept. Moreover, this paper contributes to the theoretical development by proposing a new concept of solutions for linguistic bimatrix games and by proving the existence of this new concept. The linguistic expected payoffs and the Nash equilibrium concerning linguistic bimatrices are defined. Further, a nonlinear linguistic programme has been defined to compute Nash equilibrium in a linguistic bimatrix game. The proposed programme is further simplified and solved in terms of linguistics. The existence

and computation of mixed Nash equilibrium are provided in Theorem 4.6. The theorem guarantees the necessary and sufficient conditions required for the existence and computation of mixed Nash equilibrium in linguistic bimatrix games. Additionally, the proposed methodology is used to solve two-person constant-sum linguistic matrix games, too. Thus, it can be said that the developed approach may be used to solve any arbitrary two-person game, be it a constant-sum or a non-zero sum game. Technically, this theory contributes to provide Nash equilibrium and the value of the game using Maple software to provide better and more accurate results. Unlike traditional games, the values of both players in this paper are evaluated in a single Maple programme reducing the time and computation complexity. The whole computation procedure to obtain mixed Nash equilibrium is provided in the computation procedure 4.3. Also, the computation complexity of the defined methodology is provided in Theorem 5.3.

The entire paper is laid out in five sections as follows. Section 2 provides some related works to the field of bimatrix games. Section 3 establishes fundamental definitions and concepts of two-person non-zero sum games as well as the 2-TL model, along with the subscript-symmetric linguistic computation model and the 2-TL additive operator. In Section 4, we introduce the linguistic bimatrix games with payoffs as 2-TL variables belonging to the interval $[-g, g]$. Further in this section, we define pure and mixed Nash equilibrium, and the methodology for finding the mixed Nash equilibrium used to maximise the players’ payoffs has also been defined. Two numerical examples have been presented in Section 5. Section 6 discusses the relevance of the proposed methodology in solving two-person games. Section 7 offers some conclusions of the paper.

2. Related works

Beginning with the seminal work of Neumann and Morgenstern (1944), matrix games are substantially studied and proficiently applied to a lot of fields like economics, management and advertising. Authors have done phenomenal works in the field of game theory (Barron, 2011; Myerson, 1991). Further, fuzziness in game problems has had much more influence from an extensive range of research (Aubin, 1981; Bector & Chandra, 2005; Bector et al., 2004a, 2004b; Dengfeng, 1999). Butnariu (1978) introduced the concept of fuzzy games as matrix games having payoffs as fuzzy numbers. Campos (1989) formulated a fuzzy linear programming method to solve fuzzy matrix games. The impreciseness in games was first observed by Harsanyi (1967) and was dealt with the help of probabilistic theories. But with the introduction

of fuzzy set theory (Zadeh, 1965), fuzzy games (Aristidou & Sarangi, 2006; Campos & Gonzalez, 1991; Garagic & Cruz, 2003; Krishnaveni & Ganesan, 2018; Sakawa & Nishizaki, 1994) have been studied extensively. Larbani (2009) presents the review of fuzzy matrix games in normal form. Various solution methods are implemented to solve fuzzy matrix games like defuzzification methods (Sahoo, 2017), interval parametric techniques (Sahoo, 2015), signed distance methods (Sahoo, 2015), etc. Recently, much attention has been focussed on bimatrix games also (Belhaiza et al., 2012; Firouzbakht et al., 2016; Hladik, 2010). Since the 1960's, mathematicians and computer scientists have been studying computational aspects of finding Nash equilibria in bimatrix games (Bai, 1996; Bosse et al., 2010; Gao, 2021; Lemke & Howson, 1964) as well as in fuzzy bimatrix games (Nishizaki & Sakawa, 1995 (Chen, 2002; Han et al., 2009; Li, 2012; Maeda, 2000). Gani and Kumar (2013) provided a new method to solve fuzzy bimatrix games using LCP. Larbani (2009) solved fuzzy bimatrix games by introducing Nature as a third player. Gao (2013) discussed uncertainty in bimatrix games. However, to deal with the ill-defined implementation of traditional quantitative methods, linguistic variables became essential topic of discussion. Introduced by Zadeh (1975), linguistic variables have provided useful results in the field of linguistic games (Arfi, 2006a, 2006b; Delgado et al., 1990; Sahoo, 2017). The errors involved in the process of computing with words (Zadeh, 1999) were resolved by Herrera and Martínez (2000) and Herrera and Martínez (2001) by introducing the 2-TL model in which the linguistic information is represented in the form of 2-tuples. Numerous game theories make use of the linguistic model. The implementation of this 2-TL model has been done by Singh and Gupta (2018a, 2018b) and Malhotra et al. (2019) by providing a methodology to solve two-person constant-sum matrix games having payoffs as 2-TL variables. Considering the influence of linguistic models on matrix games as well as bimatrix games, we summarise the theories concerning linguistic models and matrix games in the following Table 1.

3. Preliminaries

In this section, certain important concepts of non-zero sum games, 2-TL variables and the subscript-symmetric model are recalled to provide a foundation for our proposal in this work.

3.1. Two-person non-zero sum games

The zero-sum games consider that whatever amount is won by one player, is the same amount lost by the other

Table 1. Summary of significant contributions of linguistic models on matrix and bimatrix games.

Title	Paper	Characteristic
2-Tuple fuzzy linguistic representation model for computing with words	Herrera and Martínez (2000)	The linguistic information is expressed in form of 2-tuples, comprising of a linguistic term and a number in $[0.5, 0.5]$; overcomes the loss of information
Matrix games with 2-tuple linguistic information	Singh and Gupta (2018b)	Formulation of a linguistic linear programming (LLP) model and its use to solve matrix games consisting of 2-tuple linguistic payoffs.
Methodology for Interval-Valued Matrix Games with 2-Tuple Fuzzy Linguistic Information	Malhotra et al. (2019)	A methodology is developed to evaluate saddle point and optimal interval-valued linguistic value of a 2-player zero-sum interval-valued 2-TL matrix game
Equilibrium points of bimatrix games	Lemke and Howson (1964)	Existence of equilibrium points for bimatrix games is proven. A structured computation method for finding Nash equilibrium is provided.
Two-person Non Zero-Sum Bimatrix Game with Fuzzy Payoffs and its Equilibrium Strategy	Han et al. (2009)	Some basic results for solving bimatrix games with fuzzy payoffs are obtained based on fuzzy max order
A Methodology for Solving Bimatrix Games under 2-tuple Linguistic Environment	Present Paper	A novel linguistic equilibrium concept is defined to solve bimatrix games with payoffs as linguistic 2-tuples. A methodology based on a nonlinear programme is defined to compute Nash equilibrium.

one. However, this is way too limiting for various games, like in politics or economics where both players may win or both may lose something. There are situations where the interests of both players may not be exactly the opposite. These types of situations bring about the concept of non-zero sum games, also called bimatrix games. In two-person non-zero sum games, both players have their individual payoff matrix. Both players aim to maximise their own individual payoff.

Definition 3.1 (Barron, 2011): Suppose the payoff matrices of player I and player II be $A = [a_{ij}]_{n \times m}$ and $B = [b_{ij}]_{n \times m}$, respectively. When player I plays i^{th} row

and player II plays j^{th} column, the payoff is represented by (a_{ij}, b_{ij}) in which a_{ij} represents the payoff to player I and b_{ij} to player II.

The respective rows and columns are termed as pure strategies. Moreover, a mixed strategy $\Phi \in S^n = \{\Phi = (\phi_1, \phi_2, \dots, \phi_n) | \phi_i \geq 0, i = 1, 2, \dots, n, \sum_{i=1}^n \phi_i = 1\}$ represents the probability that player I uses row i . Similarly, a mixed strategy for player II is $\Psi \in S^m = \{\Psi = (\psi_1, \psi_2, \dots, \psi_m) | \psi_j \geq 0, j = 1, 2, \dots, m, \sum_{j=1}^m \psi_j = 1\}$. Each player will have their own expected payoffs as

$$E_I(\Phi, \Psi) = \Phi A \Psi^T \text{ for player I,}$$

$$E_{II}(\Phi, \Psi) = \Phi B \Psi^T \text{ for player II.}$$

Further, Bector and Chandra (2005) defined a bimatrix game as $BG = (S^n, S^m, A, B)$.

Next we present the Nash equilibrium defined in the case of mixed as well as pure strategies.

Definition 3.2 (Barron, 2011): $\Phi^* \in S^n$ and $\Psi^* \in S^m$ form a Nash equilibrium if

$$E_I(\Phi, \Psi^*) \leq E_I(\Phi^*, \Psi^*) \text{ for every mixed } \Phi \in S^n$$

and,

$$E_{II}(\Phi^*, \Psi) \leq E_{II}(\Phi^*, \Psi^*) \text{ for every mixed } \Psi \in S^m.$$

If (Φ^*, Ψ^*) is a Nash equilibrium, denote $v_A = E_I(\Phi^*, \Psi^*)$ and $v_B = E_{II}(\Phi^*, \Psi^*)$ as the optimal payoff to each player. In matrix form, (Φ^*, Ψ^*) is a Nash equilibrium if

$$E_I(\Phi^*, \Psi^*) = \Phi^* A \Psi^{*T} \geq \Phi A \Psi^{*T} = E_I(\Phi, \Psi^*),$$

for every $\Phi \in S^n$

$$E_{II}(\Phi^*, \Psi^*) = \Phi^* B \Psi^{*T} \geq \Phi^* B \Psi^T = E_{II}(\Phi^*, \Psi),$$

for every $\Psi \in S^m$

Moreover, a Nash equilibrium in case of pure strategies are i^* and j^* such that $a_{ij^*} \leq a_{i^*j^*}$ and $b_{i^*j} \leq b_{i^*j^*}$.

Next we provide the methodology required to solve any arbitrary two-person non-zero sum game.

Non-linear Programming Method for Non-zero Sum Two-person Games (Barron, 2011) : This method provides a way for computing all the Nash equilibrium for an arbitrary two-person non-zero sum game. This method of computing the solution of a nonlinear programme lays on the foundation of the method provided by Lemke and Howson (1964). The method proceeds as follows:

Consider the two-person game with matrices (A, B) . Then $\Phi^* \in S^n$, $\Psi^* \in S^m$ is said to be a Nash equilibrium

iff they satisfy, along with scalars p^* & q^* the nonlinear programming problem

$$\begin{aligned} \max_{\Phi, \Psi, p, q} \quad & \Phi A \Psi^T + \Phi B \Psi^T - p - q \\ \text{s.t.} \quad & A \Psi^T \leq p J_n^T \\ & B^T \Phi^T \leq q J_m^T \text{ (equivalently } \Phi B \leq q J_m) \\ & \phi_i \geq 0, \psi_j \geq 0, \Phi J_n = 1 = \Psi J_m^T \end{aligned} \quad (1)$$

where, J_k is the unit vector consisting of all 1s. Additionally, $p^* = E_I(\Phi^*, \Psi^*)$, $q^* = E_{II}(\Phi^*, \Psi^*)$.

3.2. The basics of the 2-TL model

The 2-TL model introduced by Herrera and Martínez (2000) is a strong and accurate model representing the linguistic information using the pair (ℓ_i, α) , where ℓ_i denotes a linguistic term belonging to a linguistic term set and α is a numerical value belonging to $[-0.5, 0.5]$ representing the value of the symbolic translation.

Definition 3.3 (Herrera & Martínez, 2000): Let $L = \{\ell_0, \ell_1, \dots, \ell_g\}$ be the linguistic term set and $\beta \in L$ be a numerical value resulting from a symbolic aggregation operation. Then the function which transforms β into a 2-TL variable is given by:

$$\begin{aligned} \Delta: [0, g] &\rightarrow L \times [-0.5, 0.5] \equiv \bar{L} \\ \Delta(\beta) &= (\ell_i, \alpha) \end{aligned}$$

with $i = \text{round}(\beta)$ & $\alpha = \beta - i \in [-0.5, 0.5]$ where $\text{round}(\cdot)$ assigns the closest index label to β .

The reason that the interval of translation is $[-0.5, 0.5]$ is that the symbolic translation is a numerical value in $[-0.5, 0.5]$ that holds the ‘difference of information’ between a counting of information $\beta \in [0, g]$ of L and the index of the closest linguistic term in L .

Definition 3.4 (Herrera & Martínez, 2000): Let $L = \{\ell_0, \ell_1, \dots, \ell_g\}$ be a linguistic set and (ℓ_i, α) be a 2-TL variable. Then there always exists Δ^{-1} ,

$$\begin{aligned} \Delta^{-1}: \bar{L} &\rightarrow [0, g] \\ \Delta^{-1}(\ell_i, \alpha) &= i + \alpha = \beta. \end{aligned}$$

so that it yields an equivalent number $\beta \in [0, g] \subset \mathbb{R}$.

Remark 3.1: We may convert a linguistic term into a 2-TL variable by attaching 0 as symbolic translation, that is, $\ell_i \in L \implies (\ell_i, 0) \in \bar{L}$.

Definition 3.5 (Herrera & Martínez, 2000): Negation of a 2-TL Variable

The negation of a 2-TL variable is

$$\text{Neg}(\ell_i, \alpha) = \Delta(g - (\Delta^{-1}(\ell_i, \alpha)))$$

$g+1$ being the cardinality of the linguistic term set $L = \{\ell_0, \ell_1, \dots, \ell_g\}$.

3.3. Subscript-symmetric linguistic computational model

Definition 3.6 (Xu, 2005): Let $LT = \{\ell_{-g}, \dots, \ell_0, \dots, \ell_g\}$ be a finite and totally ordered predefined linguistic term set following that

- LT is ordered set that is, $\ell_i > \ell_j \Leftrightarrow i > j$, and
- For any $\ell_i \in LT$, $\text{Neg}(\ell_i) = \ell_{-i}$.

The traditional 2-TL model introduced by Herrera and Martínez (2000) defined earlier in this section is stipulated over the interval $[0, g]$. Nevertheless, this model may also be stated simply for the domain $[-g, g]$ by using subscript-symmetric linguistic term set (SSLTS) named LT . The idea of Δ and Δ^{-1} operators, theoretically validated by Herrera and Martínez (2000) for the set LT are given below.

Definition 3.7 (Malhotra et al., 2019): Let $LT = \{\ell_{-g}, \dots, \ell_0, \dots, \ell_g\}$ be a set of finite linguistic terms with cardinality $2g+1$ and suppose $x \in [-g, g]$. Then, the function Δ that converts x into its corresponding 2-TL variable is identified as

$$\Delta : [-g, g] \rightarrow LT \times [-0.5, 0.5] \equiv \overline{LT}$$

$$\Delta(x) = (\ell_i, \alpha)$$

where $i = \text{round}(x) \in LT$ and $\alpha = x - i \in [-0.5, 0.5]$. Here, $\text{round}(\cdot)$ is the standard round operation, ℓ_i is a linguistic term nearest to β and α is the symbolic translation.

It is clear that Δ is bijective; thus, its inverse may be specified as

$$\Delta^{-1} : \overline{LT} \rightarrow [-g, g] \text{ s.t. } \Delta^{-1}(\ell_i, \alpha) = i + \alpha$$

Moreover, similar to the comparison of conventional 2-TL variables stipulated on lexicographic ordering, the comparison of the 2-TL variables based on SSLTS is defined as follows.

Definition 3.8 (Malhotra et al., 2019): Let (ℓ_i, α_i) and (ℓ_j, α_j) be two 2-TL variables based on LT . Then

- If $i < j$, then $(\ell_i, \alpha_i) < (\ell_j, \alpha_j)$.

- If $i = j$, then
 - * If $\alpha_i = \alpha_j$, then (ℓ_i, α_i) and (ℓ_j, α_j) represent the same information.
 - * If $\alpha_i < \alpha_j$, then $(\ell_i, \alpha_i) < (\ell_j, \alpha_j)$.
 - * If $\alpha_i > \alpha_j$, then $(\ell_i, \alpha_i) > (\ell_j, \alpha_j)$.

Further, Malhotra and Gupta (2020) have defined the 2-TL additive operator based on the above mentioned SSLTS. However, we'll be reviewing the following theorem in our paper as it would be useful throughout our work.

Theorem 3.9 (Malhotra & Gupta, 2020): Let $LT = \{\ell_{-g}, \dots, \ell_0, \dots, \ell_g\}$ be the SSLTS with odd cardinality $2g+1$ and let $\alpha_i \in [-0.5, 0.5]$ be the numerical value depicting the value of symbolic translation corresponding to ℓ_i . Then $\overline{LT}$ forms a direct group under the binary operation $\circ$ defined as :

$$(\ell_i, \alpha_i) \circ (\ell_j, \alpha_j) = (\ell_i \circ \ell_j, \alpha_i \circ \alpha_j)$$

where,

$$\begin{aligned} \ell_i \circ \ell_j &= \ell_i * \ell_j \\ &= \ell_{i+j} \\ &= \begin{cases} \ell_{i+j+g}, & \text{if } i+j < -g \\ \ell_{i+j}, & \text{if } -g \leq i+j \leq g \\ \ell_{i+j-g}, & \text{if } i+j > g \end{cases} \end{aligned}$$

and,

$$\begin{aligned} \alpha_i \circ \alpha_j &= \alpha_i \oplus \alpha_j \\ &= \begin{cases} \alpha_i + \alpha_j, & \text{if } \alpha_i + \alpha_j \in (-0.5, 0.5) \\ \alpha_i + \alpha_j - \text{round}(\alpha_i + \alpha_j), & \text{else} \end{cases} \end{aligned}$$

where $\ell_i, \ell_j \in LT$ and $\alpha_i, \alpha_j \in [-0.5, 0.5]$.

Proof: For detailed reference of the proof, one can refer to the paper (Malhotra & Gupta, 2020). ■

4. Linguistic bimatrix games having payoffs as 2-TL variables

In this section, we'll define a methodology to solve the linguistic bimatrix games having 2-TL payoffs where the payoffs belong to a predefined SSLTS. The present section is divided into two subsections. The details of the basic definitions and terminologies beneficial for the development of the methodology to solve two-person non-zero sum linguistic bimatrix games are stated in Subsection 4.1. The existence and evaluation of mixed Nash equilibria in such games are then provided in Subsection 4.2 followed by the computation procedure used to compute Nash equilibrium in Subsection 4.3.

4.1. Linguistic bimatrix games

In the existent game problems, the decision framework is usually identified by a number of feasible strategies, intricate connections between strategic alternatives and their complex impact on payoffs. Thus, estimating the precise evaluation of the payoff bimatrices becomes almost impossible. In this type of situation, we may identify the uncertain payoffs in the form of fuzzy linguistic variables.

Consider a game having player I and player II as two players and let $N = \{1, 2, \dots, n\}$ and $M = \{1, 2, \dots, m\}$ denote the sets of all pure strategies available for player I and II, respectively. Suppose the payoff matrix of player I is $\tilde{A} = [\tilde{a}_{ij}]_{n \times m}$ and of player II is $\tilde{B} = [\tilde{b}_{ij}]_{n \times m}$. The payoffs $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ of player I and II respectively, come up when player I selects a pure strategy $i \in N$ and player II selects a pure strategy $j \in M$. The payoffs of both players are represented together as $(\tilde{a}_{ij}, \tilde{b}_{ij})$. These payoffs $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ where $i \in N, j \in M$ are the linguistic variables belonging to a predefined SSLTS, or are 2-TL variables for any linguistic variable ℓ_i . Now, the notion of matrix games can be extended to linguistic bimatrix games in the following manner.

Definition 4.1: A 2-person non-zero sum linguistic bimatrix game may be defined as $\tilde{G} = (S^n, S^m, LT, \tilde{A}, \tilde{B})$ where S^n and S^m are the strategy sets of player I and II respectively, and the mixed strategies of player I and player II are represented by

$$\Phi = (\phi_1, \phi_2, \dots, \phi_n)^T \in S^n = \{(\phi_1, \phi_2, \dots, \phi_n) : \phi_i \geq 0, i = 1, 2, \dots, n, \sum_{i=1}^n \phi_i = 1\}$$

and,

$$\Psi = (\psi_1, \psi_2, \dots, \psi_m)^T \in S^m = \{(\psi_1, \psi_2, \dots, \psi_m) : \psi_j \geq 0, j = 1, 2, \dots, m, \sum_{j=1}^m \psi_j = 1\}$$

respectively, where T denotes the transposition. $LT = \{\ell_{-g}, \dots, \ell_0, \dots, \ell_g\}$ is the linguistic term set for players with cardinality $2g + 1$, $\tilde{A}$ and $\tilde{B}$ are the linguistic payoff matrices of player I and player II respectively, having payoffs as 2-TL variables. We write this game in matrix form as $(\tilde{A}, \tilde{B})$.

The evaluation of the expected payoffs of both players is given as:

Definition 4.2: Suppose player I and player II select the mixed strategy $\Phi \in S^n$ and $\Psi \in S^m$ respectively, then the

expected payoffs of player I and player II are:

$$E_I(\Phi, \Psi) = \Phi \tilde{A} \Psi^T$$

and,

$$E_{II}(\Phi, \Psi) = \Phi \tilde{B} \Psi^T$$

respectively. Moreover, if $(\tilde{A}, \tilde{B})$ is a bimatrix game, then the safety value of player I is $\text{value}(\tilde{A})$ and of player II is $\text{value}(\tilde{B}^T)$.

Now before proceeding towards the main theorem, we first define the scalar multiplication of 2-TL variables as follows.

Definition 4.3: For any $\lambda \in [0, 1]$ and $(\ell_i, \alpha) \in LT \times [-0.5, 0.5]$, we write

$$\lambda \cdot (\ell_i, \alpha) = (\ell_{\hat{i}}, \lambda \cdot \alpha), \text{ where } \hat{i} = \text{round}(\lambda \cdot i)$$

We next proceed towards the theorem outlining the Nash equilibrium.

Theorem 4.4: A pair of mixed strategies $(\Phi^* \in S^n, \Psi^* \in S^m)$ is a Nash equilibrium iff

$$E_I(\Phi^*, \Psi^*) = \Phi^* \tilde{A} \Psi^{*T} \geq \Phi \tilde{A} \Psi^{*T} = E_I(\Phi, \Psi^*),$$

for every $\Phi \in S^n$, (2)

$$E_{II}(\Phi^*, \Psi^*) = \Phi^* \tilde{B} \Psi^{*T} \geq \Phi^* \tilde{B} \Psi^T = E_{II}(\Phi^*, \Psi),$$

for every $\Psi \in S^m$. (3)

Proof: First suppose that the pair (Φ^*, Ψ^*) is a Nash equilibrium. Then

$$\begin{aligned} E_I(\Phi^*, \Psi^*) &= \Phi^* \tilde{A} \Psi^{*T} \\ &= [\phi_1^* \phi_2^* \dots \phi_n^*]_{1 \times n} \\ &\quad [\tilde{a}_{ij} = (\ell_{ij}, \alpha_{ij})]_{n \times m} \begin{bmatrix} \psi_1^* \\ \psi_2^* \\ \vdots \\ \psi_m^* \end{bmatrix}_{m \times 1} \\ &= \circ_{i=1}^n \circ_{j=1}^m (\ell_{\hat{ij}}, \Phi^* \alpha_{ij} \Psi^*), \end{aligned}$$

$$\text{where } \hat{ij} = \text{round}(\Phi^*_{ij} \Psi^*),$$

$$\geq \circ_{i=1}^n \circ_{j=1}^m (\ell_{\bar{ij}}, \Phi \alpha_{ij} \Psi^*),$$

$$\text{where } \bar{ij} = \text{round}(\Phi_{ij} \Psi^*)$$

$$(\because \Phi^* \text{ and } \Psi^* \text{ are Nash equilibrium})$$

$$= \Phi \tilde{A} \Psi^{*T}, \text{ for every } \Phi \in S^n$$

Similarly, $E_{II}(\Phi^*, \Psi^*) \geq E_{II}(\Phi^*, \Psi)$, for every $\Psi \in S^m$.

Here, $\circ$ denotes the 2-TL binary operator as defined in Theorem 3.9.

The converse follows trivially. ■

It is to be noted that the pure strategy Nash equilibrium would also be the row i^* and column j^* such that $\tilde{a}_{ij^*} \leq \tilde{a}_{i^*j^*}$ and $\tilde{b}_{ij^*} \leq \tilde{b}_{i^*j^*}$ where $i \in N$, $j \in M$.

The above-mentioned ordering is valid because of the lexicographic ordering of 2-TL variables. Hence, for bimatrix games, a Nash equilibrium in pure strategies is a pair that is, the largest first component in column and the largest second component in row simultaneously.

We illustrate this with the help of an example:

Example 4.5: A company owner observed during a health emergency, that the absence of employees of his company is affecting the company's profits, especially in the department of sales and the department of operations, very badly. He introduced some schemes to make profits back on track. The schemes include Work from Home (WFH), Work from Office (WFO) (this scheme involves working of all employees from office, however, this may affect the health of employees resulting in a bad environment to work) and Roster-based attendance (i.e. only 30% of employees of both departments attend office each day maintaining required safety measures). He met employees of both departments and offered them these schemes, and tried to observe improvement in profits with each scheme for a month. The linguistic bimatrix game that models this situation is as follows: The linguistic payoffs belong to the LT set

$$LT = \{\ell_{-2} : \text{Very Low}, \ell_{-1} : \text{Low}, \ell_0 : \text{Average}, \\ \ell_1 : \text{High}, \ell_2 : \text{Very High}\}$$

The payoff $\tilde{a}_{12}$ depicts that if the sales team works from home and the operations team works from the office, then the sales get high profits and the operations team get average profits.

Sales/Operations	WFH	WFO	Roster
WFH	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_0)	(ℓ_1, ℓ_2)
WFO	(ℓ_0, ℓ_1)	(ℓ_0, ℓ_0)	(ℓ_0, ℓ_2)
Roster	(ℓ_2, ℓ_1)	(ℓ_2, ℓ_0)	(ℓ_2, ℓ_2)

Sales/Operations	WFH	WFO	Roster
WFH	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_0)	(ℓ_1, ℓ_2)
WFO	(ℓ_0, ℓ_1)	(ℓ_0, ℓ_0)	(ℓ_0, ℓ_2)
Roster	(ℓ_2, ℓ_1)	(ℓ_2, ℓ_0)	(ℓ_2, ℓ_2)

Now to establish pure Nash equilibrium points, we put a bar above the first component that is the largest in each

column and the second component that is the largest in each row. In this way, the pair of linguistic terms in which both components have bar is a Nash equilibrium in pure strategies. So, in this problem we have

This means that there is a unique pure Nash equilibrium at (3, 3), which means that if both teams come to office on a Roster basis, then they can stabilise their profits without compromising their health.

Now it might be possible that a pure Nash equilibrium does not always exist, for example, consider the 2×2 game whose bimatrix is given by

$$\begin{bmatrix} (\ell_0, \ell_1) & (\ell_1, \ell_2) \\ (\ell_{-1}, \ell_2) & (\ell_2, \ell_1) \end{bmatrix}$$

In this game, there is no pair (i^*, j^*) which holds the earlier mentioned criteria for existence of pure Nash equilibrium. Therefore, this game admits no Nash equilibria in pure strategies. The problem of finding Nash equilibrium can be further removed by the introduction of mixed strategies, and thus by calculating the expected payoffs of the players.

4.2. Existence and computation of mixed Nash equilibrium

This section offers the introduction of the methodology to find mixed Nash equilibria in bimatrix games having entries as 2-TL variables. Based on the construction mentioned in previous sections, we contemplate the following theorem, that guarantees the necessary and sufficient conditions required for the existence and computation of mixed Nash equilibrium in two-person non-zero sum linguistic bimatrix games.

Theorem 4.6: Consider a linguistic two-person non-zero sum game having matrices $(\tilde{A}, \tilde{B})$ for player I and player II. $\Phi^* \in S^n$, $\Psi^* \in S^m$ is a Nash equilibrium iff they satisfy, along with linguistic variables $\tilde{p}^*$ and $\tilde{q}^*$ the non linear linguistic programme

$$\Delta \left[\max_{\Phi, \Psi, \tilde{p}, \tilde{q}} \Delta^{-1} \left\{ \Phi \tilde{A} \Psi^T \circ \Phi \tilde{B} \Psi^T - \tilde{p} - \tilde{q} \right\} \right]$$

such that

$$\tilde{A} \Psi^T \leq \tilde{p} J_n^T$$

$$\tilde{B}^T \Phi^T \leq \tilde{q} J_m^T$$

$$\phi_i \geq 0, \psi_j \geq 0, \Phi J_n = 1 = \Psi J_m^T \quad (4)$$

where J_k is the unit vector consisting of all 1s. Further, $\tilde{p}^* = E_I(\Phi^*, \Psi^*)$ and $\tilde{q}^* = E_{II}(\Phi^*, \Psi^*)$ and $\circ$ denotes the 2-TL additive operator mentioned in Theorem 3.9.

Proof: Consider the players, namely player I & player II having mixed strategy vectors $\Phi = (\phi_1, \dots, \phi_n)$, $\Psi = (\psi_1, \dots, \psi_m)$, respectively, $\tilde{A}$ and $\tilde{B}$ be payoff matrices of both players having payoffs as 2-TL variables. Player I maximises over the rows of $\tilde{A}$ while player II, over the columns of $\tilde{B}$.

According to objective of player I

$$\begin{aligned} & \max_{\Phi} \Phi \tilde{A} \Psi^T \\ & \text{s.t.} \\ & J_n^T \Phi - 1 = 0 \\ & \Phi \geq 0 \end{aligned} \quad (5)$$

and according to the objective of player II

$$\begin{aligned} & \max_{\Psi} \Phi \tilde{B} \Psi^T \\ & \text{s.t.} \\ & J_m^T \Psi - 1 = 0 \\ & \Psi \geq 0 \end{aligned} \quad (6)$$

where T denotes the transpose of the matrix.

We know that a Nash equilibrium point (Φ^*, Ψ^*) consists of a pair of strategies where the objectives (5) and (6) are satisfied at the same time. More precisely, if (Φ^*, Ψ^*) is a Nash equilibrium, then

$$\begin{aligned} E_I(\Phi^*, \Psi^*) &= \Phi^* \tilde{A} \Psi^{*T} \\ &= \max_{\Phi} \{ \Phi \tilde{A} \Psi^{*T} \mid J_n^T \Phi - 1 = 0, \Phi \geq 0 \} \\ E_{II}(\Phi^*, \Psi^*) &= \Phi^* \tilde{B} \Psi^{*T} \\ &= \max_{\Psi} \{ \Phi^* \tilde{B} \Psi^T \mid J_m^T \Psi - 1 = 0, \Psi \geq 0 \} \end{aligned}$$

We need to show that (Φ^*, Ψ^*) is a Nash equilibrium iff they satisfy along with the scalars $\tilde{p}^*$, $\tilde{q}^*$, the non linear programme given in (4) which can be simplified as

$$\begin{aligned} & \Delta \left[\max_{\Phi, \Psi, \tilde{p}, \tilde{q}} \Phi \Delta^{-1} (\tilde{A} \circ \tilde{B}) \Psi^T - \Delta^{-1} (\tilde{p}) - \Delta^{-1} (\tilde{q}) \right] \\ & \text{such that} \\ & \Delta^{-1} (\tilde{A}) \Psi^T \leq \Delta^{-1} (\tilde{p}) J_n^T \\ & \Delta^{-1} (\tilde{B}^T) \Phi^T \leq \Delta^{-1} (\tilde{q}) J_m \\ & \phi_i \geq 0, \psi_j \geq 0 \end{aligned} \quad (7)$$

Now, according to Theorem 4.4, $E_I(\Phi^*, \Psi^*) = \Phi^* \tilde{A} \Psi^{*T} \geq \Phi \tilde{A} \Psi^{*T}$ and $E_{II}(\Phi^*, \Psi^*) = \Phi^* \tilde{B} \Psi^{*T} \geq \Phi^* \tilde{B} \Psi^T$, it can clearly be observed that $E_I(\Phi^*, \Psi^*)$ and

$E_{II}(\Phi^*, \Psi^*)$ are 2-TL numbers. In (2) we successively select $\Phi = (0, 0, \dots, 1, \dots, 0)$ having 1 at the n^{th} spots, and in (3) we select $\Psi = (0, 0, \dots, 1, \dots, 0)$ having 1 at the m^{th} spots. Doing so, we obtain that $E_I(\Phi^*, \Psi^*) \geq E_I(i, \Psi^*) = i \tilde{A} \Psi^{*T}$ and $E_{II}(\Phi^*, \Psi^*) \geq E_{II}(\Phi^*, j) = \Phi^* \tilde{B} j$. In matrix form, it can be written as

$$E_I(\Phi^*, \Psi^*) J_n^T = \Phi^* \tilde{A} \Psi^{*T} J_n^T \geq \tilde{A} \Psi^{*T} \quad (8)$$

$$E_{II}(\Phi^*, \Psi^*) J_m = \Phi^* \tilde{B} \Psi^{*T} J_m \geq \Phi^* \tilde{B} \quad (9)$$

Since (8) and (9) are true for the pair (Φ^*, Ψ^*) , then these strategies must be a Nash point. This means that (2) and (3) hold true (due to definition). So pre-multiplying $\Phi \in S^n$ and post-multiplying $\Psi \in S^m$ in Equations (8) and (9), respectively.

$$\left. \begin{aligned} E_I(\Phi^*, \Psi^*) \Phi J_n^T &= \Phi^* \tilde{A} \Psi^{*T} \Phi J_n^T \geq \Phi \tilde{A} \Psi^{*T} \\ E_{II}(\Phi^*, \Psi^*) J_m \Psi^T &= (\Phi^* \tilde{B} \Psi^{*T}) J_m \Psi^T \geq \Phi^* \tilde{B} \Psi^T \end{aligned} \right\} \\ \left[\because \Phi J_n^T = J_m \Psi^T = 1 \right]$$

Next, we select the scalars $\tilde{p}^*$ and $\tilde{q}^*$ as

$$\tilde{p}^* = E_I(\Phi^*, \Psi^*) = \Phi^* \tilde{A} \Psi^{*T}$$

$$\tilde{q}^* = E_{II}(\Phi^*, \Psi^*) = \Phi^* \tilde{B} \Psi^{*T}$$

Then, by using the equivalent characterisation of a Nash point, we get by Equations (8) and (9) that

$$\Phi^* \tilde{A} \Psi^{*T} J_n^T = \tilde{p}^* J_n^T \geq \tilde{A} \Psi^{*T}$$

Also,

$$(\Phi^* \tilde{B} \Psi^{*T}) J_m = \tilde{q}^* J_m \geq \Phi^* \tilde{B}$$

Other constraints are fulfilled directly because $\Phi^* \in S^n$ and $\Psi^* \in S^m$. Hence, $(\Phi^*, \Psi^*, \tilde{p}^*, \tilde{q}^*)$ is a required feasible point.

Next, we show that $(\Phi^*, \Psi^*, \tilde{p}^*, \tilde{q}^*)$ maximises the function

$$\tilde{f}(\Phi, \Psi, \tilde{p}, \tilde{q}) = \Phi \tilde{A} \Psi^T \circ \Phi \tilde{B} \Psi^T - \tilde{p} - \tilde{q}$$

over the set of feasible points.

Note that, here we use Δ^{-1} function for further computations as ‘ $\circ$ ’ is not a predefined/valid operator in 2-TL domain due to its non-closure property. As Δ is an isomorphic function, suppose $p = \Delta^{-1}(\tilde{p})$, $q = \Delta^{-1}(\tilde{q})$.

Since every feasible solution satisfies the constraints

$$\Delta^{-1}(\tilde{A}) \Psi^T \leq p J_n^T, \quad \Phi \Delta^{-1}(\tilde{B}) \leq q J_m$$

By multiplying first inequality by Φ on the left and by second by Ψ^T on the right, we obtain

$$\Delta^{-1}(\Phi \tilde{A} \Psi^T) \leq p \Phi J_n^T, \quad \Delta^{-1}(\Phi \tilde{B} \Psi^T) \leq q J_m \Psi^T$$

$$\begin{aligned} \implies \Delta^{-1}(\Phi \tilde{A} \Psi^T) &\leq p, \quad \Delta^{-1}(\Phi \tilde{B} \Psi^T) \leq q \\ \implies \Delta^{-1}(\Phi \tilde{A} \Psi^T \circ \Phi \tilde{B} \Psi^T) &- p - q \leq 0 \end{aligned}$$

(By adding and rearranging)

So, $f(\Phi, \Psi, p, q) \leq 0$ for any feasible point. However, with $p^* = \Delta^{-1}(\Phi^* \tilde{A} \Psi^{*T})$ and $q^* = \Delta^{-1}(\Phi^* \tilde{B} \Psi^{*T})$, it can be concluded that $(\Phi^*, \Psi^*, p^*, q^*)$ is a feasible solution of (7) and

$$\begin{aligned} f(\Phi^*, \Psi^*, p^*, q^*) &= \Delta^{-1}(\Phi \tilde{A} \Psi^T \circ \Phi \tilde{B} \Psi^T) - \Delta^{-1}(\tilde{p}) \\ &\quad - \Delta^{-1}(\tilde{q}) = 0 \end{aligned}$$

by the definition of p^* and q^* . The above step is justified as Δ is a bijective function, hence invertible and further Δ satisfies the linearity property.

Hence, $(\Phi^*, \Psi^*, p^*, q^*)$ is feasible and offers the maximum value of the objective, and so is a solution of the nonlinear programme. Therefore we conclude that if there is a Nash equilibrium, it must solve the nonlinear programming problem.

Note : This must be noted that for the final linguistic results, we have to apply Δ to the maximised objective function. So we can say that $(\Phi^*, \Psi^*, p^*, q^*)$ satisfies the problem $\Delta[\max_{\Phi, \Psi, \tilde{p}, \tilde{q}} \Delta^{-1}\{\Phi \tilde{A} \Psi^T \circ \Phi \tilde{B} \Psi^T - \tilde{p} - \tilde{q}\}]$ also.

Now for the converse, we need to show that any solution of the nonlinear programme (4) must be a Nash point. Let $\Phi_1, \Psi_1, \tilde{p}_1, \tilde{q}_1$ be any arbitrary solution of (4). Let (Φ^*, Ψ^*) be a Nash point, and set $\tilde{p}^* = \Phi^* \tilde{A} \Psi^{*T}$, $\tilde{q}^* = \Phi^* \tilde{B} \Psi^{*T}$. We need to show that (Φ_1, Ψ_1) must be a Nash point of the game.

Since Φ_1, Ψ_1 are a solution, they must satisfy the constraints of the problem (4), that is

$$\begin{aligned} \tilde{A} \Psi_1^T &\leq \tilde{p}_1 J_n^T \quad \text{and} \quad \Phi_1 \tilde{B} \leq \tilde{q}_1 J_m \\ \implies \tilde{A} \psi_1^T &\leq (\phi_1 \tilde{A} \psi_1^T) J_n^T \quad \text{and} \\ \phi_1 \tilde{B} &\leq (\phi_1 \tilde{B} \psi_1^T) J_m \end{aligned}$$

This makes (Φ_1, Ψ_1) a Nash point. Hence, it is proved that any solution of the nonlinear programme should give a Nash equilibrium, and the scalars should be the expected payoffs. ■

To solve the proposed nonlinear programme, we follow the below-mentioned computation procedure.

4.3. Computation procedure

The solution process to obtain the Nash equilibrium points and the maximised expected payoffs of both players of a two-person non-zero sum game is introduced below.

Step 1 : Input two payoff matrices $(\tilde{A}, \tilde{B})$ corresponding to player I and player II, respectively.

Step 2 : Put $\tilde{A}$ and $\tilde{B}$ in the nonlinear programme (4) given in Theorem 4.6.

Step 3 : Δ being a monotonic and linear bijection, apply Δ^{-1} to both sides of the constraints of programme (4).

The problem becomes

$$\Delta \left[\max_{\Phi, \Psi, \tilde{p}, \tilde{q}} \Phi \Delta^{-1}(\tilde{A} \circ \tilde{B}) \Psi^T - \Delta^{-1}(\tilde{p}) - \Delta^{-1}(\tilde{q}) \right]$$

such that

$$\Delta^{-1}(\tilde{A}) \Psi^T \leq \Delta^{-1}(\tilde{p}) J_n^T$$

$$\Delta^{-1}(\tilde{B}^T) \Phi^T \leq \Delta^{-1}(\tilde{q}) J_m^T$$

$$\phi_i \geq 0, \quad \psi_j \geq 0$$

Step 4 : Set $\Delta^{-1}(\tilde{A}) = A$, $\Delta^{-1}(\tilde{B}) = B$, $\Delta^{-1}(\tilde{p}) = p$ and $\Delta^{-1}(\tilde{q}) = q$, the nonlinear programme becomes

$$\Delta \left[\max_{\Phi, \Psi, p, q} \Phi \Delta^{-1}(\tilde{A} \circ \tilde{B}) \Psi^T - p - q \right]$$

such that

$$A \Psi^T \leq p J_n^T$$

$$B^T \Phi^T \leq q J_m^T$$

$$\phi_i \geq 0, \quad \psi_j \geq 0, \quad \Phi J_n = 1 = \Psi J_m^T \quad (10)$$

Step 5 : Use the Maple commands mentioned in Figure 1 to solve the nonlinear programme (7).

Step 6 : Step 5 gives the values of p, q, Φ, Ψ . Substituting Δ function back to these variables, the solution of original linguistic nonlinear programme may be obtained.

Note: On changing the initial point in code, we may achieve the pure Nash solution also, but it is in fact a waste as pure Nash points may be evaluated directly from the game unless it is a very large game. There are two ways of solving this problem using Maple commands. Firstly, we may use the command QPSolve if the objective function is a quadratic function. Secondly, NLPsolve may be used for a general nonlinear objective function. If the expected payoffs of the game might be negative, then we have to drop the 'assume = nonnegative' part and mention the nonnegativity of the strategies to constraints.

In the next section, we proceed towards illustrating the practicality of the proposed approach of maximising the objective and finding Nash equilibrium.

```

> with(LinearAlgebra):
> A := Matrix([ [a11, a12, ..., a1m], [a21, a22, ..., a2m], ..., [an1, an2, ..., anm] ]);
> B := Matrix([ [b11, b12, ..., b1m], [b21, b22, ..., b2m], ..., [bn1, bn2, ..., bnm] ]);
> Φ := < φ[1], φ[2], ..., φ[n] >;
> Ψ := < ψ[1], ψ[2], ..., ψ[m] >;
> Cnst := {seq((A.Ψ)[i] <= p, i = 1..n), seq((Transpose(Φ).B)[i] <= q, i = 1..m) add(φ[i], i = 1..n) = 1, add(ψ[i], i = 1..m) = 1};
> with(Optimization):
> objective := expand(Transpose(Φ).A.Ψ + Transpose(Φ).B.Ψ - p - q);
> QPSolve(objective, Cnst, assume=nonnegative, maximize);
> QPSolve(objective, Cnst, assume=nonnegative, maximize, initialpoint=({q = 1, p = 2}));
> NLPsolve(objective, Cnst, assume=nonnegative, maximize);

```

Figure 1. Maple commands for solving the Nonlinear programme.

5. Numerical illustration

In the present section, we consider two examples to demonstrate the effectiveness of the proposed nonlinear linguistic programme for solving 2-TL bimatrix game problems. The Example 5.1 considers a general linguistic bimatrix game, whose payoff entries belong to a predefined LT set belonging to the interval $[-g, g]$. Apart from this, Example 5.2 includes linguistic bimatrices of two players whose payoffs belong to an LT set in the interval $[0, g]$.

Example 5.1: Consider a general linguistic bimatrix game having payoffs belonging to a symmetrical linguistic terms set

$LT = \{\ell_{-2} : \text{Very Bad (VB)}, \ell_{-1} : \text{Bad (B)}, \ell_0 :$

Average (AVG), $\ell_1 : \text{Good (G)}, \ell_2 : \text{Very Good (VG)}\}$

and suppose the payoff matrices corresponding to player I and player II be

$$\tilde{A} = \begin{bmatrix} G & G & (AVG, 0.3) & (VG, -0.1) & VG \\ VG & (B, 0.4) & (G, 0.8) & G & AVG \\ VB & (AVG, 0.2) & G & AVG & VG \\ VG & B & AVG & (G, 0.2) & (AVG, 0.4) \end{bmatrix}$$

$$= \begin{bmatrix} \ell_1 & \ell_1 & (\ell_0, 0.3) & (\ell_2, -0.1) & \ell_2 \\ \ell_2 & (\ell_{-1}, 0.4) & (\ell_1, 0.8) & \ell_1 & \ell_0 \\ \ell_{-2} & (\ell_0, 0.2) & \ell_1 & \ell_0 & \ell_2 \\ \ell_2 & \ell_{-1} & \ell_0 & (\ell_1, 0.2) & (\ell_0, 0.4) \end{bmatrix}$$

and,

$$\tilde{B} = \begin{bmatrix} AVG & VG & (AVG, 0.3) & G & VG \\ VG & (G, 0.1) & B & VG & (VB, 0.1) \\ B & (VB, 0.4) & VG & G & (AVG, 0.4) \\ G & VG & (B, 0.4) & (VB, 0.2) & AVG \end{bmatrix}$$

$$= \begin{bmatrix} \ell_0 & \ell_2 & (\ell_0, 0.3) & \ell_1 & \ell_2 \\ \ell_2 & (\ell_1, 0.1) & \ell_{-1} & \ell_2 & (\ell_{-2}, 0.1) \\ \ell_{-1} & (\ell_{-2}, 0.4) & \ell_2 & \ell_1 & (\ell_0, 0.4) \\ \ell_1 & \ell_2 & (\ell_{-1}, 0.4) & (\ell_{-2}, 0.2) & \ell_0 \end{bmatrix}$$

respectively.

We first look for pure Nash points in this game and these points are given in Table 2. Now applying the operator “ $\circ$ ” as in Theorem 3.9 to $\tilde{A}$ and $\tilde{B}$,

$$\tilde{A} \circ \tilde{B} = \begin{bmatrix} \ell_1 & \ell_1 & (\ell_0, -0.4) & (\ell_1, -0.1) & \ell_2 \\ \ell_2 & (\ell_0, -0.5) & (\ell_0, -0.2) & \ell_3 & (\ell_{-2}, 0.1) \\ \ell_{-1} & (\ell_{-2}, -0.4) & \ell_3 & \ell_1 & (\ell_2, 0.4) \\ \ell_1 & \ell_1 & (\ell_{-1}, 0.4) & (\ell_{-1}, 0.4) & (\ell_0, 0.4) \end{bmatrix}$$

and solving the nonlinear programming problem 4, the problem becomes,

$$\Delta \left[\max_{\Phi, \Psi, \tilde{p}, \tilde{q}} \Delta^{-1} \left\{ \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix} \begin{bmatrix} \ell_1 & \ell_1 & (\ell_0, -0.4) & (\ell_1, -0.1) & \ell_2 \\ \ell_2 & (\ell_0, -0.5) & (\ell_0, -0.2) & \ell_3 & (\ell_{-2}, 0.1) \\ \ell_{-1} & (\ell_{-2}, -0.4) & \ell_3 & \ell_1 & (\ell_2, 0.4) \\ \ell_1 & \ell_1 & (\ell_{-1}, 0.4) & (\ell_{-1}, 0.4) & (\ell_0, 0.4) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{bmatrix} - \tilde{p} - \tilde{q} \right\} \right]$$

such that

$$\tilde{A}\Psi^T \leq \tilde{p}J_n^T$$

$$\tilde{B}^T\Phi^T \leq \tilde{q}J_m^T$$

$$\phi_i \geq 0, \psi_j \geq 0, \Phi J_n = 1 = \Psi J_m^T$$

Table 2. Table for pure Nash points.

Φ	Ψ	$E_I(\Phi, \Psi)$	$E_{II}(\Phi, \Psi)$
(1, 0, 0, 0)	(0, 1, 0, 0, 0)	ℓ_1	ℓ_2
(1, 0, 0, 0)	(0, 0, 0, 0, 1)	ℓ_2	ℓ_2
(0, 1, 0, 0)	(1, 0, 0, 0, 0)	ℓ_2	ℓ_2

Then for the linguistic variables $\tilde{p}$ and $\tilde{q}$, let $\Delta^{-1}(\tilde{p}) = p$ and $\Delta^{-1}(\tilde{q}) = q$. So we write the Maple code given in Figure 1 for achieving the solution after applying Δ^{-1} to constraints. Solving this, we get the result as $p = 1.11$, $q = 1.05$, $\phi[1] = 0.47$, $\phi[2] = 0.05$, $\phi[3] = 0.48$, $\phi[4] = 0$ and $\psi[1] = 0$, $\psi[2] = 0$, $\psi[3] = 0.51$, $\psi[4] = 0.19$, $\psi[5] = 0.3$. This implies that the expected payoffs of players I and II are $\tilde{p} = E_I(\Phi, \Psi) = \Delta(p) = (\ell_1, 0.11)$ and $\tilde{q} = E_{II}(\Phi, \Psi) = \Delta(q) = (\ell_1, 0.05)$. The results indicate that with mixed Nash points $\Phi = (0.47, 0.05, 0.48, 0)$ and $\Psi = (0, 0, 0.51, 0.19, 0.3)$, player I gets an expected payoff which is 'good' (with translational value 0.11) and player II also gets a 'good' expected payoff (with translational value 0.05).

Next we solve the following example which is taken from the paper (Singh & Gupta, 2018b).

Example 5.2: Consider a constant-sum linguistic game with payoffs from the linguistic terms

$$\begin{aligned}
 LT = \{ & \ell_0 : \text{Very Poor (VP)}, \ell_1 : \text{Poor(P)}, \\
 & \ell_2 : \text{Moderately Poor(MP)}, \ell_3 : \text{Fair(F)}, \\
 & \ell_4 : \text{Moderately Good(MG)}, \ell_5 : \text{Good (G)}, \\
 & \ell_6 : \text{Very Good (VG)} \}
 \end{aligned}$$

Suppose

$$\tilde{A} = \begin{bmatrix} MG & MG & F & F & (MG, 0.3) \\ MG & (F, -0.4) & (VG, -0.1) & MG & VP \\ (F, 0.4) & VG & MG & MG & MG \end{bmatrix}$$

is the payoff matrix of player I, and since it is a constant-sum (more precisely, zero-sum) game, the payoff matrix for player II would be $\tilde{B} = \text{neg}(\tilde{A})$. This means that,

$$\tilde{B} = \text{neg}(\tilde{A}) = \begin{bmatrix} MP & MP & F & F & (MP, -0.3) \\ MP & (F, 0.4) & (VP, 0.1) & MP & VG \\ (F, -0.4) & VP & MP & MP & MP \end{bmatrix}$$

Applying Δ^{-1} to the matrices $\tilde{A}$ and $\tilde{B}$ and based on the commands given in Figure 1, we get the optimal solution of the above problem as presented in Table 3. This implies that the expected payoffs of players I and II are $\tilde{p} = E_I(\Phi, \Psi) = \Delta(p) = (\ell_4, -0.3343)$ and $\tilde{q} = E_{II}(\Phi, \Psi) = \Delta(q) = (\ell_2, 0.3343)$. Thus, the lower value of the game would be $\tilde{p} = (\ell_4, -0.3343)$ and the upper value of the game would be $\text{Neg}(\tilde{q}) = (\ell_4, -0.3343)$.

Theorem 5.3: The computation time required to compute the Nash equilibrium of a linguistic two-person non-zero sum game having entries as linguistic 2-tuples using the proposed method has complexity of $\mathcal{O}(m + n + mn + \min(m^2, n^2))$.

Proof: In this proof, we derive the computation time to solve a linguistic bimatrix game using computation procedure given in Section 4.3.

Consider the payoff matrices of player I and II as $\tilde{A}_{ij}$ and $\tilde{B}_{ij}$, respectively, where i runs from 1 to n and j runs from 1 to m . Step 1 takes the input of two matrices making $2nm$ operations. Step 5 solves the nonlinear programme with the help of Maple, and its complexity is as given in Table 4. In total complexity of the defined methodology is $\mathcal{O}(2mn) + \mathcal{O}(mn) + \mathcal{O}(m) + \mathcal{O}(n) + \mathcal{O}(mn) + \mathcal{O}(m) + \mathcal{O}(n) + \mathcal{O}(\min(m^2, n^2)) = \mathcal{O}(m + n + mn + \min(m^2, n^2))$. ■

6. Comparison and discussion analysis

The present section aims to discuss the comparative analysis of the proposed model by discussing the simplicity of our methodology with the existing methods given in the literature.

6.1. Comparison of the proposed approach with the approach used to solve 'matrix games with linguistic payoffs'

Singh and Gupta (2018b) discussed a two-player constant-sum 2-TL matrix game in their paper, in which such constant-sum matrix games are solved that have payoffs consisting of 2-TL variables. The authors constituted a linguistic linear programming (LLP) approach

Table 3. Optimal solution of the problem given in Example 5.2.

Φ	Ψ	$E_I(\Phi, \Psi)$ (p)	$E_{II}(\Phi, \Psi)$ (q)
(0.3343, 0.1086, 0.5571)	(0.5571, 0, 0, 0.3593, 0.0836)	3.6657	2.3343

Table 4. Computational time complexity of Maple programme.

Constraints	$\mathcal{O}(mn) + \mathcal{O}(mn) + \mathcal{O}(m) + \mathcal{O}(n) = \mathcal{O}(mn) + \mathcal{O}(m) + \mathcal{O}(n)$
Objective	$\mathcal{O}(mn) + \mathcal{O}(mn) - \mathcal{O}(1) - \mathcal{O}(1) = \mathcal{O}(mn) + \mathcal{O}(m) + \mathcal{O}(n)$
QPSolve/NLPSolve	$\mathcal{O}(\min(m^2, n^2))$

to solve these types of games. It is interesting to know that, since constant-sum games are a particular case of non-zero sum games, we can also use our proposed approach to solve the games introduced by them. According to the results obtained in the corresponding paper, the optimal values, that is, the expected payoffs of player I and player II of Example 5.2 are $(\ell_4, -0.3343)$ and $(\ell_2, 0.3343)$ respectively. Moreover, the value of the game is $(\ell_4, -0.3343)$. The obtained optimal solution is analogous to the results shown in our paper. This exhibits the consistency of the developed approach used for solving the non-zero sum as well as zero-sum matrix games having 2-TL payoffs in the interval $[0, g]$ as well.

Besides, our methodology proffers a generalisation of the technique developed by the authors in the paper (Singh & Gupta, 2018b). For solving complex decision-making problems, the methodology of our paper are more simpler and practical than the other general game solution models. Despite the fact that there are several methodologies available in the literature to solve constant-sum linguistic matrix games, the approach proposed by us is an improvement to offer acceptable results at a faster rate. This could happen because one can find the results using a single nonlinear programme only, for the task of forming two separate optimisation problems for both the players have now been reduced and the safety values of both players as well as the value of the game may now be computed using one nonlinear programme, which we can solve easily using Maple code provided in the text. Moreover, the proposed approach is capable enough to solve comprehensive game problems. If such a nonlinear programme was not constructed, in that case, we had to construct two nonlinear optimisation problems, which might be somewhat tedious for solving large-scale matrix game problems.

6.2. Comparison of the proposed approach with the approach used to solve 'fuzzy bimatrix games'

Larbani (2009) introduced bimatrix games having fuzzy payoffs and provided a solution method to solve such games. For this, the game is first defuzzified by taking α -cuts of the payoffs. The resulting game is then treated as the game with uncertainty. This concept of defuzzification converts the fuzzy payoffs into numerical ones by using the process of α -cut. This approach is limited by its representation and the solution methods on linguistic terms. This limitation is the loss of information occurred because of expressing the results in the initial expression domain, and this loss implicates a lack of accuracy occurred in the results due to aggregation of linguistic information. However in our work, this limitation

is overcome by expressing the linguistic information as 2-tuples that allow a continuous form of the linguistic data in its domain. Along with the 2-tuple representation of payoffs, we establish a solution method for computing with words in bimatrix games causing no loss of information. The loss of information caused by the need to express the results in the initial expression domain is discrete in the former paper. This loss is easily dealt with by the Δ function in our paper, which describes the conclusion in the original domain using Δ^{-1} function, without causing a single information loss in a continuous domain.

Hence to conclude, this study provides the concepts of linguistic bimatrix games, which is considered to be a new study in the literature addressing such types of games. Considering the importance of the methodology proposed and its relevance in the theory of bimatrix games in the real-time scenario, it emphasises us to take forward the present theory to solve games with incomplete information to handle the uncertainty and vagueness present in the real-world problems.

7. Conclusions

In this paper, we introduced a new explanation as well as the existence of Nash equilibrium in linguistic bimatrix games. Here, the elements of payoff matrices are in the form of descriptive words that characterise 2-TL variables. For this purpose, a methodology to solve linguistic bimatrix games is suggested to establish the solution. This approach is used to find the mixed Nash equilibrium as well as the maximised expected payoffs of both players at the same time. Instead of forming two auxiliary equations for both players separately, we have constructed a single nonlinear programme which is supporting the computations of both players simultaneously. Moreover, the computations become easier to perform because of the Maple programming code used for solving the nonlinear programming problem. The payoffs belong to a predefined LT set defined in the interval $[-g, g]$. The proposed methodology would be very much helpful in solving two-person constant-sum linguistic matrix games with payoffs in $[0, g]$ also and this is depicted with the help of an example in the paper as well. Moreover, our method is beneficial and supportive for solving large-scale two-person game problems.

Future Scope. The proposed approach is advantageous in solving two-person non-zero sum linguistic games, where the payoffs belong to the linguistic term set having balanced or symmetrical distribution data. So, in the near future, we aim to extend this methodology to linguistic games where the data belongs to an asymmetrical or unbalanced linguistic term set. We also envisage studying more advanced topics, for instance, accomplishing

a more efficient algorithm to solve cooperative games incurring linguistic variables to implement the ideas in more general cases. Moreover, further study can also be focussed on constructing dynamics in repeated bimatrix games. A dynamic game is played over many time phases. A bimatrix game requires solving at every possible system state in every phase. As a result, many bimatrix games must be computed over various time phases. This originates the demand for an efficient computational method that can be used to determine Nash equilibrium points of dynamic bimatrix games. These problems are worth being explored from the beginning.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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A novel cloud model based on multiplicative unbalanced linguistic term set

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Abstract

The cloud decision support model is an effective technique for describing an ambiguous mapping between qualitative notions and their quantitative manifestations. However, the existing research on cloud models is based on balanced linguistic term set and cannot address information in an unbalanced linguistic form. Henceforth, to thoroughly capture the intrinsic fuzziness and vagueness of decision information an accurate decision model is required to acquire good solution. In this direction, the proposed work aims to introduce a cloud model based on multiplicative unbalanced linguistic term sets. To match our research requirements, we initially propose a multiplicative unbalanced point-based cloud model. The innovation of the model overcomes the inherent limitation of the existing cloud and is competent of producing new solution according to the cloud model theory. Furthermore, a generalized multiplicative unbalanced linguistic scale function is introduced. The proposed generalized function creates a map between the linguistic data and numerical interval values. Based on this rational conversion method, interval cloud variables are transformed, which overcomes the loss of information existing in the point-based cloud models where linguistic variables are transformed into a specific numerical value. The process is then used to demonstrate the accuracy and efficacy of the suggested model using a numerical example, which is then followed by a comparison study. The overall results typically show that the cloud model has excellent performance and provides better solutions.

Keywords Normal cloud model · Asymmetric trapezoidal clouds · Multiplicative unbalanced interval cloud · Unbalanced linguistic term set

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1 Introduction

Several decision analysis problems under uncertain conditions encounter the challenge of ultimately representing decision-makers' (DMs) opinions and preferences. Given the vague information incorporated within those problems or the high computational cost required to express precise numerical data, the DMs do not describe their evaluations in terms of quantitative descriptors but rather in linguistic form. The requirement for linguistic data is excellently suited because it is simple to distinguish uncertainty with a non-probabilistic nature. As a result, numerous researchers have researched it [1, 2]. In order to demonstrate a pertinent idea involved in expressing issues in terms of natural languages, Zadeh [3] introduced linguistic variables. As the basis of human thought and the embodiment of human intellect, natural languages are crucial to understanding difficulties. In the present specialized literature on linguistic decision analysis and Computing with Words (CWW)-related approaches, the utility of linguistic ideas implies a broad outlook. Many linguistic computational models [4, 5] were created based on symmetrical and balanced linguistic data to prevent the information loss occurring in CWW operations.

Nevertheless, as the complications within the practical decision-making issues increase, it is arduous to reflect the qualitative information in a balanced linguistic term set. Taking investment returns as an example where investors care about the distribution of investment return data. Empirically, the investment return data follow an asymmetric distribution scheme. This is because investment performance is usually skewed by periods of high market volatility or unusual fiscal and monetary policies during which returns growth are abnormally high or low. Considering the irregularity of the data, it is envisioned that the existing linguistic models are not capable enough to represent asymmetric information effectively. To resolve this issue, a series of research is conducted, for instance, [6–14] to proficiently describe qualitative concept having an irregular spacing of linguistic labels. Henceforth, the developed content related to unbalanced linguistic information has made a significant impact on linguistic decision-making theories. As a popular tool for modelling the qualitative assessment information, the theories [15–21] allow the decision makers to work under uncertain linguistic environment. The use of uncertain linguistic information enriches the flexibility of DMs in decision-making [22–25] scenarios. Considering the existing literature of unbalanced linguistic computational models, it is identified that the representation of unbalanced information is presented in terms of a uniformly distributed set of linguistic terms. A new consensus-reaching approach for MCGDM with multi-granular unbalanced linguistic information, for instance, is developed by Zhang et al. [26] that simultaneously takes into account the adjustment amount and limited confidence levels of DMs. To do this, the authors first use the model provided in [9] to determine the numerical scales of multi-granular unbalanced linguistic terms, and then, they convert these terms into linguistic 2-tuples defined on the same balanced linguistic term set. A clustering approach based on LDAs is subsequently developed to categorize decision-makers in the paper [27], which also converts all multi-granular unbalanced hesitant fuzzy linguistic data into

LDAs defined on a balanced linguistic word set. The results are then used to compute the overall ratings for each cluster. In this way, the accurate sense of “unbalanced” is not well represented. To remedy this, Malhotra and Gupta [28] introduced a novel 2-tuple multiplicative unbalanced linguistic computational model addressing a substantial amount of linguistic information in terms of a globally inconsistent multiplicative unbalanced linguistic scale set. This set is primarily based on the notion of minimum distance measure. It is anticipated that under complicated decision-making circumstances, the proposed model is computationally less expensive and reduces time complexity.

Addressing decision-making issues with linguistic information to apprehend the uncertainty has received much interest. However, with rapid societal development, linguistic decision-making problems have become increasingly complex and challenging. In such a scenario, the cognitive cloud models proposed by Li et al. [29–31] are lucrative for defining linguistic labels as they can synthetically explain the notion of randomness and fuzziness of the decision information simultaneously.

Since the introduction of cloud models, several researchers have devoted their efforts to enhance cloud specialized studies such as integrated cloud [32], risk assessment [33, 34], water quality assessment [35] along with more wide-ranging applications [36–43]. The experts introduce some methods for applying cloud models to MCDM [44–49]. The cloud models outline a linguistic concept using three numerical properties, namely *EX* (expectation), *EN* (entropy), and *HE* (hyper-entropy). To realize the bidirectional cognitive transformation of a notion [50–52], two cloud transformations, forward cloud transformation and backward cloud transformation, are used. In addition to propose a second-order generic forward normal cloud transformation technique, Wang et al. [53] provided a generic cloud model that established a connection between the normal cloud and the normal distribution. This transformation algorithm reveals that the second-order normal cloud [54] and the normal distribution are both special cases of the second-order generic normal cloud. A linguistic assessment scale is defined by Wang et al. [49] to perform the transformation among the clouds and the linguistic variables. Although it is envisioned that the existing cloud models can retain the original information as much as possible, they still cannot create a good representation for the concepts where linguistic labels are asymmetrically distributed. To resolve this issue, Wang et al. [55] proposed an extended cloud model, namely the Asymmetric Trapezoidal Clouds (ATCs) model. The *EN* and *HE* of the ATCs are different on the left and right sides in this model; thus, the authors gave expectation as an interval rather than a precise number. The authors claim that ATCs are excellent quantitative information representations. However, unlike unbalanced linguistic assessments, ATCs are not promptly used in decision-making due to their non-existence in a practical scenario. Consequently, the authors link ATCs to unbalanced linguistic distribution assessments (ULDAs). The authors also provided an extended linguistic scale function (LSF), which indicates variations in spacing rather than symmetrical spacing among linguistic terms, to map linguistic terms to interval values. The expectation interval value of ATCs can be computed using the extended LSF. Also, a method is suggested for calculating the entropy and hyper-entropy values of ATCs developed using the 3σ concept.

Throughout the past, researchers working on cloud models representing balanced concepts were acknowledged for enhancing the literature on cloud models. As a kind of uncertain information representation in complex decision-making issues, these models are not competent to express unbalanced concepts well. In the light of the latter point, a number of computationally challenging unbalanced linguistic models have been proposed in the literature; as a result, solving complex decision problems becomes more time-consuming. Henceforth, the primary purpose of this current study is to develop an accurate and innovative cloud decision support model to address unbalanced linguistic concepts by emphasizing the succeeding momentous research gaps.

1.1 Research gap

- The ULDA utilized in the existing ATC model is primarily based on the hierarchical unbalanced linguistic model which is a very complex and computationally expensive approach.
- The existing theories apply to additive linguistic data only.
- It is crucial to choose a string of appropriate linguistic cardinalities in real fuzzy decision-making problems using linguistic variables. To the best of our knowledge, the majority of the existing literature uses linguistic cardinality with a uniformly distributed scale pattern, which means that any two neighbouring linguistic cardinalities have the same level of granularity. A symmetrically distributed linguistic term (LT) set, thus, provides the representation of the unbalanced linguistic judgements. As a result, the uneven linguistic information lacks its literal meaning.
- The use of the translation function to convert a balanced LT set into an unbalanced LT set leads to the complexity of the model.

To address above-mentioned practical issues, in our research, we provide two kinds of cloud models, i.e. point based and generalized interval cloud models where linguistic information is defined on a multiplicative scale set. The advantage of using a multiplicative scale set is that the set perfectly maintains unbalanced information throughout the process. As a result, it is computationally cheaper and simpler than other linguistic computational models [1, 9]. Further, the proposed model manages the linguistic information effortlessly, reducing the time and cost complexity. Additionally, the innovative model proposed in this paper reduces the risk of information loss occurring in the process of unbalanced decision-making. We next summarize the differences and similarities of the uncertain cloud models, and the characteristics of a few existing models in Table 1.

1.2 Contribution

The key contributions of this work can be summed up as follows in order to fulfil the aforementioned objectives.

Table 1 A summary of similarity and differences of various cloud models

Paper	Linguistic model implemented	Multiplicative	Unbalanced	Point based/interval
[29]	N/A	No	No	Point based
[49]	N/A	No	No	Point based
[47]	N/A	No	No	Point based
[44]	N/A	No	No	Partially interval (Expectation values are interval based, rest values point based)
[55]	Hierarchy model [1]	No	Yes	Both
Current paper	Unbalanced 2-tuple linguistic model [28]	Yes	Yes	Both

- The research gap is filled by introducing a multiplicative unbalanced linguistic point-based and interval cloud model that makes use of an unbalanced LT set which is computationally less expensive than other existing computational models.
- Extended multiplicative unbalanced linguistic scale functions have been developed to transform an unbalanced linguistic term into an interval value. The so-developed extended multiplicative unbalanced linguistic scale function efficiently embodies the differences in spacing of linguistic terms, both in terms of representation and meaning.
- A method based on the 3σ principle is provided to determine entropy and hyper-entropy of the extended multiplicative unbalanced interval cloud model. The entropy and hyper-entropy thus defined differ from the ones defined in [49] as the former is based on multiplicative unbalanced linguistic data.
- Based on the concept of generalized multiplicative unbalanced linguistic interval cloud, our research develops the comparison method to rank the cloud variables.
- The proposed theory contributes to the existing decision-making theories [48, 56, 57] by solving a decision-making real-life problem.

The rest of the paper is organized as follows. Section 2 introduces some basic concepts related to normal cloud model and unbalanced linguistic computational model. In Sect. 3, a method for transformation between the unbalanced linguistic terms and clouds is proposed. Section 4 introduces the generalized multiplicative unbalanced interval cloud model. In Sect. 5, some numerical example along with comparison analysis is given to demonstrate the consistency of the defined model. Section 6 offers the conclusions of the paper.

Before proceeding further in this article, we introduce some common abbreviations used in the paper in Table 2.

2 Preliminaries

In this section, we first review some basic concepts of the normal cloud model along with the transformation of linguistic variables into normal clouds. Then, we outline the ATC model and some essential LSFs that are required in our paper. Finally, the multiplicative unbalanced linguistic term set is presented along with its properties.

2.1 Basic concepts of the normal cloud model

The normal cloud model, as described by Li et al. [29], is a cognitive model that implements the uncertain translation between linguistic concepts and quantitative values and describes the uncertainties of linguistic concepts, particularly the randomness and fuzziness.

Table 2 Abbreviation and symbols used throughout the paper

Abbrevia- tion/ Symbol	Definition/Meaning	Characteristic
CWW	Computing with Words	A methodology in which words are used in place of numbers for computing & reasoning
DM	Decision maker	Anybody who makes decisions or takes part in decision making process
MCDM	Multi-Criteria Decision Making	Decision-making problems based on multiple criteria
MCGDM	Multi-Criteria Group Decision Making	Decision-making problems based on multiple criteria involving multiple groups
LDA	Linguistic Distribution Assessments	Linguistic information represented in terms of symbolic proportions assigned to all linguistic terms
EX	Expectation	Mathematical expectation of the cloud drops belonging to a concept in the universe
EN	Entropy	The uncertainty measurement of a qualitative concept
HE	Hyper-entropy	The uncertain degree of entropy EN
ATC	Asymmetric Trapezoidal Cloud	A numerical concept developed to describe linguistic information
LSF	Linguistic Scale Function	A function that maps linguistic terms to numerical values
MULSF	Multiplicative Unbalanced Linguistic Scale Function	A function that maps multiplicative unbalanced linguistic terms to numerical values
LT	Linguistic Term	A term whose value is not number but words or sentences in a natural or artificial language
ULTS	Unbalanced Linguistic Term Set	A set whose members are linguistic terms distributed in an unbalanced or asymmetric manner
2-TLCM	2-tuple linguistic computational model	A novel 2-tuple model for unbalanced linguistic information based upon a notion of minimum distance measures
MUIC	Multiplicative Unbalanced Interval Cloud	A cloud in interval form transformed from multiplicative unbalanced linguistic terms
t_d	Multiplicative Unbalanced Linguistic Term	Notation of LT
δ_d	A numerical value corresponding to t_d	Belongs to the interval $[0, \frac{1}{\log a}]$
$\tilde{\mathfrak{U}}$	LSF	A function that transforms t_d to δ_d
$\underline{\delta}_d$	Lower value of δ_d in the interval $[\underline{\delta}_d, \bar{\delta}_d]$	–
$\bar{\delta}_d$	Upper value of δ_d in the interval $[\underline{\delta}_d, \bar{\delta}_d]$	–
$\tilde{\mathfrak{C}}$	LSF	A function that transforms t_d to $\underline{\delta}_d, \bar{\delta}_d$
EX_d	Lower value of expectation (Entropy/ hyper-entropy) in	–

Table 2 (continued)

Abbrevia- tion/ Symbol	Definition/Meaning	Characteristic
$(\overline{EN}_{dt}/\overline{HE}_{dt})$	the expectation (entropy/hyper-entropy) interval	
$\overline{EX}_{dt}$	Upper value of expectation (entropy/hyper-entropy)	–
$(\overline{EN}_{dt}/\overline{HE}_{dt})$	in the expectation (entropy/hyper-entropy) interval	
var_{Int}	Variance of interval cloud droplets	Given by a formula
sd_{Int}	Standard deviation of interval cloud droplets	Given by a formula
sc_{Int}	Interval score function	A coefficient to capture the attitude towards standard deviation

Definition 1 [29] Let U be the universe of discourse, and $\tilde{T}$ be a qualitative concept defined over U . If $x \in U$ is a random instantiation of the concept $\tilde{T}$ that satisfies $x \sim N(EX, EN'^2)$, $EN' \sim N(EN, HE^2)$, and $y \in [0, 1]$ defined as

$$y = e^{-\frac{(x-EX)^2}{2(EN')^2}}$$

be the certainty degree of x , then the distribution of x in the universe U is called as a normal cloud and (x, y) is called a normal cloud drop. Here,

- EX is the mathematical expectation of the cloud drops.
- Entropy EN represents the uncertainty measurement of the qualitative concept.
- Hyper-entropy HE is the uncertain degree of entropy EN .

Given the numerical attributes (EX, EN, HE) of a cloud, the forward generator of this cloud is given in the following algorithm:

Forward normal cloud generator

Input: three parameters EX, EN, HE and the number of cloud drops c .

Output: c cloud drops and their certainty degree.

Steps:

1. Generate a normally distributed random number EN'_i with expectation EN and variance HE^2 .
 2. Generate a normally distributed random number x_i with expectation EX and variance EN'_i .
 3. Calculate $y_i = e^{-\frac{(x_i-EX)^2}{2(EN'_i)^2}}$.
 4. x_i is a cloud drop in the universe, and y_i is the degree of certainty of x_i belonging to the concept $\tilde{T}$.
 5. Repeat steps 1–4 until c cloud drops have been generated.
-

Next, the transformation between linguistic variables and clouds is presented by Wang et al. in [49]. The challenging problem of transforming the linguistic variables into normal clouds was first dealt with by Wang et al. [48] based on the golden ratio. But this method had a limitation associated that it could transform an LT set possessing five labels only. To overcome this limitation, Wang et al. [49] introduced a new method as given below.

Definition 2 [49] Let $\mathbb{T} = \{t_i | i = -n, \dots, 0, \dots, n, n \in N^*\}$ be a finite subscript-symmetric LT set with odd cardinality, where N^* = the set of positive integers, t_i denotes the value of a linguistic variable, t_{-n} and t_n are the lower and upper limits of LT set. The LSF running the mapping from t_i to θ_i ($i = -n, \dots, 0, \dots, n$) is given as:

$$f : t_i \rightarrow \theta_i$$

$$\theta_i = \begin{cases} \frac{a^n - a^{-i}}{2a^n - 2}, & -n \leq i \leq 0 \\ \frac{a^n + a^i - 2}{2a^n - 2}, & 0 < i \leq n \end{cases} \quad (1)$$

Here, $\theta_i \in [0, 1]$ is a numeric value. Also, the value of a can be calculated with the help of experiments. Based on various experimental studies, the value of a is mostly obtained in the interval $[1.36, 1.4]$. The procedure for transforming linguistic labels into $2n + 1$ number of clouds is given as follows. Assume the subscript-symmetric LT set to be $\mathbb{T} = \{t_i | i = -n, \dots, 0, \dots, n, n \in N^*\}$, and a valid universe $[X_{\min}, X_{\max}]$ as provided by the experts. When the cloud model is used to express linguistic values, a corresponding cloud variable $Y_i(EX_i, EN_i, HE_i)$ is formulated for each arbitrary $t_i \in \mathbb{T}$. The following is the process for creating cloud variables.

1. Calculate θ_i . Map t_i to θ_i using Equation (1).
2. Calculate EX_i as $EX_i = X_{\min} + \theta_i(X_{\max} - X_{\min})$.
3. Calculate EN_i as

$$EN_{-i} = EN_i = \begin{cases} \frac{(\theta_{|i-1|} + \theta_{|i|} + \theta_{|i+1|})(X_{\max} - X_{\min})}{9} & 0 < |i| \leq n - 1 \\ \frac{(\theta_{|i|} + \theta_{|i-1|})(X_{\max} - X_{\min})}{9} & |i| = n \\ \frac{(\theta_i + 2\theta_{i+1})(X_{\max} - X_{\min})}{9} & i = 0 \end{cases}$$

4. Calculate HE_i . Due to $EN'_i \sim N(EN_i, HE_i^2)$, the hyper-entropy must follow the 3σ principle of the normal distribution curve. Hence, $HE_{-i} = HE_i = (EN'^+ - EN_i)/3$, where $EN'^+ = \max_k \{EN'_k\}$.

Next, we introduce the asymmetric trapezoidal cloud (ATC) method as defined by Wang et al. [55] in which the authors describe the values of expectation, entropy, and hyper-entropy as interval values instead of specific ones. Moreover, this model bridges ULDAs (common information representations in group decision-making) and ATCs. Also, the ATC model overcomes the limitations of the normal cloud model.

Definition 3 [55] The ATC model is denoted by z and is given by

$$z = (\underline{HE}, \underline{EN}, \underline{EX}, \overline{EX}, \overline{EN}, \overline{HE})$$

The expectation interval $[\underline{EX}, \overline{EX}]$, the left half cloud $(\underline{HE}, \underline{EN}, \underline{EX}, 0, 0)$, and the right half cloud $(0, 0, \overline{EX}, \overline{EN}, \overline{HE})$ constitute this model. $\underline{EX}, \underline{EN}, \underline{HE}$ and $\overline{EX}, \overline{EN}, \overline{HE}$ are the expectation, entropy and hyper-entropy of the left and right half cloud, respectively. Also, the range of the qualitative notions is explained by the interval $[\underline{EX}, \overline{EX}]$ explains the range of the qualitative notions. The level of uncertainty of a qualitative concept is shown by the $\underline{EN}$ and $\overline{EN}$, respectively, while the entropies' level of uncertainty is shown by $\underline{HE}$ and $\overline{HE}$.

In comparison with the normal cloud model, the ATC model has novel advantages instinctively represented in Fig. 1. For detailed knowledge of the ATC model, one can refer to the paper [55].

It is well known that the operations cannot be carried out directly when numbers combine with linguistic terms. So there is a need to define LSFs that assign

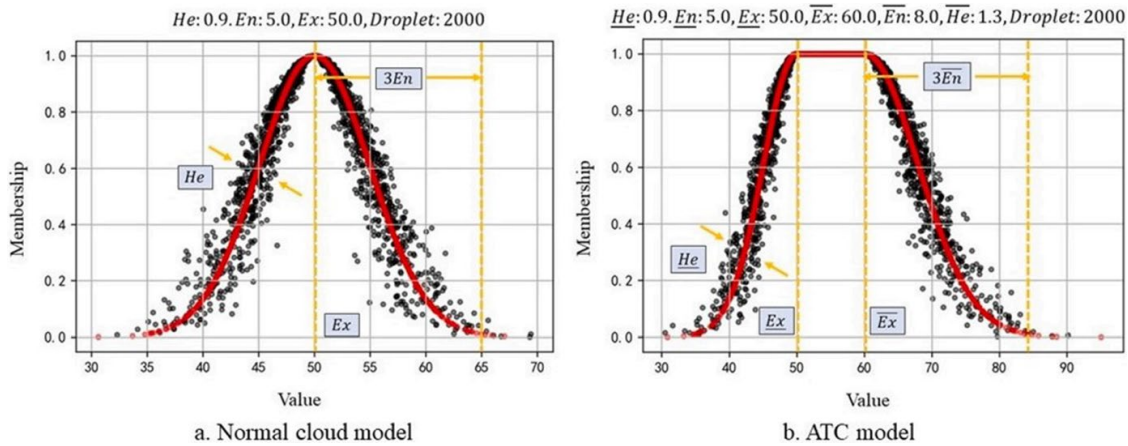


Fig. 1 Comparison between normal cloud model and ATC model [55]

different semantic values to linguistic terms under certain situations. Wang et al. [43] defined different LSFs for different cases, but their research includes a transformation from a linguistic term to a single number. However, it does not sufficiently outline the uncertain information in some conditions. Hence, to avoid this information loss, Wang et al. [42] defined some LSFs that convert linguistic variables to interval numbers. These LSFs are described below.

Definition 4 [42] Let $\mathbb{L} = \{t_i | i = 0, 1, \dots, 2n, n \in \mathbb{N}\}$ be an LT set and $t_i \in \mathbb{L}$ be a linguistic variable. When $[\underline{\theta}_i, \bar{\theta}_i]$ is an interval fuzzy number, the LSF runs a mapping given as:

$$f : t_i \rightarrow [\underline{\theta}_i, \bar{\theta}_i], i = 0, 1, \dots, 2n$$

$[\underline{\theta}_i, \bar{\theta}_i]$ satisfies the relationship $0 \leq \underline{\theta}_0 \leq \bar{\theta}_0 \leq \underline{\theta}_1 \leq \bar{\theta}_1 \leq \dots \leq \underline{\theta}_{2n} \leq \bar{\theta}_{2n} \leq 1$. To meet this property, the mapping rule is improved and shown below:

$$\begin{aligned} H : t_0 &\rightarrow [\underline{\theta}_0, \bar{\theta}_0]; \\ t_i &\rightarrow [\max\{\bar{\theta}_{i-1}, \underline{\theta}_i\}, \bar{\theta}_i], i = 1, 2, \dots, n; \\ t_i &\rightarrow [\underline{\theta}_i, \min\{\bar{\theta}_i, \underline{\theta}_{i+1}\}], i = n + 1, \dots, 2n - 1; \\ t_{2n} &\rightarrow [\underline{\theta}_{2n}, \bar{\theta}_{2n}]. \end{aligned}$$

Hence, the improved LSF is:

$$LSF : H_2(t_i) = [\underline{\theta}_i, \bar{\theta}_i]$$

$$\underline{\theta}_i = \begin{cases} \frac{(a^n - a^{n-i})}{2(a^n - 1)}, & i = 0, 1, \dots, n \\ \frac{(a^n + a^{i-n} - 2 - \lambda_3(i-n))}{2(a^n - 1)}, & i = n + 1, \dots, 2n \end{cases} \quad (2)$$

$$\bar{\theta}_i = \begin{cases} \frac{(a^n - a^{(n-i)} + \lambda_4(n-i))}{2(a^n - 1)}, & i = 0, 1, \dots, n \\ \frac{(a^n + a^{i-n} - 2)}{2(a^n - 1)}, & i = n + 1, \dots, 2n \end{cases} \quad (3)$$

where λ_3 and λ_4 are adjustable parameters. Based on the above-mentioned LSF, the transformation method is conducted to map linguistic information into numerical values. For detailed understanding of the given transformation procedure, one may refer to the corresponding paper [55].

2.2 The unbalanced 2-tuple linguistic computational model (2-TLCM)

The unbalanced 2-TLCM is introduced by Malhotra, and Gupta [28] to cope with unbalanced linguistic information. To do so, an efficient method is designed to characterize the interpretation of the linguistic terms belonging to the unbalanced LT set.

Definition 5 [28] Let $\mathbb{LS} = \{t_{a^{-n}}, t_{a^{-(n-1)}}, \dots, t_{a^0}, \dots, t_{a^m}\}$, $a > 1$, $n, m \in \mathbb{Z}^+$ be a multiplicative unbalanced LT set having cardinality $n + m + 1$ such that n and m may not be equal. Moreover, the set $\mathbb{LS}$ is ordered if the given properties hold:

- $\mathbb{LS}$ is an ordered set: $t_{a^i} \geq t_{a^j}$ for $i \geq j$;
- $\max(t_{a^i}, t_{a^j}) = t_{a^i}$ for $i \geq j$;
- $\min(t_{a^i}, t_{a^j}) = t_{a^j}$ for $i \leq j$.

Further, the negation operator for such 2-tuples is defined as:

$$\text{Neg}(t_{a^i}, \phi) = (\text{Neg}(t_{a^i}), -\phi)$$

where $\text{Neg}(t_{a^i})$ is defined as follows:

- If $n = m$, then $\text{Neg}(t_{a^i}) = t_{a^{-i}}$
- If $n < m$, then $\text{Neg}(t_{a^i}) = t_{a^{-i}}$, for $i \leq n$ and $\text{Neg}(t_{a^i}) = t_{a^{m-i}}$ otherwise.
- If $n > m$, then $\text{Neg}(t_{a^{-i}}) = t_{a^i}$, if $i \leq m$ and $\text{Neg}(t_{a^{-i}}) = t_{a^{i-n}}$ otherwise.

For further explanation of the operators, one may refer to Algorithm 2 of Malhotra and Gupta [28].

Definition 6 [28] Let $\mathbb{LS}$ be the multiplicative unbalanced LT set as defined above and $\mathbb{R}$ be the set of real numbers. Then the numerical scale function

$$NS : \mathbb{LS} \rightarrow \mathbb{R}$$

such that

$$NS(t_{a^i}) = a^i$$

In the practical world scenario, several situations exist involving imprecise information. The uncertainty of such information is too complicated to explain with the help of quantitative expressions, but linguistic descriptors describe reality much better. However, the transformation between quantitative and qualitative concept values has always been a hurdle in developing linguistic decision-making methods. Modern researchers need a perfect way for CWWs. Our next section introduces a novel transformation method between the quantitative and unbalanced qualitative concept.

3 Method for transformation between multiplicative unbalanced linguistic variables and clouds

The present section introduces a novel multiplicative unbalanced linguistic scale function to transform multiplicative unbalanced linguistic variables into point-based unbalanced clouds.

Definition 7 Let $\mathbb{LS} = \{t_{a^i} | i = -n, \dots, 0, \dots, m\}$, $a > 1$ (n and m may or may not be equal) be any finite ordered multiplicative unbalanced LT set,

and a valid universe $[X_{\min}, X_{\max}]$. When the defined cloud model is applied to represent the multiplicative unbalanced linguistic variables, the corresponding clouds are represented as follows:

$$\begin{aligned} t_{a^{-n}} &\rightarrow (EX_{a^{-n}}, EN_{a^{-n}}, HE_{a^{-n}}) \\ &\vdots \\ t_{a^0} &\rightarrow (EX_{a^0}, EN_{a^0}, HE_{a^0}) \\ &\vdots \\ t_{a^m} &\rightarrow (EX_{a^m}, EN_{a^m}, HE_{a^m}) \end{aligned}$$

Definition 8 Let $\mathbb{LS} = \{t_{a^i} | i = -n, \dots, 0, \dots, m\}$, $a > 1$ (n and m may or may not be equal) be any finite ordered multiplicative unbalanced LT set. The LSF converting a linguistic term t_{a^i} to a certain number δ_{a^i} , ($i = -n, \dots, 0, \dots, m$) can be expressed as follows:

$$\tilde{g} : t_{a^i} \rightarrow \delta_{a^i} \quad (i = -n, \dots, 0, \dots, m)$$

where

$$\delta_{a^i} = \begin{cases} \left(\frac{b^n - b^{-i}}{2b^n - 2} \right) \frac{1}{\log a}, & -n \leq i \leq 0 \\ \left(\frac{b^n + b^i - 2}{2b^n - 2} \right) \frac{1}{\log a}, & 0 < i \leq n \end{cases} \quad (4)$$

Here, $\delta_{a^i} \in [0, 1/\log a] \subseteq [0, 1]$ is a numeric value. The value of the parameter b can be obtained through experiments such as $b \in [1.36, 1.4]$ [58].

We next describe the following theorem giving the specific properties of the above-mentioned multiplicative unbalanced LSF.

Theorem 1 *Based on the multiplicative unbalanced linguistic scale function (MULSF), the following properties hold:*

1. If $-n \leq i \leq 0$, then $0 \leq \delta_{a^i} \leq \frac{1}{2 \log a}$
2. If $0 < i \leq n$, then $\frac{1}{2 \log a} < \delta_{a^i} \leq \frac{1}{\log a}$
3. If $i + j = 0$, then $\delta_{a^i} + \delta_{a^j} = \frac{1}{\log a}$.

Proof

1. For given $-n \leq i \leq 0$, we have

$$\delta_{a^i} = \left(\frac{b^n - b^{-i}}{2b^n - 2} \right) \frac{1}{\log a}$$

By setting $i = -n$ in the above equation, we have

$$\delta_{a^{-n}} = \left(\frac{b^n - b^n}{2b^n - 2} \right) \frac{1}{\log a} = 0 \text{ as } n \rightarrow \infty$$

By setting $i = 0$, we have

$$\delta_{a^0} = \left(\frac{b^n - b^0}{2b^n - 2} \right) \frac{1}{\log a} = \frac{1}{2 \log a} \text{ as } n \rightarrow \infty$$

Next, by setting $-n < i < 0$, we have

$$\delta_{a^i} = \left(\frac{b^n - b^{-i}}{2b^n - 2} \right) \frac{1}{\log a}$$

Taking limits on both sides, we get

$$\lim_{n \rightarrow \infty} \delta_{a^i} = \lim_{n \rightarrow \infty} \left(\frac{b^n - b^{-i}}{2b^n - 2} \right) \frac{1}{\log a} = \frac{1}{2 \log a}$$

Therefore, the limiting value of the function δ_{a^i} is obtained as

$$\lim_{n \rightarrow \infty} \delta_{a^i} = \begin{cases} 0, & i = -n \\ \frac{1}{2 \log a}, & -n < i \leq 0 \end{cases}$$

2. Based on Definition 8, for $0 < i \leq n$, we have

$$\delta_{a^i} = \left(\frac{b^n + b^i - 2}{2b^n - 2} \right) \frac{1}{\log a}$$

By setting $i = 0$,

$$\delta_{a^0} = \left(\frac{b^n + 1 - 2}{2b^n - 2} \right) \frac{1}{\log a} = \frac{1}{2 \log a} \text{ as } n \rightarrow \infty.$$

By setting $i = n$

$$\delta_{a^n} = \left(\frac{b^n + b^n - 2}{2b^n - 2} \right) \frac{1}{\log a} = \frac{1}{\log a} \text{ as } n \rightarrow \infty.$$

By setting $0 < i < n$, then

$$\delta_{a^i} = \left(\frac{b^n + b^i - 2}{2b^n - 2} \right) \frac{1}{\log a}$$

On rearranging the terms and taking limit on both sides, we get

$$\lim_{n \rightarrow \infty} \delta_{a^i} = \frac{1}{2 \log a}$$

3. Based on Definition 8,

$$\delta_{a^i} = \begin{cases} \left(\frac{b^n - b^{-i}}{2b^n - 2} \right) \frac{1}{\log a}, & -n \leq i \leq 0 \\ \left(\frac{b^n + b^i - 2}{2b^n - 2} \right) \frac{1}{\log a}, & 0 < i \leq n \end{cases}$$

For $i + j = 0$, we have $j = -i$. Therefore,

$$\delta_{a^i} + \delta_{a^j} = \left(\frac{b^n - b^{-i}}{2b^n - 2} \right) \frac{1}{\log a} + \left(\frac{b^n + b^i - 2}{2b^n - 2} \right) \frac{1}{\log a}$$

Adding and subtracting the term b^{-i} in the above expression, and by rearranging the terms, we obtain

$$\delta_{a^i} + \delta_{a^j} = \frac{1}{\log a}$$

Hence the proof. $\square$

We will next introduce the process of generating cloud droplets based on the rational transformation method of unbalanced linguistic variables into clouds.

Let $t_{a^i} \in \mathbb{LS}$ be any arbitrary unbalanced linguistic term corresponding to the cloud drop $z_{a^i}(EX_{a^i}, EN_{a^i}, HE_{a^i})$. Then, the working cloud generation procedure is given as follows:

Step 1: Calculate δ_{a^i} using Eq. 4.

Step 2: Calculate EX_{a^i} .

$$EX_{a^i} = X_{\min} + \delta_{a^i}(X_{\max} - X_{\min})$$

Step 3: Calculate EN_{a^i} . Let us represent a cloud droplet as (x, y) , since $x \sim N(EX_{a^i}, EN_{a^i}^2)$, then as per the 3σ principle of the normal distribution curve, we have

$$3EN_{a^i}' = \max(X_{\max} - EX_{a^i}, EX_{a^i} - X_{\min})$$

Because of $EN' \sim N(EN, HE^2)$, EN_{a^i} may be considered the mean value of EN_{a^i}' corresponding to the clouds in consideration and their adjacent clouds.

$$EN_{a^{-i}} = EN_{a^i} = \frac{EN_{a^{i-1}}' + EN_{a^i}' + EN_{a^{i+1}}'}{3 \log a} \text{ for } |i| = 0, 1, \dots, n-1, \quad (5)$$

and for $|i| = n$,

$$EN_{a^{-n}} = EN_{a^n} = \frac{EN_{a^n}' + EN_{a^{n-1}}'}{2 \log a} \quad (6)$$

These calculations may be rewritten as the following formulas:

$$EN_{a^i}' = \begin{cases} \frac{(1-\delta_{a^i})(X_{\max}-X_{\min})}{3 \log a}, & -n \leq i \leq 0 \\ \frac{\delta_{a^i}(X_{\max}-X_{\min})}{3 \log a}, & 0 < i \leq n \end{cases} \quad (7)$$

$$EN_{a^{-i}} = EN_{a^i} = \begin{cases} \frac{(\delta_{a^{|i-1|}} + \delta_{a^{|i|}} + \delta_{a^{|i+1|}})(X_{\max}-X_{\min})}{9 \log a}, & 0 < |i| < n-1 \\ \frac{(\delta_{a^{|i|}} + \delta_{a^{|i-1|}})(X_{\max}-X_{\min})}{6 \log a}, & |i| = n \\ \frac{(\delta_{a^i} + 2\delta_{a^{i+1}})(X_{\max}-X_{\min})}{9 \log a}, & i = 0 \end{cases} \quad (8)$$

Step 4: Calculate HE_{a^i} . Due to $EN_{a^i}' \sim N(EN_{a^i}, HE_{a^i}^2)$, the hyper-entropy satisfies the 3σ principle. In particular,

$$HE_{a^{-i}} = HE_{a^i} = \frac{EN_{a^i}'^+ - EN_{a^i}}{3 \log a} \quad (9)$$

where, $EN_{a^i}'^+ = \sup_k \{EN_{a^i}'\}$.

Next, we present the working example to explain the methodology mentioned above.

Example 1 Consider the universe $[X_{\min}, X_{\max}] = [0, 9]$ and let the unbalanced LT set can be given as

$$\begin{aligned} \mathbb{L}\mathbb{S} = \{ & t_{2^{-4}} : \text{None}, t_{2^{-3}} : \text{extremely poor}, t_{2^{-2}} : \text{very poor}, \\ & t_{2^{-1}} : \text{poor}, t_{2^0} : \text{avg}, t_{2^1} : \text{good}, t_{2^2} : \text{very good}, \\ & t_{2^3} : \text{extremely good}, t_{2^4} : \text{excellent} \} \end{aligned}$$

The procedure of transforming unbalanced linguistic variables into clouds is presented in the following manner:

1. Based on Definition 8, setting $b = 1.4$, the value for $\delta_{2^i} (i = -4, -3, \dots, 0, 1, 2, \dots, 4)$ can be computed as

$$\begin{aligned} \delta_{2^{-4}} &= 0, \delta_{2^{-3}} = 0.6416, \delta_{2^{-2}} = 1.0998, \delta_{2^{-1}} = 1.4271, \\ \delta_{2^0} &= 1.6609, \delta_{2^1} = 1.8948, \delta_{2^2} = 2.2221, \\ \delta_{2^3} &= 2.6804, \delta_{2^4} = 3.3219 \end{aligned}$$

- 2.

$$\begin{aligned} Ex_{2^{-4}} &= 0.0625, Ex_{2^{-3}} = 10.2875, Ex_{2^{-2}} = 17.5910, \\ Ex_{2^{-1}} &= 22.8078, Ex_{2^0} = 26.5341, Ex_{2^1} = 30.2604, \\ Ex_{2^2} &= 35.4772, Ex_{2^3} = 42.7808, Ex_{2^4} = 53.0057 \end{aligned}$$

- 3.

$$\begin{aligned} En'_{2^{-4}} &= 1.5992, En'_{2^{-3}} = 0.5732, \\ En'_{2^{-2}} &= -0.1596, En'_{2^{-1}} = -0.6831, En'_{2^0} = -1.0570, \\ En'_{2^1} &= 3.0302, En'_{2^2} = 3.5536, En'_{2^3} = 4.2865, En'_{2^4} = 5.3125 \end{aligned}$$

- 4.

$$\begin{aligned} En_{2^{-4}} &= 4.7995 & En_{2^{-3}} &= 4.042 & En_{2^{-2}} &= 3.3791 \\ En_{2^{-1}} &= 2.9056 & En_{2^0} &= 0.6840 & En_{2^1} &= 2.9056 \\ En_{2^2} &= 3.3791 & En_{2^3} &= 4.042 & En_{2^4} &= 4.7995 \end{aligned}$$

- 5.

$$\begin{aligned} He_{2^{-4}} &= 0.0515 & He_{2^{-3}} &= 0.1275 \\ He_{2^{-2}} &= 0.1940 & He_{2^{-1}} &= 0.2415 & He_{2^0} &= 0.4644 \\ He_{2^1} &= 0.2415 & He_{2^2} &= 0.1940 \\ He_{2^3} &= 0.1275 & He_{2^4} &= 0.0515 \end{aligned}$$

Thus, the 9-asymmetric linguistic clouds obtained are given as follows:

$$\begin{aligned} t_{2^{-4}} &\longrightarrow (0.0625, 4.7995, 0.0515), & t_{2^{-3}} &\longrightarrow (10.2875, 4.042, 0.1275), \\ t_{2^{-2}} &\longrightarrow (17.5910, 3.3791, 0.1940), & t_{2^{-1}} &\longrightarrow (22.8078, 2.9056, 0.2415), \\ t_{2^0} &\longrightarrow (26.5341, 0.684, 0.4644), & t_{2^1} &\longrightarrow (30.2604, 2.9056, 0.2415), \\ t_{2^2} &\longrightarrow (35.4772, 3.3791, 0.1940), & t_{2^3} &\longrightarrow (42.7808, 4.042, 0.1275), \\ t_{2^4} &\longrightarrow (53.0057, 4.7995, 0.0515). \end{aligned}$$

Figure 2 presents the graphical representation of the obtained nine asymmetric clouds. Here, EN and HE reflect the extent of dispersion of the cloud droplets.

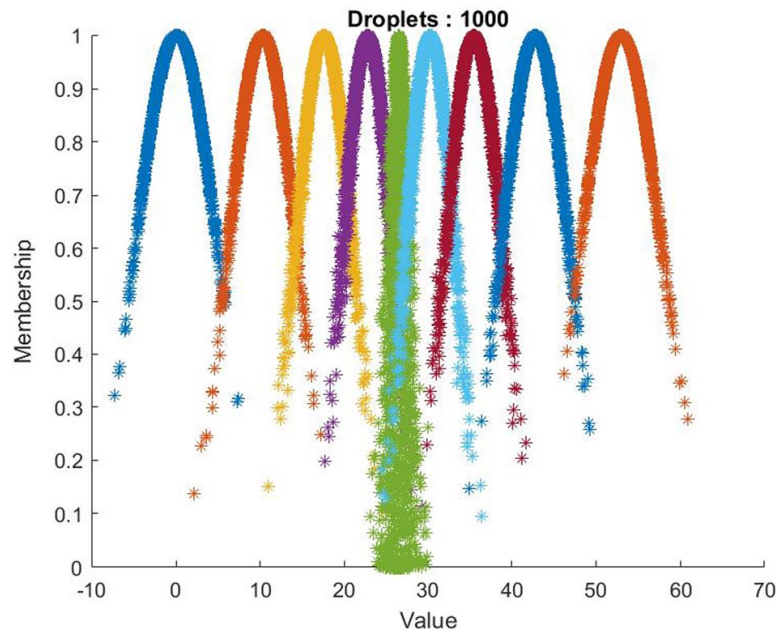


Fig. 2 Evaluation information for asymmetric point-based cloud model with respect to unbalanced linguistic terms

Note 1 It should be noted that the point-based cloud-generating procedure described above can be used in the scenario $n \neq m$ if dummy variables are added to the multiplicative unbalanced linguistic term set in a way that $n = m$.

Although the above-defined methodology is a well-defined model; it has some limitations associated, as the expected value of the unbalanced linguistic term is a single numeric value. This causes information loss in representing the semantics of linguistic terms to an extent. Thus, to overcome this information loss, we define a generalized multiplicative unbalanced interval cloud model in the next section, to effectively describe unbalanced linguistic terms that entail interval concepts instead of specific ones.

4 Generalized multiplicative unbalanced interval cloud model

In this section, we first develop a generalized MULSF to transform an unbalanced linguistic variable into an interval numerical value. Then, we provide a detailed procedure to generate a generalized multiplicative unbalanced interval cloud, followed by suggesting a novel interval score function to rank interval clouds.

Definition 9 Let $\mathbb{LS} = \{t_{a^i} | i = -n, -(n-1), \dots, 0, \dots, m, a > 1, n, m \in \mathbb{Z}^+\}$ (where n and m may or may not be equal) be a finitely ordered unbalanced LT set. Then, the extended MULSF can be conducted as a mapping from unbalanced linguistic label t_{a^i} to interval numbers $[\underline{\delta}_{a^i}, \bar{\delta}_{a^i}]$ and the mapping can be expressed as below

$$\tilde{\mathbb{G}} : t_{a^i} \rightarrow [\underline{\delta}_{a^i}, \bar{\delta}_{a^i}], i = -n, \dots, 0, \dots, m$$

where $[\underline{\delta}_{a^i}, \bar{\delta}_{a^i}] \in [0, 1/\log a] \subseteq [0, 1]$.

Usually, the correspondence between unbalanced linguistic term t_{a^i} and the subscript a^i , $a > 1$ is strictly increasing, and therefore, $[\underline{\delta}_{a^i}, \bar{\delta}_{a^i}]$ needs to satisfy

$$0 \leq \underline{\delta}_{a^{-n}} < \bar{\delta}_{a^{-n}} \leq \dots \leq \underline{\delta}_{a^0} < \bar{\delta}_{a^0} \leq \dots \leq \underline{\delta}_{a^m} < \bar{\delta}_{a^m} \leq 1/\log a$$

To strictly meet this property, an improved mapping rule is defined below

$$\begin{aligned} \tilde{\mathbb{G}} : t_{a^{-n}} &\rightarrow [\underline{\delta}_{a^{-n}}, \bar{\delta}_{a^{-n}}]; \\ t_{a^i} &\rightarrow [\sup\{\bar{\delta}_{a^{i-1}}, \underline{\delta}_{a^i}\}, \bar{\delta}_{a^i}], i = -n, \dots, 0; \\ t_{a^i} &\rightarrow [\underline{\delta}_{a^i}, \inf\{\bar{\delta}_{a^i}, \underline{\delta}_{a^{i+1}}\}], i = 1, 2, \dots, m-1; \\ t_{a^m} &\rightarrow [\underline{\delta}_{a^m}, \bar{\delta}_{a^m}] \end{aligned}$$

Here, t_{a^0} is the mid-term of the set $\mathbb{LS}$ and $I(t_{a^i}) = a^i$.

The preceding mapping rules state that $\tilde{\mathbb{G}}$ is a strictly and monotonically increasing function with respect to a^i . Based on the knowledge of function translation, the extended MULSF is represented as follows:

$$MULSF : \tilde{\mathbb{G}}(t_{a^i}) = [\underline{\delta}_{a^i}, \bar{\delta}_{a^i}]$$

where

$$\underline{\delta}_{a^i} = \begin{cases} \frac{1}{\log a} \left| \frac{b^{\#\mathbb{LS}_{left}} - b^{(\#\mathbb{LS}_{left} - a^i)}}{2(b^{\#\mathbb{LS}_{left}} - 1)} \right|, & -n \leq i \leq 0 \\ \frac{1}{\log a} \left| \frac{b^{\#\mathbb{LS}_{right}} + b^{(a^i + C - \#\mathbb{LS}_{right})} - 2 - \lambda_3(a^i + C - \#\mathbb{LS}_{right})}{2(b^{\#\mathbb{LS}_{right}} - 1)} \right|, & 0 < i \leq m \end{cases} \quad (10)$$

And,

$$\bar{\delta}_{a^i} = \begin{cases} \frac{1}{\log a} \left| \frac{b^{\#\mathbb{LS}_{left}} - b^{(\#\mathbb{LS}_{left} - a^i)} + \lambda_4(\#\mathbb{LS}_{left} - a^i)}{2(b^{\#\mathbb{LS}_{left}} - 1)} \right|, & -n \leq i \leq 0 \\ \frac{1}{\log a} \left| \frac{b^{\#\mathbb{LS}_{right}} + b^{(a^i + C - \#\mathbb{LS}_{right})}}{2(b^{\#\mathbb{LS}_{right}} - 1)} \right|, & 0 < i \leq m \end{cases} \quad (11)$$

The meaning of other symbols remains the same, where λ_3 and λ_4 are adjustable parameters. Further, $C = |\#\mathbb{LS}_{right} - \#\mathbb{LS}_{left}|$. We define the adjustable parameters λ_3 and λ_4 for the sake of calculation convenience as follows:

$$\lambda_3 = \frac{1}{a^i + C - \#\mathbb{L}\mathbb{S}_{right} + 1}$$

and,

$$\lambda_4 = \frac{1}{\#\mathbb{L}\mathbb{S}_{left} - a^i + 1}$$

For a fit with the psychological value during the evaluation process, *MULSF* makes use of the bidirectional rise in the geometric progression of the scale value to accurately depict the cognitive behaviour of experts. In this situation, the absolute deviation between adjacent linguistic subscripts increases with an extension from the central portion of linguistic terms to both ends. After considerable experiments, parameter ‘*b*’ is demonstrated to belong to the interval [1.36, 1.4] [58]. Without loss of generality, *b* is taken to be equal to 1.4 throughout the paper for numerical computation.

Next, we proceed to define the concept of the multiplicative unbalanced interval cloud model.

Definition 10 The multiplicative unbalanced interval cloud (MUIC) model, denoted by *A*, is defined as

$$A = (\underline{HE}, \underline{EN}, \underline{EX}, \overline{EX}, \overline{EN}, \overline{HE})$$

The MUIC model involves three numerical sections, i.e. the expectation interval $[\underline{EX}, \overline{EX}]$, left half cloud $(\underline{HE}, \underline{EN}, \underline{EX}, 0, 0)$ and right half cloud $(0, 0, \overline{EX}, \overline{EN}, \overline{HE})$. Here, $\underline{EX}$, $\underline{EN}$, $\underline{HE}$ are the expectation, entropy, and hyper-entropy of the left half cloud, also known as lower expectation, lower entropy, and lower hyper-entropy, respectively. On the other hand, $\overline{EX}$, $\overline{EN}$, $\overline{HE}$ are the expectation, entropy, and hyper-entropy of the right half cloud, also called upper expectation, upper entropy, and upper hyper-entropy, respectively.

We next proceed to present the procedure proficient in converting unbalanced linguistic term into unbalanced linguistic interval cloud variables based on the aforementioned *MULSF*.

Let $\mathbb{L}\mathbb{S} = \{t_{a^i} | i = -n, \dots, 0, 1, \dots, m \text{ and } a > 1, n, m \in \mathbb{Z}^+\}$ be a multiplicative unbalanced LT set, where t_{a^i} is plausible unbalanced linguistic term. The relationship between the unbalanced linguistic term and MUIC is presented in the following steps.

Step 1: Initially calculate $[\underline{\delta}_{a^i}, \overline{\delta}_{a^i}]$ using *MULSF* defined earlier, over the universe $[X_{\min}, X_{\max}]$.

Step 2: Determine the expectation interval $[\underline{EX}_{a^i}, \overline{EX}_{a^i}]$ where $\underline{EX}_{a^i}$ and $\overline{EX}_{a^i}$ are calculated as

$$\begin{aligned}\underline{EX}_{a^i} &= X_{\min} + \underline{\delta}_{a^i}(X_{\max} - X_{\min}) \\ \overline{EX}_{a^i} &= X_{\min} + \overline{\delta}_{a^i}(X_{\max} - X_{\min})\end{aligned}\quad (12)$$

Step 3: Determine the entropy $\underline{EN}_{a^i}$ and $\overline{EN}_{a^i}$. Based on the 3σ principle,

$$\begin{aligned}\underline{EN}_{a^{-n}} &= 0 \\ \underline{EN}_{a^i} &= \left| \frac{\underline{EX}_{a^i} - \overline{EX}_{a^{i-1}}}{3 \log a} \right|, \quad -(n-1) \leq i \leq m \\ \overline{EN}_{a^i} &= \left| \frac{\underline{EX}_{a^{i+1}} - \overline{EX}_{a^i}}{3 \log a} \right|, \quad -n \leq i \leq m-1 \\ \overline{EN}_{a^m} &= 0\end{aligned}\quad (13)$$

Step 4: Determine the hyper-entropies $\underline{HE}_{a^i}$ and $\overline{HE}_{a^i}$. Since $EN' \sim N(EN, HE^2)$, $\underline{HE}$ also satisfies the 3σ principle. Hence, the hyper-entropies $\underline{HE}_{a^i}$ and $\overline{HE}_{a^i}$ of MUIC may be obtained using the formulas:

$$\begin{aligned}\underline{HE}_{a^{-n}} &= 0 \\ \underline{HE}_{a^i} &= \frac{\max\{\max\{\underline{EN}_{a^i}\} - \underline{EN}_{a^i}, \underline{EN}_{a^i} - \min\{\underline{EN}_{a^i}\}\}}{3 \log a}, \quad -(n-1) \leq i \leq m \\ \overline{HE}_{a^i} &= \frac{\max\{\max\{\overline{EN}_{a^i}\} - \overline{EN}_{a^i}, \overline{EN}_{a^i} - \min\{\overline{EN}_{a^i}\}\}}{3 \log a}, \quad -n \leq i \leq m-1 \\ \overline{HE}_{a^m} &= 0\end{aligned}\quad (14)$$

In the literature, Li et al. [29] proposed the concept of variance of a normal cloud. Analogously, we introduce the notion of variance for MUIC.

Definition 11 Let $A = (\underline{HE}, \underline{EN}, \underline{EX}, \overline{EX}, \overline{EN}, \overline{HE})$ be a MUIC with c generated interval cloud droplets, the variance of c interval cloud droplets can be calculated as follows:

$$var_{Int} = [\underline{EN}, \overline{EN}] \times [\underline{EN}, \overline{EN}] + [\underline{HE}, \overline{HE}] \times [\underline{HE}, \overline{HE}]$$

then, the standard deviation of c interval cloud droplets is

$$sd_{Int} = \sqrt{[\underline{EN}, \overline{EN}] \times [\underline{EN}, \overline{EN}] + [\underline{HE}, \overline{HE}] \times [\underline{HE}, \overline{HE}]}$$

Here, the standard deviation sd_{Int} estimates the dispersal of interval cloud droplets that can be utilized to address the existing uncertainty and alteration of an unbalanced qualitative concept. A low sd_{Int} specifies that an unbalanced linguistic label tends to be relatively stable. In contrast, a high sd_{Int} implies the existence of a high level of uncertainty and alteration about the linguistic label. In this scenario, the

relationship between $EX_{Int} = [\underline{EX}, \overline{EX}]$ and sd_{Int} can be considered as the expected utility and the utility dispersion in the utility theory [59]. Then, the preference made for a choice can be presented by a linear trade-off between expected utility and utility dispersal. Motivated by the notion, we define an interval score function below to rank MUICs.

Definition 12 Given a MUIC $A = (\underline{HE}, \underline{EN}, \underline{EX}, \overline{EX}, \overline{EN}, \overline{HE})$, an interval score function can be stated as

$$sc_{Int} = EX_{Int} - \gamma \cdot sd_{Int}$$

where $\gamma \in [0, 1]$ reflects a coefficient to capture the attitude towards the standard deviation.

The γ value is estimated based on the real-world decision-making scenario. If $\gamma = 0$, it means that the DMs simply take into account the EX_{Int} and ignore uncertainty and informational modification.

The monotonicity property is satisfied by the defined interval score function. When compared to EX_{Int} , the score function increases, but when compared to EN_{Int} or HE_{Int} , it decreases.

Next, based on Definition 12, the following method can be proposed to rank two MUICs.

Definition 13 Let A_1 and A_2 be any two MUICs, then

- If $sc_{Int1} > sc_{Int2}$, then $A_1 > A_2$;
- If $sc_{Int1} = sc_{Int2}$, then $A_1 = A_2$;
- If $sc_{Int1} < sc_{Int2}$, then $A_1 < A_2$.

Note 2 It is noted that the inequality stated in the above definition is crisply defined. Henceforth, the case of a nested interval is neglected.

5 Illustrative example and comparative analysis

This section demonstrates the complete process of transforming unbalanced linguistic variables into MUICs via numerical illustration. Subsequently, the feasibility and rationality of the proposed methodology are given via comparative analyses from existing studies in the literature.

5.1 Illustrative example

Here, we propose a practical decision-making example to demonstrate the feasibility and advantages of the developed methodology for both the cases, $n = m$ and $n \neq m$.

Table 3 Five IT companies considered for investing purpose

IT1	IT2	IT3	IT4	IT5
Infosys	Wipro	TCS	HCL Technologies	Larsen & Toubro Infotech

Table 4 The assessment representation of the linguistic terms by MUICs for case $n = m$

Linguistic terms	MUICs
t_{2-2}	$A_1 = (0, 0, 1.3675, 6.873, 4.7032, 2.1314)$
t_{2-1}	$A_2 = (2.1314, 3.2918, 2.626, 7.817, 4.7032, 3.6454)$
t_{20}	$A_3 = (1.563, 0.9584, 4.8445, 9.171, 3.2918, 2.5841)$
t_{21}	$A_4 = (2.1314, 0.9584, 8.3055, 8.3055, 2.7785, 2.5841)$
t_{22}	$A_5 = (0, 0, 10.8145, 16.611, 2.7785, 3.6454)$

Example 2 We first consider the case where $n = m$, i.e. $\mathbb{L}\mathbb{S} = \{t_{a^{(-n)}}, \dots, t_{a^0}, \dots, t_{a^{(n)}}\}$.

Suppose there is a DM who is willing to invest his sum of money in IT companies by carefully observing the average closing stock values for the past 5 consecutive year, i.e. 2017-21. There are five possible alternatives DM is considering for investing the money. Table 3 shows the alternatives of the IT companies.

Since the stock values of companies get affected by linguistic factors. Henceforth, it facilitates to evaluate stock prices for the IT companies in terms of the predefined unbalanced linguistic scale set given as:

$$\mathbb{L}\mathbb{S} = \{t_{2-2} : \text{Very Low (VL)}, t_{2-1} : \text{Low (L)}, \\ t_{20} : \text{Average (Avg)}, t_{21} : \text{High (H)}, t_{22} : \text{Very High (VH)}\}$$

The variations incurred in the companies' average stock values for the past five years are expressed in an unbalanced linguistic scale set $\mathbb{L}\mathbb{S}$. In this scenario, analysing the uncertainty and fluctuation of stock values expressed by the dispersion of cloud drops is significant. Henceforth, we utilize our methodology to transform unbalanced linguistic variables into clouds so that the preference for selecting a particular alternative is possible. To do this, we follow similar steps 1–4 as given in the last section.

Based on the defined methodology in Steps, 1–4, the assessment information presented by MUICs is shown in Table 4 and their graphical representation is illustrated in Figs. 3 and 4. Next, based on Definition 12, the scoring values obtained are given as follows: $sc_{Int1} = [1.3675, 1.7094]$, $sc_{Int2} = [-1.2956, 1.8665]$,

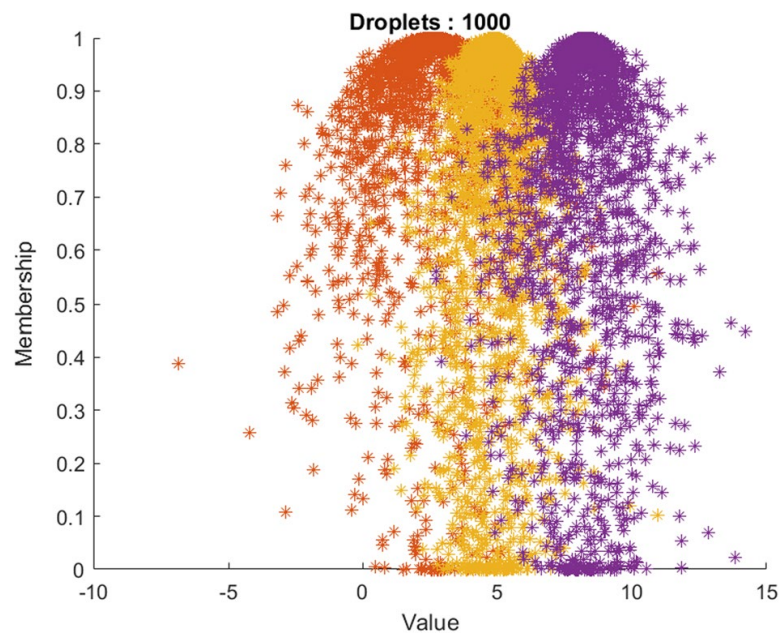


Fig. 3 Evaluation information for asymmetric lower half cloud with respect to unbalanced linguistic terms for Case $n = m$

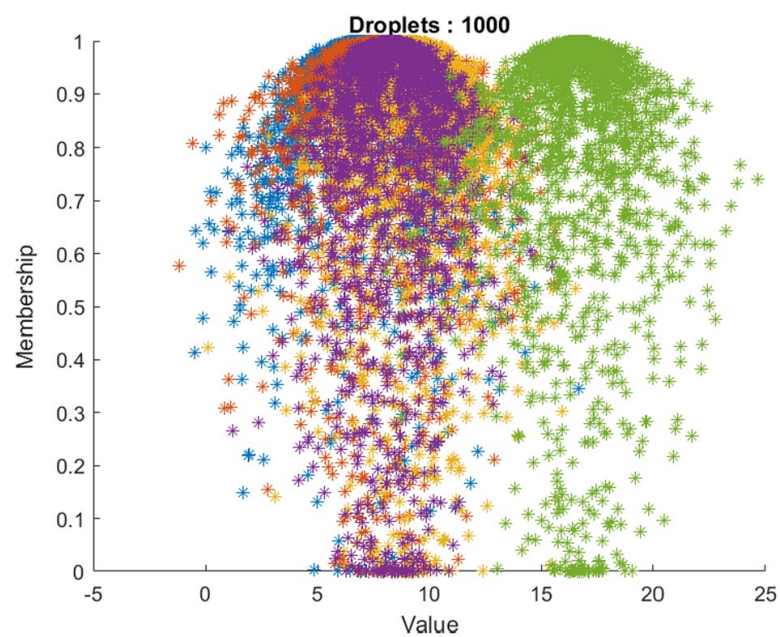


Fig. 4 Evaluation information for asymmetric upper half cloud with respect to unbalanced linguistic terms for Case $n = m$

$sc_{Int3} = [3.0111, 4.9861]$, $sc_{Int4} = [4.5111, 5.9685]$ and $sc_{Int5} = [10.8145, 12.0274]$. According to the score values, the ranking scheme of the alternatives can be fixed, and they are obtained as follows: $A_5 > A_4 > A_3 > A_1$. Hence, IT5 has a stable cloud and is preferred over other alternatives.

Table 5 The assessment representation of the linguistic terms by MUICs for the case $n \neq m$

Linguistic terms	MUICs
$t_{2^{-2}}$	$A_1 = (0, 0, 1.9145, 9.6222, 6.5853, 31.234)$
$t_{2^{-1}}$	$A_2 = (5.1017, 4.6069, 3.6757, 10.9424, 6.5853, 31.234)$
t_{2^0}	$A_3 = (3.6166, 1.3411, 6.7823, 12.838, 4.6069, 33.425)$
t_{2^1}	$A_4 = (3.6166, 1.3411, 11.627, 11.627, 2.14, 36.1572)$
t_{2^2}	$A_5 = (2.7319, 2.14, 13.5597, 18.0264, 34.79, 36.1572)$
t_{2^3}	$A_6 = (3.6454, 0, 49.4424, 55.1565, 34.79, 0)$

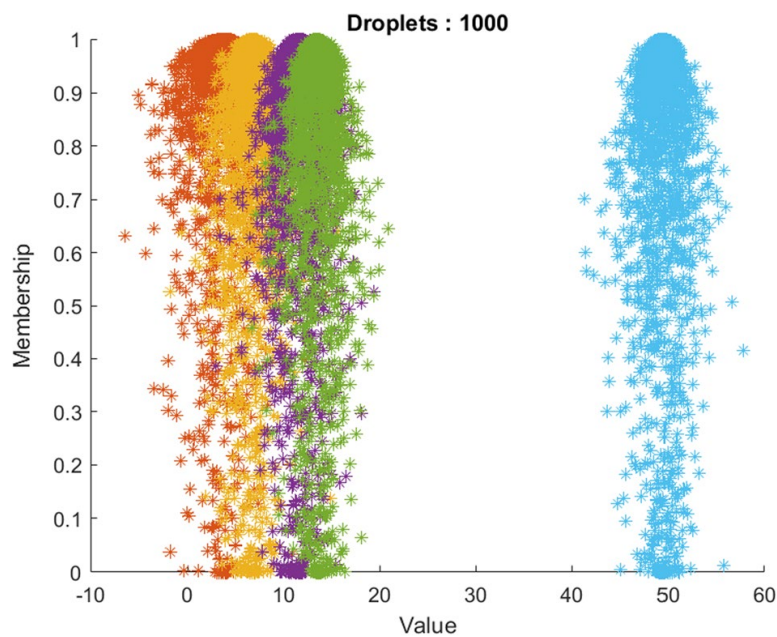


Fig. 5 Evaluation information for asymmetric lower half cloud with respect to unbalanced linguistic terms for Case $n \neq m$

Example 3 Next, we discuss the case where $n \neq m$, i.e. $\mathbb{L}\mathbb{S} = \{t_{a^{(-n)}}, \dots, t_{a^0}, \dots, t_{a^{(m)}}\}$. Consider a multiplicative unbalanced set

$$\mathbb{L}\mathbb{S} = \{t_{a^{-2}} : \text{Terrible}, t_{a^{-1}} : \text{Poor}, t_{a^0} : \text{Middle}, \\ t_{a^1} : \text{Satisfactory}, t_{a^2} : \text{Good}, t_{a^3} : \text{Excellent}\}$$

In this particular instance, $\#\mathbb{L}\mathbb{S}_{\text{left}} = n = 2$, $\#\mathbb{L}\mathbb{S}_{\text{right}} = m = 3$. Let $a = 2$, $b = 1.4$, $[X_{\min}, X_{\max}] = [0, 7]$. The assessment representation of linguistic terms by MUICs is given in Table 5, and their graphical representation is illustrated in Figs. 5 and 6.

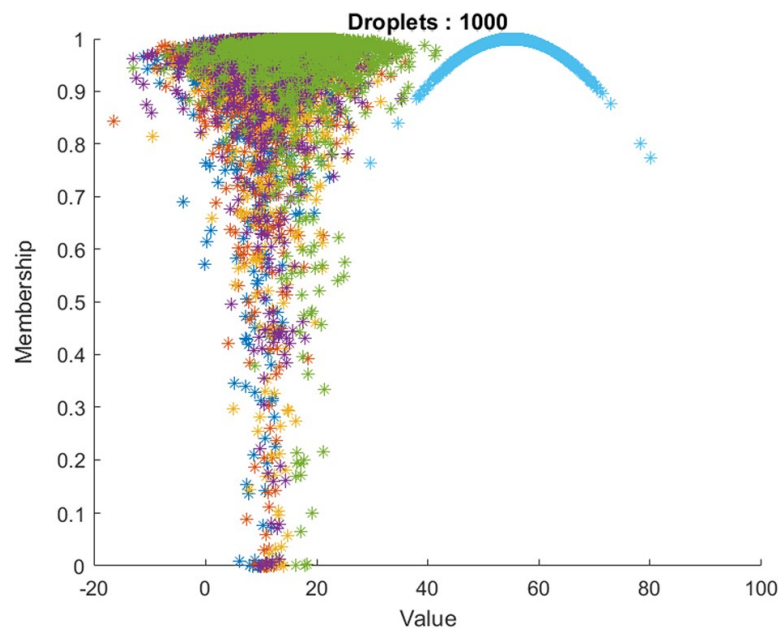


Fig. 6 Evaluation information for asymmetric upper half cloud with respect to unbalanced linguistic terms for Case $n \neq m$

5.2 Comparative analysis

Here, we analyse the numerical results presented in the previous subsection and demonstrate the viability and superiority of the proposed cloud model by comparing it to the existing studies from [55] and [44].

To confirm that the ranking outcomes are comparable, numerical example 3 given in the previous subsection is taken from the paper [55]. Since the final ranking outcome obtained is the same as that of the ATC model [55], therefore the proposed model offers analogous results. However, the difference lies in the information representation and the evaluation method to solve decision problems. The unbalanced linguistic computational model put forth by Herrera et al. [11] serves as the foundation for the model proposed by Wang et al. [55]. The scale values could be mapped to a number of hierarchically constructed uniformly distributed scale sets, which is effective for unbalanced linguistic cardinalities. Then, in information aggregation procedures, the uniformly dispersed scale values are used to replace the original imbalanced ones. This whole process altogether increases the complexity of the ATC model. Moreover, the unbalanced linguistic information presented in the model is given by a symmetrically distributed LT set. Consequently, the literal meaning of the unbalanced LT set is missing. On the contrary, the superiority of our work lies in the fact that it develops a new concept, namely a multiplicative unbalanced linguistic cloud model, to overcome the limitation of the existing cloud model as a qualitative information representation model. Our proposed model is based on the multiplicative unbalanced linguistic

model developed by Malhotra and Gupta [28] which is computationally less expensive and preserves the literal meaning of the unbalanced LT set. In addition, the existing ATC cloud model has difficulty describing Type II unbalanced linguistic concepts; however, our proposed model is suitable for both Type I and Type II unbalanced LT sets. The MULSF proposed in our work is based on a multiplicative scale set and improves the related mapping rules. Moreover, the existing cloud model [55] has difficulty in describing a continuous concept. Taking the concept of investment return, where the data distribution of investment return follows an irregular pattern, our proposed model is more suitable for description as it preserves the meaning and prevents information loss.

In the literature, the Fuzzy Trapezium Cloud model [44] is a subjective and effective cloud model that transforms different types of linguistic fuzzy sets and trapezium clouds. Calculations of weights for decision-making problems are also done by some defined operations. However, this model has a limitation that an optimization method is used for solving weight vectors which is a very complex and time taking procedure. Moreover, the model is based on a linguistic term set which is a purely balanced and symmetric term set. On the contrary, our model defined in the paper works for unbalanced linguistic data along with balanced linguistic data. Furthermore, the unbalanced linguistic set used in our paper is not transformed from a balanced linguistic set. This property reduces the time and evaluation steps of the transforming method. Taking the example picked in the Fuzzy Trapezium Cloud model [44], the battlefield situation is analysed and expressed in terms of different levels. Further, these levels correspond to linguistic terms that are symmetrically distributed. Nevertheless, in a real-life scenario, this situation might occur irregularly and the level of performance may not be symmetrically distributed. Hence, multiplicative unbalanced information could provide improved results.

Apart from the aforementioned examples, there are various real-life examples such as picking a multiplicative unbalanced cloud model in paper [38] for improving power grid failures analysis of the shortage of power supply and the pressure of dual control of energy consumption in [46], etc., provides significant results. Finally, on calculating the complexity of the proposed model given in Table 6, it is foreseen that the complexity of the defined model is linear, i.e. $4\mathcal{O}(n + m + 1) \equiv \mathcal{O}(n + m + 1)$, whereas other existing cloud model [55] has a nonlinear complexity. Hence, our model is applicable to handle complex decision problems.

Table 6 Computation complexity of the defined methodology

Step	Complexity
δ_{a^i}	$\mathcal{O}(n + m + 1)$
EX_{a^i}	$\mathcal{O}(n + m + 1)$
EN_{a^i}	$\mathcal{O}(n + m + 1)$
HE_{a^i}	$\mathcal{O}(n + m + 1)$

where $n + m + 1$ is the cardinality of the multiplicative ULTS

6 Conclusions

The normal cloud model is an excellent tool for converting qualitative concepts to quantitative information in linguistic decision-making problems. But the existing cloud models either cannot adequately describe the linguistic concepts or include information loss in the conversion process. Our work defines a comprehensive information representation model, the MUIC model, that can handle the multiplicative unbalanced data well. This model forms a bridge enabling efficient conversions between unbalanced linguistic information and clouds. A MULSF has been defined for this purpose. However, the LSFs existing in the literature map only a linguistic term to a particular specific value, thus creating information loss in the transformation process. To overcome this defect, we have defined a generalized MULSF that can convert the unbalanced linguistic data into an interval rather than a specific point. Thus, the advantage of a MUIC model is that it overcomes the loss of information persisting in converting the unbalanced linguistic data to a specific value. Rather it maps an unbalanced linguistic term to a continuous interval. The generalized MULSF is thus stated to efficiently incorporate the disparities in spacing of LTs in terms of representation and meaning by making a mapping from unbalanced LTs to interval values. Also, a method based on the 3σ principle is proposed to determine the values of entropy and hyper-entropy of the generalized MUIC. Lastly, this research provides a comparison method to rank the MUICs. The defined methodology has been implicated practically with the help of some examples.

Our method effectively addresses the linguistic imbalance, but it also has some limitations. It is noted that one must add a dummy variable to the set $\mathbb{L}\mathbb{S}$ to proceed with calculations for the point-based unbalanced cloud model for the case where the unbalanced LT set has an unequal number of linguistic terms spread on each side of the central term. Some multi-criteria decision-making problems may not have an accurate solution after adding this dummy variable. Therefore, further extension of this model is required to get over the corresponding limitation. Moreover, to enhance the continuity of our work, we also plan to improve the proposed cloud model's efficacy and attempt to create a decision support model based on cloud variables to resolve complex decision issues and deliver efficient outcomes in fewer computation times.

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Declarations

Conflicts of interest The authors have no competing interests to declare that are relevant to the content of this article.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Code availability Not applicable.

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Matrix Games with Linguistic Distribution Assessment Payoffs



Parul Chauhan and Anjana Gupta

Abstract In this paper, we propose a new concept of two-person constant-sum matrix games having payoffs in the form of linguistic distribution assessments. Such types of payoffs allow the players to express their opinion in terms of a whole fuzzy set and thus, are not limited to just one single term. To establish an equilibrium solution of these kinds of matrix games, first, we define the maxmin and minmax strategies to apply in case the players play pure strategies. In mixed strategies, we develop a linguistic distribution linear programming (LDLP) approach to find the players' mixed strategies. The method is depicted as a generalization of that traditionally used in the solution of a classical game. The applicability of defined LDLP is illustrated with the help of an example.

Keywords Distribution assessment sets · Linguistic term sets · Two-person matrix games · Linguistic payoffs

1 Introduction

The overpowering complications of the socio-economic environment make it more uncertain and imprecised. Human beings usually make use of words in prevailing evaluation and decision-making processes without the use of a certain numeric value. Computing with Words (CW) models [1–4] play a significant role in dealing with computational processes linguistically. Basically, CW is an approach in which the aspects of computations are words inferred out of a natural language [5, 6] that emerges to imitate such human conduct. Therefore, a critical attribute of CW is that it operates linguistic inputs to achieve linguistic outputs quickly understandable

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by humans. Yager [7] indicates the significance of the translation and retranslation processes to realize the objectives of CW.

CW has played a significant role in dealing with game problems, mainly in economics, politics, and social sciences. Since the introduction of game theory by Neumann & Morgenstern [8], a lot of research is committed to solving various types of game problems in exceptional situations, e.g., a group of players, the strategy sets available to the players, and a stipulation of payoffs for every association of strategies. Traditionally, the matrix games have considered a precise or certain value of the payoffs. Nevertheless, there exist some complicated conditions in games, e.g., unbalanced information among the game's players, imprecise factors in the decision situations, and a degree of the vagueness among the players' reluctance, such that some payoffs may be uncertain and not able to be evaluated exactly by the players. In this type of conditions, it is helpful to design the problems as matrix games having fuzzy payoffs. Several researchers have utilized fuzzy sets [9] in matrix games [10–14]. Fuzzy games, however, are restricted to numbers and do not deal with the linguistic aspect of human interaction, making it difficult to express the information linguistically. Hence, to facilitate the players' straightforward selection of utilities, Arfi [15, 16] suggested matrix games based on a linguistic approach. Delgado et al. [17] have also focused on this non-conventional side of matrix games. Sahoo [18] has also solved matrix games having linguistic payoffs. In addition to these contributions, Singh et al. [19] amplify the dimension of matrix games under a 2-tuple linguistic structure. In that paper, the authors have implemented the 2-tuple linguistic model [1, 20] to solve two-player constant-sum linguistic matrix games.

The application of the 2-tuple model only allows players to express information in terms of linguistic terms that belong to a balanced or symmetrical linguistic term (LT) set. However, in a more realistic viewpoint, it is not always possible to express information in the form of balanced linguistic terms. So to address the uncertainty well, methodologies implementing unbalanced LT sets are incorporated in various fields [2, 21–24]. Moreover, Wang and Hao [25] established an advanced form of the linguistic representation model for unbalanced sets, namely the symbolic proportional model. This newly designed so-called symbolic proportional model is interestingly easy-going and is flexible in a computation stage for CW allowing decision-makers to evaluate their alternatives via proportional 2-tuples of the form $(\delta u_i, \gamma u_{i+1})$, where u_i, u_{i+1} are represented as two successive linguistic terms, such that $\delta, \gamma \in [0, 1]$, $\delta + \gamma = 1$. However, this model has a limitation associated with it as the symbolic proportions are allocated only to 2 consecutive members of an LT set. To overcome this limitation, Zhang et al. [26] presented the notion of distribution assessments in an LT set. In this concept, symbolic proportions are assigned to all the linguistic terms, unlike the symbolic proportional approach.

In this paper, we consider matrix games having payoffs in the form of linguistic distribution assessments. This means that this study considers that the players may express their opinion in the form of distribution assessments. For example, if $LS = \{\ell_{-2} = \text{very poor}, \ell_{-1} = \text{poor}, \ell_0 = \text{average}, \ell_1 = \text{good}, \ell_2 = \text{very good}\}$ is an LT set, and a cricket captain uses the members of the set LS to assess a player's performance. The player takes part in 10 cricket games, in which

7 times was evaluated as ℓ_2 , 1 time as ℓ_1 , and 2 times as ℓ_{-1} . Then, the captain's evaluations may be interpreted in the form of a distribution assessments of LS as $\{(\ell_{-2}, 0), (\ell_{-1}, 0.2), (\ell_0, 0), (\ell_1, 0.1), (\ell_2, 0.7)\}$. The novelty of the paper lies in the fact that the defined methodology allows the decision-makers to evaluate performances in which the players' opinions do not have to be restricted to one or two terms. To deal with such types of game problems, first, we define what matrix games having payoffs as linguistic distribution assessments mean. Next, we define the methodology in case the players apply the pure strategies in the game. For the case of mixed strategies, we define the linguistic distribution linear programming (LDLP) approach to evaluate matrix games. This methodology gives information about what strategy would be beneficial for the player to win the game.

The paper is set out as follows. In Sect. 2, we present some brief concepts of linguistic distribution assessment sets and two-person matrix games. Section 3 defines the idea of linguistic matrix games having payoffs as distributed assessment sets. In Sect. 4, we introduce the LDLP approach to solve matrix games having payoffs as distributed assessment sets, in which the players chose to apply mixed strategies to win the game. Section 5 analyzes the results obtained in previous sections. Section 6 provides some conclusions of the paper and recommends the future works.

2 Preliminaries

This section reviews some fundamental notations, definitions, and properties of distributed assessment sets. Further, this section discusses the concept of two-person games.

2.1 Distribution Assessment Sets[26]

Definition 1 Let $LS = \{\ell_{-t}, \dots, \ell_t\}$ be an LT set. Let $m = \{(\ell_s, \beta_s) | s = -t, \dots, t\}$, where $\ell_s \in LS$, $\beta_s \geq 0$, $\sum_{s=-t}^t \beta_s = 1$ and β_s is the symbolic proportion of ℓ_s . Then, m is said to be a distribution assessment of LS .

Definition 2 Let $m = \{(\ell_s, \beta_s) | s = -t, \dots, t\}$, $\ell_s \in LS$, $\beta_s \geq 0$, $\sum_{s=-t}^t \beta_s = 1$, be a distribution assessment of LS . The expectation of m is defined as follows:

$$E(m) = \sum_{s=-t}^t \beta_s \Delta^{-1}(\ell_s). \quad (1)$$

In the computational model developed by Zhang et al. [26], the following properties hold for distribution assessment set:

1. Comparison Operator: If m_1 & m_2 are two distribution assessments of LS . Then

- (a) If $E(m_1) < E(m_2)$, then m_1 is less important than m_2 .
 - (b) If $E(m_1) = E(m_2)$, then m_1 and m_2 are equivalent.
2. A Negation Operator of distribution assessments:
 $\text{Neg}(\{(\ell_s, \beta_s), s = -t, \dots, t\}) = \{(\ell_s, \beta_{-s}), s = -t, \dots, t\}.$
3. The weighted average aggregation operator of distribution assessments is defined in the subsequent definition.

Definition 3 Let $\{m_1, \dots, m_n\}$ be a set of distribution assessments of LS , where $m_i = \{(\ell_s, \beta_s^i) | s = -t, \dots, t\}$, $i = 1, 2, \dots, n$. Let $\{w_1, w_2, \dots, w_n\}$ be the associated weights satisfying $w_i > 0$ and $\sum_{i=1}^n w_i = 1$. The weighted averaging operator of distribution assessments is:

$$\text{DAWA}_w(m_1, \dots, m_n) = \{(\ell_s, \beta_s), s = -t, \dots, t\}, \text{ where } \beta_s = \sum_{i=1}^n w_i \beta_s^i \quad (2)$$

2.2 Two Person Matrix Games

A crisp two-player zero-sum matrix game is represented as a triplet (S^n, S^m, A) , where

- (i) S^n is the set of ' n ' possible strategies of player I.
- (ii) S^m is the set of ' m ' possible strategies of player II.
- (iii) A is the payoff matrix of order $n \times m$ of player I corresponding to player II, and $\text{Neg}(A) = -A$ refers to the matrix consisting of payoffs of player II.

Definition 4 [27] A matrix game having $A = (a_{ij})_{n \times m}$ as the matrix of payoffs has lower value

$$v^- = \max_{i=1, \dots, n} \min_{j=1, \dots, m} a_{ij}$$

and upper value

$$v^+ = \min_{j=1, \dots, m} \max_{i=1, \dots, n} a_{ij}$$

v^- is that minimum quantity that player I is secured to get, and v^+ is that assured maximum quantity that player II might lose. We say that the game has a value if the upper value equals the lower value, and we express it as $v^+ = v^- = v$, and the row and column (i^*, j^*) giving payoff $a_{i^*j^*} = v^- = v^+$ are called optimal, or more precisely a saddle point in case the pure strategies are applied. Further, $a_{ij^*} \leq a_{i^*j^*} \leq a_{i^*j}$, $\forall$ rows $i = 1, 2, \dots, n$ and columns $j = 1, 2, \dots, m$.

Definition 5 [27] A mixed strategy is a vector $\Phi = (\phi_1, \phi_2, \dots, \phi_n)$ for player I and $\Psi = (\psi_1, \psi_2, \dots, \psi_m)$ for player II, where

$$\phi_i \geq 0, \sum_{i=1}^n \phi_i = 1 \quad \& \quad \psi_j \geq 0, \sum_{j=1}^m \psi_j = 1.$$

ϕ_i represents the probability of using row ‘ i ’ by player I, and ψ_j , the probability of using column ‘ j ’ by player II. We represent the set of mixed strategies having ‘ k ’ components as

$$S^k = \{(z_1, z_2, \dots, z_k) | z_r \geq 0, r = 1, 2, \dots, k, \sum_{r=1}^k z_r = 1\}. \quad (3)$$

In this terminology, elements $\Phi \in S^n$ and $\Psi \in S^m$ represent the mixed strategies for player I and player II, respectively.

Now, in case of mixed strategies, the expected payoff of players is defined as follows:

Definition 6 [27] Let $\Phi = (\phi_1, \phi_2, \dots, \phi_n)$ and $\Psi = (\psi_1, \psi_2, \dots, \psi_m)$ be the given choices of strategies of player I and II, respectively, then chosen independently, the expected payoff to player I is

$$E(\Phi, \Psi) = \sum_{i=1}^n \sum_{j=1}^m a_{ij} \text{Prob(I selects } i \text{ \& II selects } j) = \sum_{i=1}^n \sum_{j=1}^m a_{ij} \text{Prob(I selects } i) \text{ Prob(II selects } j) \quad (4)$$

$$= \sum_{i=1}^n \sum_{j=1}^m \phi_i a_{ij} \psi_j \quad (5)$$

$$= \Phi A \Psi^T \quad (6)$$

In the next section, we discuss the matrix games based on the information underlying in an unbalanced LT set having distribution assessment payoffs.

3 Linguistic Games with Payoffs as Distributed Assessment Sets

In this section, we discuss the matrix games consisting of the information from linguistic distribution assessment sets. The matrix games involved have elements of a distribution assessment set. The advantage of this method is that the payoffs are a set that includes every single term of the original linguistic term set with some proportions (weightage).

Definition 7 Let us consider a two-player matrix game $\tilde{G}$ with payoffs as distributed assessment sets, where the payoff matrix of player I is given as $\tilde{A} = (\tilde{a}_{ij})_{n \times m}$, whose elements $\tilde{a}_{ij}$ are distributed assessment sets of a pre-defined LT set. Moreover, considering only zero-sum games, the payoff matrix of player II would be $\tilde{B} = \text{Neg}(\tilde{A})$,

where the negation operator is same as defined earlier. The strategy sets of player I and player II are

$$S^n = \{\Phi = (\phi_1, \phi_2, \dots, \phi_n) | \phi_i \geq 0, i = 1, 2, \dots, n, \sum_{i=1}^n \phi_i = 1\} \quad (7)$$

$$S^m = \{\Psi = (\psi_1, \psi_2, \dots, \psi_m) | \psi_j \geq 0, j = 1, 2, \dots, m, \sum_{j=1}^m \psi_j = 1\} \quad (8)$$

respectively.

Definition 8 The amount that player I is assured to receive in the worst possible situation is

$$\tilde{v}^{1-} = \max_{i=1, 2, \dots, n} \min_{j=1, 2, \dots, m} \tilde{a}_{ij}$$

Similarly, the amount that player II is assured to receive is

$$\tilde{v}^{2-} = \max_{j=1, 2, \dots, m} \min_{i=1, 2, \dots, n} \text{neg}(\tilde{a}_{ij})$$

where each $\tilde{a}_{ij}$ is a distribution assessment set of original LT set.

The lower and upper values of the game are defined as $\tilde{v}^- = \tilde{v}^{1-}$ and $\tilde{v}^+ = \text{neg}(\tilde{v}^{2-})$, respectively. The game has value $\tilde{v}$ if $\tilde{v} = \tilde{v}^+ = \tilde{v}^-$. So the smallest maximum and the biggest minimum should be same, and the row and column i^*, j^* yielding $\tilde{a}_{i^*j^*} (= \tilde{v}^+ = \tilde{v}^-)$ are optimal (a saddle point in case of pure strategies).

It is to be noted that

$$\begin{aligned} \tilde{v}^+ &= \text{neg}(\tilde{v}^{2-}) = \text{neg}\left(\max_{j=1, 2, \dots, m} \min_{i=1, 2, \dots, n} \text{neg}(\tilde{a}_{ij})\right) \\ &= \min_{j=1, 2, \dots, m} \max_{i=1, 2, \dots, n} (\tilde{a}_{ij}) \end{aligned}$$

Let us try to look at this with the help of an example.

Note. The Examples 1 and 2 consider game problems in which the payoffs are of the form of linguistic distribution assessment sets. These distribution assessments are taken from Example 1 of Zhang et al. [26].

Let $LS = \{\ell_{-2}: \text{very poor}, \ell_{-1}: \text{poor}, \ell_0: \text{average}, \ell_1: \text{good}, \ell_2: \text{very good}\}$ be the pre-defined LT set. Suppose that m_1, m_2, m_3, m_4, m_5 represents a set of distribution assessments of the set LS given as

$$\begin{aligned}
m_1 &= \{(\ell_{-2}, 0.2), (\ell_{-1}, 0.2), (\ell_0, 0.2), (\ell_1, 0.2), (\ell_2, 0.2)\} \\
m_2 &= \{(\ell_{-2}, 0), (\ell_{-1}, 0.3), (\ell_0, 0.7), (\ell_1, 0), (\ell_2, 0)\} \\
m_3 &= \{(\ell_{-2}, 0), (\ell_{-1}, 0), (\ell_0, 0.8), (\ell_1, 0.2), (\ell_2, 0)\} \\
m_4 &= \{(\ell_{-2}, 0.4), (\ell_{-1}, 0.4), (\ell_0, 0.2), (\ell_1, 0), (\ell_2, 0)\} \\
m_5 &= \{(\ell_{-2}, 0), (\ell_{-1}, 0), (\ell_0, 0.2), (\ell_1, 0.2), (\ell_2, 0.6)\}
\end{aligned}$$

The payoff $\tilde{a}_{12} = m_3$ demonstrates that if P_1 invests his money in F_1 and if P_2 invests her money in F_2 , then P_1 gets a payoff m_3 , i.e., a payoff which is 80% Average and 20% Good (since the proportions for other linguistic terms are zero in m_3).

Next,

$$\begin{aligned}
E(m_1) &= \sum_{t=-2}^2 \beta_t \ell_t \\
&= \beta_{-2} \ell_{-2} + \beta_{-1} \ell_{-1} + \beta_0 \ell_0 + \beta_1 \ell_1 + \beta_2 \ell_2 \\
&= 0.2 \times (\ell_{-2}) + 0.2 \times (\ell_{-1}) + 0.2 \times (\ell_0) + 0.2 \times (\ell_1) + 0.2 \times (\ell_2) \\
&= \ell_{-0.4-0.2+0+0.2+0.4} \text{ (using operational laws defined by Xu 23 } \mu \ell_t = \ell_{t\mu} \text{)} \\
&= \ell_0 \text{ (or precisely 0 for simplicity)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
E(m_2) &= 0 \times \ell_{-2} + 0.3 \times \ell_{-1} + 0.7 \ell_0 + 0 \times \ell_1 + 0 \times \ell_2 \\
&= \ell_{-0.3} \text{ (or 0.3)}
\end{aligned}$$

In the similar manner,

$$E(m_3) = \ell_{0.2} \text{ (or 0.2)}, E(m_4) = \ell_{-0.2} \text{ (or } -0.2), E(m_5) = \ell_{1.4} \text{ (or 1.4)}.$$

$$\therefore E(m_5) > E(m_3) > E(m_1) > E(m_4) > E(m_2)$$

$$\implies m_5 > m_3 > m_1 > m_4 > m_2 \text{ (By Definition 2).}$$

Now, we proceed toward using the given distribution assessment sets in game problems.

Example 1 Two partners P_1 and P_2 have two choices for where to invest their money, say F_1 and F_2 , where F stands for firm. Based on their choices of investing the money, we model the payoff matrix for P_1 corresponding to P_2 as:

$$\tilde{A} = \begin{matrix} & \begin{matrix} F_1 & F_2 \end{matrix} \\ \begin{matrix} F_1 \\ F_2 \end{matrix} & \begin{bmatrix} m_1 & m_3 \\ m_2 & m_1 \end{bmatrix} \end{matrix}$$

where the distribution assessments m_i , $i = 1, 2, 3$ are as defined above.

The guaranteed amount of P_1 is:

$$\begin{aligned}\tilde{v}^{1-} &= \max_{i=1,2} \min_{j=1,2} (\tilde{a}_{ij}) = \max \{\min(m_1, m_3), \min(m_2, m_1)\} \\ &= \max \{m_1, m_2\} \\ &= m_1\end{aligned}$$

$\therefore P_1$ receives a payoff which has equal share of 20% of all linguistic terms if he invests in firm F_1 .

Similarly, the payoff matrix for P_2 in correspondence to P_1 is given by:

$$\text{Neg}(\tilde{A}) = \begin{matrix} & \begin{matrix} F_1 & F_2 \end{matrix} \\ \begin{matrix} F_1 \\ F_2 \end{matrix} & \begin{bmatrix} \text{Neg}(m_1) & \text{Neg}(m_3) \\ \text{Neg}(m_2) & \text{Neg}(m_1) \end{bmatrix} \end{matrix}$$

The negation of distribution assessment sets m_1 , m_2 and m_3 are:

$\text{Neg}(m_1) = \{(\ell_{-2}, 0.2), (\ell_{-1}, 0.2), (\ell_0, 0.2), (\ell_1, 0.2), (\ell_2, 0.2)\}$

$\text{Neg}(m_2) = \{(\ell_{-2}, 0), (\ell_{-1}, 0), (\ell_0, 0.7), (\ell_1, 0.3), (\ell_2, 0)\}$

$\text{Neg}(m_3) = \{(\ell_{-2}, 0), (\ell_{-1}, 0.2), (\ell_0, 0.8), (\ell_1, 0), (\ell_2, 0)\}$

So by using Comparison Operator, we can deduce that $\text{Neg}(m_3) < \text{Neg}(m_1) < \text{Neg}(m_2)$.

Hence, the assured amount received by P_2 is:

$$\begin{aligned}\tilde{v}^{2-} &= \max_{i=1,2} \min_{j=1,2} (\text{Neg} \tilde{a}_{ij}) = \max \{\text{Neg}(m_3), \text{Neg}(m_1)\} \\ &= \text{Neg}(m_1)\end{aligned}$$

The lower value of the game is

$$\tilde{v}^- = \tilde{v}^{1-} = m_1 = \{(\ell_{-2}, 0.2), (\ell_{-1}, 0.2), (\ell_0, 0.2), (\ell_1, 0.2), (\ell_2, 0.2)\}$$

and the upper value is

$$\tilde{v}^+ = \text{neg}(\tilde{v}^{2-}) = \text{Neg}(m_1) = m_1$$

$$\therefore \tilde{v}^- = \tilde{v}^+$$

$\implies (1, 1)$ is the saddle point and $\{(\ell_{-2}, 0.2), (\ell_{-1}, 0.2), (\ell_0, 0.2), (\ell_1, 0.2), (\ell_2, 0.2)\}$ is the value of the game, i.e., the players have equal chances of getting very poor, poor, average, good, and very good payoff if they both invest in firm F_1 .

Now, the problem arises when $\tilde{v}^-$ is not equal to $\tilde{v}^+$. If the optimal pure strategies do not exist, then we have to mix up the choices. According to Neumann [8], if mixed strategies are applied in a game, then the game would always retain a value and optimal strategies. In the case where the players apply mixed strategies, then their payoffs may be calculated by calculating what is “expected” by the player to receive. So next, we define the linguistic distributed expected payoff received by the player.

Definition 9 If $\Phi = (\phi_1, \phi_2, \dots, \phi_n) \in S^n$ and $\Psi = (\psi_1, \psi_2, \dots, \psi_m) \in S^m$ are the mixed strategies corresponding to player I and player II, respectively, and $\tilde{a}_{ij} = m_k$ (distribution assessment sets), $k = 1, 2, \dots, n$ be the distributed payoffs of player I, then the linguistic distribution expected payoff received by player I is given as

$$\tilde{E}_A(\Phi, \Psi) = \bigoplus_{i=1}^n \left(\phi_i \left(\bigoplus_{j=1}^m \tilde{a}_{ij} \psi_j \right) \right). \quad (9)$$

and in similar manner, the linguistic distribution expected payoff received by player II is

$$\tilde{E}_{\text{neg}A}(\Phi, \Psi) = \bigoplus_{i=1}^n \left[\phi_i \left(\bigoplus_{j=1}^m \text{neg}(\tilde{a}_{ij}) \psi_j \right) \right]. \quad (10)$$

So now, we introduce an LDLP approach for solving matrix games with distribution assessments as payoffs.

4 Linguistic Distribution Linear Programming Method for Matrix Games with Distribution Assessments as Payoffs

In the following section, we formulate the LDLP method to solve matrix games (involving games using mixed strategies) by using the expected payoff to evaluate the players' payoffs.

Let $LS = \{\ell_{-t}, \dots, \ell_t\}$ be the linguistic pre-established LT set and $m = \{(\ell_k, \beta_k) | k = -t, \dots, t\}$ be the distribution assessment set. Consider the linguistic distribution matrix $A = (a_{ij})_{n \times m}$, where each $\tilde{a}_{ij}$ represents the payoff to player I received after opting i^{th} strategy while player II opts for j^{th} strategy. Let $\Phi = (\phi_1, \phi_2, \dots, \phi_n) \in S^n$ and $\Psi = (\psi_1, \psi_2, \dots, \psi_m) \in S^m$ be the mixed strategies of player I & player II, respectively. The “expected payoff” of player I corresponding to the j^{th} strategy chosen by player II is given by $\tilde{a}_{1j}\phi_1 \oplus \tilde{a}_{2j}\phi_2 \oplus \dots \oplus \tilde{a}_{nj}\phi_n$. We need to evaluate the mixed strategy $\Phi \in S^n$ of player I such that the value of player I, i.e., $\tilde{v}^{1-}$ is maximized.

So, the required linguistic distribution linear programming for player I is:

$$\begin{aligned}
 & \max \quad \tilde{v}^{1-} \\
 & \text{subject to} \\
 & \tilde{a}_{11}\phi_1 \oplus \tilde{a}_{21}\phi_2 \oplus \cdots \oplus \tilde{a}_{n1}\phi_n \geq \tilde{v}^- \\
 & \tilde{a}_{12}\phi_1 \oplus \tilde{a}_{22}\phi_2 \oplus \cdots \oplus \tilde{a}_{n2}\phi_n \geq \tilde{v}^- \\
 & \vdots \\
 & \tilde{a}_{1m}\phi_1 \oplus \tilde{a}_{2m}\phi_2 \oplus \cdots \oplus \tilde{a}_{nm}\phi_n \geq \tilde{v}^- \\
 & \phi_1 + \phi_2 + \cdots + \phi_n = 1 \\
 & \phi_1, \phi_2, \dots, \phi_n \geq 0. \\
 & \text{where each } \tilde{a}_{ij} \text{ is linguistic distribution assessment set.}
 \end{aligned} \tag{P1}$$

As the expectation E (defined in Definition 2) is monotonically increasing function, we apply E to each constraint of (P1) and the constraints take the form:

$$\begin{aligned}
 & E(\tilde{a}_{11})\phi_1 + E(\tilde{a}_{21})\phi_2 + \cdots + E(\tilde{a}_{n1})\phi_n \geq E(\tilde{v}^-) \\
 & E(\tilde{a}_{12})\phi_1 + E(\tilde{a}_{22})\phi_2 + \cdots + E(\tilde{a}_{n2})\phi_n \geq E(\tilde{v}^-) \\
 & \vdots \\
 & E(\tilde{a}_{1m})\phi_1 + E(\tilde{a}_{2m})\phi_2 + \cdots + E(\tilde{a}_{nm})\phi_n \geq E(\tilde{v}^-) \\
 & \phi_1 + \phi_2 + \cdots + \phi_n = 1 \\
 & \phi_1, \phi_2, \dots, \phi_n \geq 0. \\
 & \text{Put } E(\tilde{a}_{ij}) = a_{ij} \\
 & E(\tilde{v}^{1-}) = v.
 \end{aligned}$$

$\Rightarrow$ constraints of (P1) become

$$\begin{aligned}
 & a_{11}\phi_1 + a_{21}\phi_2 + \cdots + a_{n1}\phi_n \geq v \\
 & a_{12}\phi_1 + a_{22}\phi_2 + \cdots + a_{n2}\phi_n \geq v \\
 & \vdots \\
 & a_{1m}\phi_1 + a_{2m}\phi_2 + \cdots + a_{nm}\phi_n \geq v \\
 & \phi_1 + \phi_2 + \cdots + \phi_n = 1 \\
 & \phi_1, \phi_2, \dots, \phi_n \geq 0. \\
 & \text{Let } X_i = \frac{\phi_i}{v}; i = 1, 2, \dots, n \text{ and } \frac{1}{v} = V
 \end{aligned}$$

Then, the problem reduces to

$$\begin{aligned}
 & \min \quad V = X_1 + X_2 + \cdots + X_n \\
 & \text{subject to} \\
 & a_{11}X_1 + a_{21}X_2 + \cdots + a_{n1}X_n \geq 1 \\
 & a_{12}X_1 + a_{22}X_2 + \cdots + a_{n2}X_n \geq 1 \\
 & \vdots \\
 & a_{1m}X_1 + a_{2m}X_2 + \cdots + a_{nm}X_n \geq 1 \\
 & X_1, X_2, \dots, X_n \geq 0.
 \end{aligned} \tag{P2}$$

and then, we may solve P2 as usual linear programming problem.

Similarly, linguistic distribution linear programming for player II would be

$$\begin{aligned}
 & \max \quad \tilde{v}^{2-} \\
 & \text{subject to} \\
 & \text{neg}(\tilde{a}_{11})\psi_1 \oplus \text{neg}(\tilde{a}_{12})\psi_2 \oplus \cdots \oplus \text{neg}(\tilde{a}_{1m})\psi_m \geq \tilde{v}^{2-} \\
 & \text{neg}(\tilde{a}_{21})\psi_1 \oplus \text{neg}(\tilde{a}_{22})\psi_2 \oplus \cdots \oplus \text{neg}(\tilde{a}_{2m})\psi_m \geq \tilde{v}^{2-} \\
 & \vdots \\
 & \text{neg}(\tilde{a}_{n1})\psi_1 \oplus \text{neg}(\tilde{a}_{n2})\psi_2 \oplus \cdots \oplus \text{neg}(\tilde{a}_{nm})\psi_m \geq \tilde{v}^{2-} \\
 & \psi_1 + \psi_2 + \cdots + \psi_m = 1 \\
 & \psi_1, \psi_2, \dots, \psi_m \geq 0.
 \end{aligned} \tag{P3}$$

We have to find the mixed strategy $\Psi \in S^m$ for player II.

Let $\text{neg}(\tilde{v}^{2-}) = \tilde{v}^+$

$\Rightarrow$ From the properties of negation operator, the problem becomes

$$\begin{aligned}
 & \min \quad \tilde{v}^+ \\
 & \text{subject to} \\
 & \tilde{a}_{11}\psi_1 \oplus \tilde{a}_{12}\psi_2 \oplus \cdots \oplus \tilde{a}_{1m}\psi_m \leq \tilde{v}^+ \\
 & \tilde{a}_{21}\psi_1 \oplus \tilde{a}_{22}\psi_2 \oplus \cdots \oplus \tilde{a}_{2m}\psi_m \leq \tilde{v}^+ \\
 & \vdots \\
 & \tilde{a}_{n1}\psi_1 \oplus \tilde{a}_{n2}\psi_2 \oplus \cdots \oplus \tilde{a}_{nm}\psi_m \leq \tilde{v}^+ \\
 & \psi_1 + \psi_2 + \cdots + \psi_m = 1 \\
 & \psi_1, \psi_2, \dots, \psi_m \geq 0.
 \end{aligned} \tag{P4}$$

Now, we apply E operator in P4.

Also, since earlier $E(\tilde{v}^{1-}) = v$ and if v is the value of the game, then $E(\tilde{v}^+) = v$. Put $Y_j = \frac{\psi_j}{v}; j = 1, 2, \dots, m$ and $E(\tilde{a}_{ij}) = a_{ij}$, the problem for player II reduces to the problem

$$\begin{aligned}
 & \max \quad V = Y_1 + Y_2 + \dots + Y_m & (P5) \\
 & \text{subject to} \\
 & a_{11}Y_1 + a_{12}Y_2 + \dots + a_{1m}Y_m \leq 1 \\
 & a_{21}Y_1 + a_{22}Y_2 + \dots + a_{2m}Y_m \leq 1 \\
 & \vdots \\
 & a_{n1}Y_1 + a_{n2}Y_2 + \dots + a_{nm}Y_m \leq 1 \\
 & Y_1, Y_2, \dots, Y_m \geq 0.
 \end{aligned}$$

So, by solving P2 and P5, we get the optimal solution $X = (X_1, X_2, \dots, X_n)$ and $Y = (Y_1, Y_2, \dots, Y_m)$ and the optimal value V . Hence, the optimal strategies Φ and Ψ of player I and player II, respectively, could be achieved and the value v of the game follows.

Example 2 Consider a payoff matrix with payoffs as distribution assessments m_i defined above in Sect. 3, where the linguistic terms are from the set LS .

$$\tilde{A} = \begin{bmatrix} m_3 & m_5 & m_2 \\ m_1 & m_2 & m_4 \\ m_5 & m_4 & m_5 \end{bmatrix}$$

Here, $\tilde{v}^{1-} = m_4$ and $\tilde{v}^+ = m_3$. So we proceed by mixing up the choices. Let $\Phi = (\phi_1, \phi_2, \phi_3)$, $\phi_i \geq 0$ for $i = 1, 2, 3$ such that $\sum_{i=1}^3 \phi_i = 1$ be the mixed strategy for player I, and $\Psi = (\psi_1, \psi_2, \psi_3)$, $\psi_j \geq 0, j = 1, 2, 3$ and $\sum_{j=1}^3 \psi_j = 1$ be mixed strategy for player II. We formulate the LDLP for player I as in (P1) in following manner:

$$\begin{aligned}
 & \max \quad \tilde{v}^- \\
 & \text{subject to} \\
 & m_3\phi_1 \oplus m_1\phi_2 \oplus m_5\phi_3 \geq \tilde{v}^- \\
 & m_5\phi_1 \oplus m_2\phi_2 \oplus m_4\phi_3 \geq \tilde{v}^- \\
 & m_2\phi_1 \oplus m_4\phi_2 \oplus m_5\phi_3 \geq \tilde{v}^- \\
 & \phi_1 + \phi_2 + \phi_3 = 1 \\
 & \phi_1, \phi_2, \phi_3 \geq 0.
 \end{aligned}$$

Applying E function in the constraints, we get

$$\begin{aligned} 0.2\phi_1 + 0\phi_2 + 1.4\phi_3 &\geq E(\tilde{v}^-) \\ 1.4\phi_1 - 0.3\phi_2 - 0.2\phi_3 &\geq E(\tilde{v}^-) \\ -0.3\phi_1 - 0.2\phi_2 + 1.4\phi_3 &\geq E(\tilde{v}^-) \\ \phi_1 + \phi_2 + \phi_3 &= 1 \\ \phi_1, \phi_2, \phi_3 &\geq 0. \end{aligned}$$

Substituting $E(\tilde{v}^-) = v$, $X_i = \frac{\phi_i}{v}$ and $V = \frac{1}{v}$, the problem reduces in the similar manner as P2 like

$$\begin{aligned} \min \quad & V = X_1 + X_2 + X_3 \\ \text{subject to} \quad & \\ & 0.2X_1 + 0X_2 + 1.4X_3 \geq 1 \\ & 1.4X_1 - 0.3X_2 - 0.2X_3 \geq 1 \\ & -0.3X_1 - 0.2X_2 + 1.4X_3 \geq 1 \\ & X_1, X_2, X_3 \geq 0. \end{aligned}$$

The optimal solution of this problem is (0.84, 0, 0.89) and $V = 1.74$.

$\therefore v = 0.5747$ (approx.)

Hence, the optimal strategies of player I are (0.4827, 0, 0.51149).

This means that player I uses row 1, 48.27% and primarily, row 3, 51.14% times to win the game.

Similarly, the LDLP for player II as in P4 is

$$\begin{aligned} \min \quad & \tilde{v}^+ \\ \text{subject to} \quad & \\ & m_3\psi_1 \oplus m_5\psi_2 \oplus m_2\psi_3 \leq \tilde{v}^+ \\ & m_1\psi_1 \oplus m_2\psi_2 \oplus m_4\psi_3 \leq \tilde{v}^+ \\ & m_5\psi_1 \oplus m_4\psi_2 \oplus m_5\psi_3 \leq \tilde{v}^+ \\ & \psi_1 + \psi_2 + \psi_3 = 1 \\ & \psi_1, \psi_2, \psi_3 \geq 0. \end{aligned}$$

By substitution as given in P5 , the problem becomes

$$\begin{aligned}
 &\max \quad V = Y_1 + Y_2 + Y_3 \\
 &\text{subject to} \\
 &0.2Y_1 + 1.4Y_2 - 0.3Y_3 \leq 1 \\
 &0Y_1 - 0.3Y_2 - 0.2Y_3 \leq 1 \\
 &1.4Y_1 - 0.2Y_2 + 1.4Y_3 \leq 1 \\
 &Y_1, Y_2, Y_3 \geq 0.
 \end{aligned}$$

The optimal solution for this LPP is (0, 0.89, 0.84) and $V = 1.74$

$\implies v = 0.5747(\text{approx.})$.

Hence, the optimal strategies for player II are (0, 0.511483, 0.4827).

This implies that column 2 is used 51.14% times, and column 3 is used 48.27% times by player II to win the game. Moreover, since $v = 0.5747$ for both the player, v is the value of the game.

5 Result Analysis

In this section, we discuss about the results for the value of the game and the best strategies chosen by the players, obtained in previous sections.

In Sect. 3, games having pure strategies have been discussed. In Example 1, the matrix $\tilde{A}$ exhibits optimal pure strategies as $\tilde{v}^-$ comes out to be equal to $\tilde{v}^+$. So the value of the game $\tilde{v}$ is obtained as $\tilde{v}^- = \tilde{v}^+ = \tilde{v}$. Also, the rows and columns i, j for which $\tilde{v}^- = \tilde{v}^+$ are called the optimal strategies of player I corresponding to player II. Further, the optimal strategies of player II may also be obtained by repeating the same process for the matrix $\text{Neg}(\tilde{A})$ instead of $\tilde{A}$.

Example 2 in Sect. 4 depicts the case when the pure strategies cannot be applied, i.e., when $\tilde{v}^- \neq \tilde{v}^+$. In this case of mixed strategies, the LDLP is formed for both the players. This LDLP consists of linguistic distributed payoffs, which are then converted into numeric terms by doing required substitutions. This converts the aforementioned LDLP into a crisp LPP, which can be solved directly. Solution of this LPP provides the best strategies to be played by both the players in order to win the game.

Notes and Comments

In the LDLP approach, since the function E is not a bijective function, it is not possible to always re-translate the game value v into its corresponding linguistic distribution assessment set. So we consider the value of the game as a numeric value

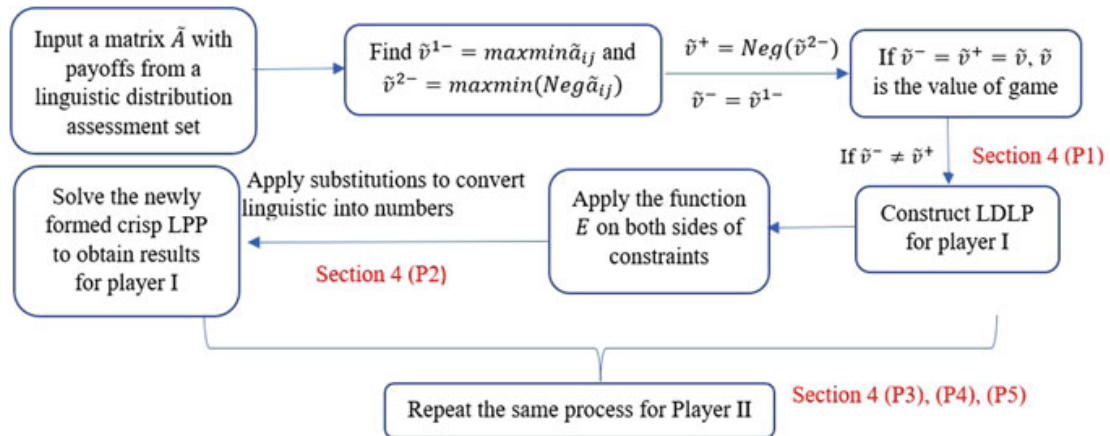


Fig. 1 Brief methodology for finding the solution of the game

only. On the other hand, we can deduce from this approach what strategy is the best for a player to win the game. For a better understanding of the whole methodology, one can refer to the Fig. 1.

6 Conclusions and Recommendations

This paper has developed a new theoretical approach to solve matrix games with payoffs as linguistic distribution assessments. This process helps us evaluate the game problems in which players express their responses in the form of a distribution assessment set, which considers all the values of an LT set despite picking a single term or two adjacent terms. We developed an LDLP approach which forms a linguistic linear program for both the players. Moreover, doing required substitutions can convert the LDLP into an easy and casual linear programming problem that is easily solvable.

Recommendations for future work include the development of a bijective function in place of the existing function ‘ E ’ so that the value of the game may be obtained in terms of the linguistic distribution assessment set only. Furthermore, incorporating the concept of linguistic distribution assessment payoffs in non-zero sum games may yield beneficial results in the future, which is desirable for situations involving complex decision-making problems in the field of bimatrix games.

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Matrix games with proportional linguistic payoffs

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Abstract

This paper aims to present a novel approach for computing two-player constant-sum matrix games laid on the notion of a “symbolic proportional” linguistic term set. It is not always possible to lay assessments based on uniformly and symmetrically distributed linguistic term sets; hence, the defined concept motivates the decision-makers to represent their opinions using 2-tuples composed of two proportional linguistic terms. The proportional 2-tuple linguistic representation of payoffs concerns linguistic labels, which do not certainly have to be symmetrically distributed or do not have the conventional prerequisite of having uniform distance among them. This representation confers an opportunity to describe the linguistic payoffs of a matrix game using members of a continuous linguistic scale domain. In our work, we have defined a two-player constant-sum proportional linguistic matrix game and proposed an approach of Proportional Linguistic Linear Programming (PLLP) to evaluate these game problems. The framed PLLP problem is then transformed into a crisp LPP that can be easily solved, decreasing the computation complexities involved in solving the linguistic decision-making problems. This perspective of proportional 2-tuples provides decision-makers an approach to represent their opinions not by just using one label, rather by proportional linguistic labels of the form $(\delta u_i, \gamma u_{i+1})$, where u_i and u_{i+1} are two successive linguistic terms, with $0 \leq \delta, \gamma \leq 1$ and $\delta + \gamma = 1$. Besides, some test examples are also presented to show the consistency of our designed approach. Further, the PLLP formulation is utilized to solve a Multi-Criteria Decision-Making problem based on actual-time linguistic data.

Keywords 2-tuple linguistic term set · Proportional linguistic term set · Linguistic linear programming problems · Proportional linguistic matrix games · MCDM problems

1 Introduction

Game Theory emerges in almost every aspect of human interaction. Therefore, it is considered a powerful tool for showcasing the interactions of two or more independent decision-makers having conflicting interests. The innovative work of Von Neumann pioneers the concept of game theory in the early twentieth century. In 1944, Neumann and Morgenstern (1944) presented preliminary concepts of game theory in their published work named “Theory of Games and Economic Behavior,” which forms a foundation for the present-day theories. The ultimate goal of game theory is to find the optimal solution(s) for players. This optimization means that we look for such values which maximize or minimize the payoffs of respective players. Considering

the importance of optimization in different research domains, several optimization methods have been presented so far. The optimization algorithms, which cover both types of maximization and minimization issues, have many applications in quantum differential evolution Cai et al. (2021); Deng et al. (2020a, b), decision-making, differential evolution algorithms Deng et al. (2021a, b); Song et al. (2021), etc. Game theory and optimization have a conceptual overlap. In other words, game theory is a kind of approximating optimization.

The game theory framework is designed to study the strategic interactions among two or more players, who form their strategies to win the game. The outcome of these interactions typically depends on the current player’s behavior and other players’ conduct in the game. In a more general scenario, the result often depends on each player’s expected payoff before making decisions, which measures each player’s satisfaction from making the decision. Thus, the players must carry out actions and take decisions that will give them maximum payoff. However, it is envisioned

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that several practical situations are concerned with data or elements involving uncertainty or impreciseness. Consequently, using the classical game-theoretic approaches in modeling such problems may offer results with imprecision and vagueness. In these cases, problems modeled via fuzzy subsets introduced by Zadeh (1965) seem appropriate and proffer results with high accuracy. As a result, several authors have studied fuzzy matrix games Butnariu (1978); Zaichenko (2020); Vijay et al. (2005a, b) in the literature and have proposed methodologies highly flexible to model game problems comprising uncertainty within the problem. For a detailed explanation of the methodologies, one may refer to the papers Bector et al. (2004a, b); Bector and Chandra (2005); Campos (1989); Dengfeng (1999).

Since the concept of fuzzy sets is uniquely stipulated in terms of membership functions associated with the elements, these memberships contribute significantly to fuzzy game theory. Nevertheless, the societal developments give rise to the complexity within the problems; as a result, representing the elements of payoffs in terms of fuzzy numbers is a challenging task. Moreover, in real decision-making problems, it is appropriate to estimate some outlooks using natural language. This implies the need to incline toward the linguistic aspect of a problem. Thus, Zadeh (1974, 1975a, b) introduced the concept of a linguistic variable to overcome this constraint. Till then, several linguistic computational models have been implemented to solve the problems having linguistic information Herrera and Martínez (2000, 2001a); Jiang et al. (2008); Herrera and Martínez (2001b). Further, to facilitate the players' straightforward selection of utilities, this newer form of ambiguous matrix game in terms of linguistics is suggested by Arfi (2006a, b) on the grounds of fuzzy linguistic approach. Delgado et al. (1990) have also focused on this non-conventional side of games where the players express their opinions in the forms of linguistic labels. Moreover, Sahoo (2017) has also solved matrix games having linguistic payoffs. In addition to these contributions, Singh et al. (2018) amplify the dimension of matrix games under a 2-tuple linguistic structure. In that paper, the authors have implemented the 2-tuple linguistic model Herrera and Martínez (2000) to solve a two-player linguistic matrix game having a constant sum. They have discussed the evaluation of saddle points and further formulated a Linear Programming approach involving linguistic terms to attain the respective optimal strategies of the game players. This approach has also been adopted for solving various Multi-Criteria Decision-Making (MCDM) problems. Furthermore, Singh and Gupta (2018), and Malhotra et al. (2019) extended the notion of matrix games to a newer approach of the 2-tuple linguistic model consisting of interval-valued games to increase the uncertainty incorporated within the game problem.

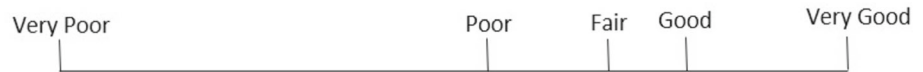
Considering the above-mentioned 2-tuple linguistic game problems, it is envisaged that the linguistic term set uti-

lized for addressing the information is a balanced linguistic term set (BLTS) because its terms are symmetrically and uniformly distributed. However, in a more logical viewpoint, some problems are not well suited to deal with BLTS, instead incorporating linguistic information that is non-uniformly distributed. A linguistic term set possessing a non-symmetrical distribution of terms is named an unbalanced linguistic term set (ULTS). According to the literature, there are two types of ULTS. First, in which the linguistic terms are asymmetrically distributed, and thus the distance between them is unequal (described in Fig. 1). Furthermore, second, in which the linguistic set possesses equal terms on either side of the middle term, but the spacing among terms is uneven (as described in Fig. 2).

Several methodologies have been developed in the literature to deal with these two types of ULTS, such as Abchir and Truck (2013); Bartczuk et al. (2012); Cai et al. (2016); Cai and Gong (2017); Herrera et al. (2008); Pei and Zheng (2016); Zou et al. (2012). Dong et al. (2009) have developed a 2-tuple linguistic model based on the notion of numerical scale. Further, Wang et al. (2015) have presented a normalized numerical scaling method for dealing with ULTS. Moreover, in order to extend the existing 2-tuple model so that it might address ULTS, Wang and Hao (2006) have established an advanced form of the linguistic representation model for unbalanced sets, namely the symbolic proportional model. This newly designed so-called symbolic proportional model is interestingly easy-going and is flexible in a computation stage for CWWs allowing decision-makers to evaluate their alternatives via proportional 2-tuples of the form $(\delta u_i, \gamma u_{i+1})$ where u_i, u_{i+1} are represented as two successive linguistic terms, such that $\delta, \gamma \in [0, 1]$, $\delta + \gamma = 1$.

In this work, the scope of 2-tuple linguistic matrix games is extended within the aforementioned unbalanced linguistic framework introduced by Wang and Hao (2006) to manage the uncertainty to a large extent. For this, an approach is developed to solve constant-sum linguistic matrix games involving proportional linguistic information. The main highlights and contributions of this paper are described as follows:

1. The methodology introduced in this paper is an entirely new approach as it is concerned with the matrix games consisting of the payoffs of the type, say $(0.6M_1, 0.4M_2)$ where M_1 and M_2 denote month 1 and month 2, respectively. The notation represents the case when the production of a commodity is reported as 60% in month 1 and 40% in month 2. This representation of payoffs is based on the concept of "symbolic proportion."

Fig. 1 One type of ULTS**Fig. 2** Another type of ULTS

2. Under proportional 2-tuple linguistic context of payoffs, not only the process of “Computing with Words” may be carried out without any loss of information in some sense, but the linguistic labels can also be dealt with, without having to be symmetrically distributed around a central label and the without having the necessity of being equidistant.
3. A two-player constant-sum proportional matrix game is developed to model game problems having unbalanced proportional payoffs. To solve such game problems, a Proportional Linguistic Linear Programming (PLLP) methodology is proposed to assess the proportional matrix game.
4. Finally, this PLLP formulation is used to solve a multi-criteria decision-making (MCDM) problem based on actual-time linguistic information to strengthen the applicability of this new version of matrix game problems.
5. The novelty of this paper lies in the fact that the unbalanced information among the game players has been taken care of; the defined methodology takes into account the complicated game problems where the payoffs belong to an LT set in which the linguistic terms are distributed unevenly. Matrix games having payoffs from a balanced LT set have been considered in earlier studies, but to the best of our knowledge, this defined study is new to address matrix games with payoffs from an unbalanced LT set.

This PLLP model happens to be a modified version of the Linguistic Linear Programming Problem as it involves proportional coefficients in constraints belonging to a proportional linguistic term set. Moreover, this new methodology not just presents an appropriate scope of computing matrix games with payoffs in the form of “words,” but it also offers an opportunity so that the decision-makers in several decision models can assess the functioning of alternatives by using not just a single label but with proportional 2-tuples (more flexible). To the extent of our knowledge, the proposed methodology is considered to be the first study within the literature addressing unbalanced linguistic information in matrix games.

The paper is set out in the following manner. Section 2 presents a brief review of proportional linguistic variables and few required conventions of matrix games. Then, we introduce a two-player constant-sum matrix game based on the proportional linguistic model in Sect. 3. In Sect. 4, a Proportional Linguistic Linear Programming is being formu-

lated in relevance to the proportional matrix game. Section 5 illustrates the application of proportional matrix games in the MCDM problem, evaluating the perspective of doctors in first-line treatment of constipation based on several alternatives been chosen on different criteria. In Sect. 6, we analyze the work done in the paper. Finally, Sect. 7 offers the conclusions.

2 Preliminaries

In this segment, we present fundamental notations, definitions, and properties of proportional linguistic variables. Further, this section reviews the notion of two-player matrix games.

2.1 Proportional linguistic variables (PLV)

The linguistic 2-tuple model presented by Herrera and Martínez (2000) comprises of linguistic terms belonging to a linguistic term (LT) set $U = \{u_i, i = 0, 1, \dots, g\}$. Generally, we assume the index of granularity in information uncertainty to be an even positive integer.

In the literature, Wang and Hao (2006) introduced an advanced version of the 2-tuple linguistic model as given below. This model is called the “symbolic proportional model” and is proficient enough in handling linguistic information asymmetrically.

Definition 1 Wang and Hao (2006) Suppose $U = \{u_0, u_1, \dots, u_g\}$ is an ordered set, $I = [0, 1]$ and

$$IU \equiv I \times U = \{(\delta, u_i) : \delta \in [0, 1] \ \& \ i = 0, 1, \dots, g\}.$$

Consider the pair (u_i, u_{i+1}) of two succeeding members of the set U , then any two elements $(\delta, u_i), (\gamma, u_{i+1}) \in IU$ are named as a symbolic proportion pair and δ, γ are termed as pair of symbolic proportions of the pair (u_i, u_{i+1}) if $\delta + \gamma = 1$. A symbolic proportion pair $(\delta, u_i), (1 - \delta, u_{i+1})$ is represented as $(\delta u_i, (1 - \delta)u_{i+1})$, and we denote the set of all symbolic proportions as $\bar{U}$, that is

$$\bar{U} = \{(\delta u_i, (1 - \delta)u_{i+1}), \delta \in [0, 1], i = 0, 1, \dots, g - 1\}$$

Definition 2 Wang and Hao (2006) **Negation of a Proportional Variable**

The negation of a PLV is described as:

$$\text{Neg}((\delta u_i, (1 - \delta)u_{i+1})) = ((1 - \delta)u_{g-i-1}, \delta u_{g-i})$$

Definition 3 Wang and Hao (2006) **Position Index Function**

Let $U = \{u_0, u_1, \dots, u_g\}$ be an ordinal term set and $\bar{U}$ be the resulting ordinal proportional 2-tuple set.

The position index function $\pi: \bar{U} \rightarrow [0, g]$ of an ordinal 2-tuple is defined as

$$\pi((\delta u_i, (1 - \delta)u_{i+1})) = i + (1 - \delta)$$

where $i = 0, 1, \dots, g - 1$ and $\delta \in [0, 1]$.

π is a bijection under identification convention, and its inverse function $\pi^{-1}: [0, g] \rightarrow \bar{U}$ may be determined as

$$\pi^{-1}(z) = ((1 - \gamma)u_i, \gamma u_{i+1})$$

where $i = E(z)$, E is the integer part function, $\gamma = z - i$ and $(1.u_g, 0.u_{g+1}) \equiv (0.u_{g-1}, 1.u_g)$.

Definition 4 Wang and Hao (2006) **Comparison of Proportional 2-tuples.**

For any $(\delta u_i, (1 - \delta)u_{i+1}), (\gamma u_j, (1 - \gamma)u_{j+1}) \in \bar{U}$

$$\begin{aligned} &(\delta u_i, (1 - \delta)u_{i+1}) < (\gamma u_j, (1 - \gamma)u_{j+1}) \\ \iff &i + (1 - \delta) < j + (1 - \gamma) \end{aligned}$$

- When $i < j$, then

- (a) $(\delta u_i, (1 - \delta)u_{i+1}) = (\gamma u_j, (1 - \gamma)u_{j+1})$ if $i = j - 1$ and $\delta = 0$ & $\gamma = 1$.
- (b) Otherwise, $(\delta u_i, (1 - \delta)u_{i+1}) < (\gamma u_j, (1 - \gamma)u_{j+1})$.

- When $i = j$, then

- (a) If $\delta = \gamma$, then $(\delta u_i, (1 - \delta)u_{i+1}) = (\gamma u_j, (1 - \gamma)u_{j+1})$.
- (b) If $\delta < \gamma$, then $(\delta u_i, (1 - \delta)u_{i+1}) > (\gamma u_j, (1 - \gamma)u_{j+1})$.
- (c) If $\delta > \gamma$, then $(\delta u_i, (1 - \delta)u_{i+1}) < (\gamma u_j, (1 - \gamma)u_{j+1})$.

It is worth noting that the above-mentioned comparison techniques can further be characterized by using position indices. Thus, for any $(\delta u_i, (1 - \delta)u_{i+1}), (\gamma u_j, (1 - \gamma)u_{j+1}) \in \bar{U}$, we have

$$\begin{aligned} &(\delta u_i, (1 - \delta)u_{i+1}) < (\gamma u_j, (1 - \gamma)u_{j+1}) \iff \\ &\pi(\delta u_i, (1 - \delta)u_{i+1}) \\ &< \pi(\gamma u_j, (1 - \gamma)u_{j+1}) \end{aligned}$$

and

$$\text{Neg}(\delta u_i, (1 - \delta)u_{i+1})$$

$$= \pi^{-1}(g - (\pi(\delta u_i, (1 - \delta)u_{i+1})))$$

Definition 5 Wang and Hao (2006) Let $U, \bar{U}$ be as before and suppose $\Delta: [0, g] \rightarrow U \times [-0.5, 0.5)$ be the function defined by Herrera and Martinez in Herrera and Martínez (2000). Let $\bar{U}^T$ be the range of the Δ function. Define $h: \bar{U} \rightarrow \bar{U}^T$ by

$$h((\delta u_i, (1 - \delta)u_{i+1})) = \begin{cases} (u_{i+1}, -\delta), & \text{if } 0 \leq \delta \leq 1/2 \\ (u_i, 1 - \delta), & \text{if } 1/2 < \delta \leq 1 \end{cases}$$

The 2-tuples present on the right side of the above equation are traditional 2-tuples as defined by Herrera & Martínez (2000). h is a bijective function that can be used to convert traditional 2-tuples into proportional 2-tuples.

2.2 A zero-sum matrix game

A two-player zero-sum game is the most elementary one, comprising of two players only. This type of game is assessed by assuming that both the players apply strategies that could be pure, mixed, or continuous (Parthasarathy and Raghavan (1971)). We recall a zero-sum game according to Barron (2011).

Definition 6 Barron (2011) A crisp two-player zero-sum matrix game is represented as a triplet (S^n, S^m, A) , where

- (i) S^n is the set of 'n' possible strategies of player I.
- (ii) S^m is the set of 'm' possible strategies of player II.
- (iii) A is the payoff matrix of order $n \times m$ of player I corresponding to player II, and $\text{neg}(A) = -A$ refers to the matrix consisting of payoffs of player II.

Definition 7 Barron (2011) A matrix game having $A = (a_{ij})_{n \times m}$ as the matrix of payoffs has the lower value

$$v^- = \max_{i=1, \dots, n} \min_{j=1, \dots, m} a_{ij}$$

and the upper value

$$v^+ = \min_{j=1, \dots, m} \max_{i=1, \dots, n} a_{ij}$$

The lower value v^- is that minimum amount that player I is secured to get, and the upper value v^+ is that assured maximum amount that player II might lose. We say that the game has a value if the upper value equals the lower value, and we express it as $v^+ = v^- = v$, and the row and column (i^*, j^*) giving payoff $a_{i^*j^*} = v^- = v^+$ are called optimal, or more precisely a saddle point in case the pure strategies

are applied. Further, $a_{ij^*} \leq a_{i^*j^*} \leq a_{i^*j}$, $\forall i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Definition 8 Barron (2011) A mixed strategy is a vector $\Phi = (\phi_1, \phi_2, \dots, \phi_n)$ for player I and $\Psi = (\psi_1, \psi_2, \dots, \psi_m)$ for player II, where

$$\phi_i \geq 0, \sum_{i=1}^n \phi_i = 1 \quad \& \quad \psi_j \geq 0, \sum_{j=1}^m \psi_j = 1.$$

ϕ_i represents the probability of using row 'i' by player I, and ψ_j , the probability of using column 'j' by player II. We represent the set of mixed strategies having 'k' components as

$$S^k = \{(z_1, z_2, \dots, z_k) \mid z_r \geq 0, r = 1, 2, \dots, k, \sum_{r=1}^k z_r = 1\}.$$

In this terminology, elements $\Phi \in S^n$ and $\Psi \in S^m$ represent the mixed strategies for player I and player II, respectively.

Now, in case of mixed strategies, the expected payoff of players is defined as follows:

Definition 9 Barron (2011) Given a choice of mixed strategy $\Phi = (\phi_1, \phi_2, \dots, \phi_n)$ for player I and $\Psi = (\psi_1, \psi_2, \dots, \psi_m)$ for player II, selected autonomously, the expected payoff of player I is

$$\begin{aligned} E(\Phi, \Psi) &= \sum_{i=1}^n \sum_{j=1}^m a_{ij} \text{Prob(I uses } i \text{ and II uses } j) \\ &= \sum_{i=1}^n \sum_{j=1}^m a_{ij} \text{Prob(I uses } i) \text{ Prob(II uses } j) \\ &= \sum_{i=1}^n \sum_{j=1}^m \phi_i a_{ij} \psi_j \\ &= \Phi A \Psi^T \end{aligned}$$

In both classical and fuzzy environments, numerous studies have been conducted for solving matrix games. However, this setup has been extended further to the linguistic games overcoming the limitation of providing precise membership values to the payoffs. The linguistic setup in games has provided a more flexible structure for the decision-makers to evaluate the game problems. This linguistic information may come from a balanced as well as an unbalanced LT set. The next section discusses the matrix games based on the information underlying an unbalanced (proportional approach) LT set.

3 Two-player constant-sum matrix games based on proportional linguistic information

In this section, we majorly focus on the non-conventional case of matrix games in which the players are capable of expressing knowledge in terms of unbalanced linguistic labels. First, we introduce a methodology to define and solve a two-player constant-sum proportional matrix game with terms in symbolic proportions. Furthermore, we also propose conditions that show the existence of duality in such types of matrix games.

Definition 10 Consider a two-player constant-sum proportional game $\tilde{G}$ as a quadruplet $(S^n, S^m, \bar{U}, \tilde{A})$ where

$$S^n = \{\Phi = (\phi_1, \phi_2, \dots, \phi_n) \mid \phi_i \geq 0, i = 1, 2, \dots, n, \sum_{i=1}^n \phi_i = 1\}$$

and

$$S^m = \{\Psi = (\psi_1, \psi_2, \dots, \psi_m) \mid \psi_j \geq 0, j = 1, 2, \dots, m, \sum_{j=1}^m \psi_j = 1\}$$

represent the strategy sets of player I and player II, respectively. The members of S^n (S^m) of the form $\Phi = (0, 0, \dots, 1, \dots, 0)^T$ where 1 is at the i th position ($\Psi = (0, 0, \dots, 1, \dots, 0)^T$ where 1 is at the j th position) are called pure strategies of Player I (Player II). $\bar{U}$ is the set of symbolic proportions as defined in Definition 1, $\tilde{A} = [\tilde{a}_{ij}]_{n \times m}$ being the proportional payoff matrix of player I corresponding to player II given as:

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1m} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \dots & \tilde{a}_{nm} \end{pmatrix}$$

where $\tilde{a}_{ij} \in \bar{U}$ represents payoff in the form of proportional 2-tuple. Basically if Player I chooses i^{th} pure strategy and Player II chooses j^{th} pure strategy, then $\tilde{a}_{ij}$ is the amount paid by Player II to Player I. Moreover, $neg(\tilde{A})$ comprehends the payoff matrix of player II which shows the amount paid by Player I to Player II. From the matrix $\tilde{A}$, it can clearly be understood that if player I & II opt for pure strategies i and j , respectively, then the payoff of player I would be $\tilde{a}_{ij}$ and that of player II would be $neg(\tilde{a}_{ij})$, i.e., if $\tilde{a}_{ij} = (\delta u_k, (1 - \delta)u_{k+1})$ is the payoff of player I for some k , then payoff of player II would be $neg(\tilde{a}_{ij}) = ((1 - \delta)u_{g-k-1}, \delta u_{g-k})$, where $g + 1$ is the cardinality of the LT set U .

We next define the values of player I and player II and the value of the game in terms of symbolic proportions. The meaning of the game's solution is best recognized in terms of maxmin and minmax principles for Player I and Player II, respectively. As per these principles, both the players choose that strategy, resulting in the best of the worst outcomes. Player I (the maximizing player) opts to play a analogous strategy to the maximum of minimum benefits for his different approach. This is called the maxmin principle. In similar terms, Player II (the minimizing player) also likes to play safe, and in that case, he opts for that strategy which is analogous to the minimum of maximum losses for his different approach. This is called the minmax principle. Implementing these principles, we obtain the following values of players.

Definition 11 Let us consider $\tilde{v}^{1-}$ and $\tilde{v}^{2-}$ as the values representing the minimum gain (in terms of symbolic proportion) as received by player I and player II, respectively. Then, these values are defined in the following manner:

$$\tilde{v}^{1-} = \max_{i=1,\dots,n} \min_{j=1,\dots,m} \tilde{a}_{ij}$$

$$\tilde{v}^{2-} = \max_{j=1,\dots,m} \min_{i=1,\dots,n} \text{neg}(\tilde{a}_{ij})$$

where $\tilde{a}_{ij} \in \bar{U}$.

Note: The lower and upper values of the game are defined as $\tilde{v}^- = \tilde{v}^{1-}$ and $\tilde{v}^+ = \text{neg}(\tilde{v}^{2-})$, respectively. Also,

$$\begin{aligned} \tilde{v}^+ &= \text{neg}(\tilde{v}^{2-}) = \text{neg}\left(\max_{j=1,\dots,m} \min_{i=1,\dots,n} \text{neg}(\tilde{a}_{ij})\right) \\ &= \min_{j=1,\dots,m} \max_{i=1,\dots,n} (\tilde{a}_{ij}) \end{aligned}$$

The game has value $\tilde{v}$ if $\tilde{v} = \tilde{v}^+ = \tilde{v}^-$.

Remark 1 Since it can be proved that π is a linear operator and is a monotonic function, we can easily conclude that if

$$\begin{aligned} \tilde{a} &= (\delta u_i, (1 - \delta)u_{i+1}) \\ \tilde{b} &= (\gamma u_j, (1 - \gamma)u_{j+1}) \end{aligned}$$

are two proportional linguistic variables (PLV's), then

- (I) $\text{neg}(\text{neg}(\tilde{a})) = \tilde{a}$
- (II) $\min(\text{neg}(\tilde{a}), \text{neg}(\tilde{b})) = \text{neg}(\max(\tilde{a}, \tilde{b}))$
- (III) $\max(\text{neg}(\tilde{a}), \text{neg}(\tilde{b})) = \text{neg}(\min(\tilde{a}, \tilde{b}))$

Remark 2 The maximum and minimum of two PLV's may be evaluated with the help of a comparison of proportional 2-tuples as described in Definition 4. That is, for every $j =$

$0, 1, \dots, g-1$, we define maximum of proportional 2-tuples as

$$\begin{aligned} \max((\delta u_i, (1 - \delta)u_{i+1})) &= (\gamma u_j, (1 - \gamma)u_{j+1}) \\ \iff (\delta u_i, (1 - \delta)u_{i+1}) &\leq (\gamma u_j, (1 - \gamma)u_{j+1}) \forall \\ i &= 0, 1, \dots, g-1. \end{aligned}$$

and similarly,

$$\begin{aligned} \min((\delta u_i, (1 - \delta)u_{i+1})) &= (\gamma u_j, (1 - \gamma)u_{j+1}) \\ \iff (\delta u_i, (1 - \delta)u_{i+1}) &\geq (\gamma u_j, (1 - \gamma)u_{j+1}) \forall \\ i &= 0, 1, \dots, g-1. \end{aligned}$$

Next, for the proportional linguistic values of the game defined in Definition 11, the following relation holds for any linguistic proportional game.

Theorem 1 Suppose $\tilde{v}^-$ and $\tilde{v}^+$ be the proportional lower and upper values of the game, respectively. Then, $\tilde{v}^-$ and $\tilde{v}^+$ are PLV's and $\tilde{v}^- \leq \tilde{v}^+$.

Proof Using Remark 2, it can easily be deduced that the minimum of two PLV's is further a PLV, and so is the maximum of minimum. Hence, max-min of any PLV comes out to be proportional.

$$\implies \tilde{v}^- = \tilde{v}^{1-} = \max_{i=1,\dots,n} \min_{j=1,\dots,m} \tilde{a}_{ij} \text{ is a PLV.}$$

Similar argument follows for $\tilde{v}^+$.

Now, suppose $\tilde{a}_{ij} \in \bar{U}$. Then, for any column j and a fixed row i , $\min_{j=1,\dots,m} \tilde{a}_{ij} \leq \tilde{a}_{ij}$ (using Comparison defined in Definition 4), thus taking maximum of both the sides of rows $i = 1, \dots, n$, we get

$$\tilde{v}^- = \max_i \min_j \tilde{a}_{ij} \leq \max_i \tilde{a}_{ij}$$

This holds for any column $j = 1, 2, \dots, m$. This means that $\tilde{v}^- \leq \min_j \max_i \tilde{a}_{ij} = \tilde{v}^+$, and hence the proof. $\square$

Definition 12 We call a particular row i^* and column j^* a saddle point in pure strategies of the game if

$$\tilde{a}_{ij^*} \leq \tilde{a}_{i^*j^*} \leq \tilde{a}_{i^*j}$$

$\forall$ rows $i = 1, 2, \dots, n$ and columns $j = 1, 2, \dots, m$.

Lemma 1 A proportional linguistic matrix game will have a saddle point in pure strategies if and only if

$$\tilde{v}^- = \max_i \min_j \tilde{a}_{ij} = \min_j \max_i \tilde{a}_{ij} = \tilde{v}^+.$$

Proof Using Definition 12,

$$\begin{aligned}\tilde{v}^+ &= \min_j \max_i \tilde{a}_{ij} \leq \max_i \tilde{a}_{ij^*} \leq \tilde{a}_{i^*j^*} \\ &\leq \min_j \tilde{a}_{i^*j} \leq \max_i \min_j \tilde{a}_{ij} = \tilde{v}^-\end{aligned}$$

But $\tilde{v}^- \leq \tilde{v}^+$ (Theorem 1), so we exhibit equality all round and $\tilde{v} = \tilde{v}^- = \tilde{v}^+ = \tilde{a}_{i^*j^*}$.

Further, if $\tilde{v}^- = \tilde{v}^+$, then

$$\min_j \max_i \tilde{a}_{ij} = \max_i \min_j \tilde{a}_{ij}$$

Suppose i^* be such that $\tilde{v}^+ = \max_j \tilde{a}_{i^*j}$ and j^* be such that $\tilde{v}^- = \min_i \tilde{a}_{ij^*}$. Then,

$$\tilde{a}_{i^*j} \leq \tilde{v}^+ = \tilde{v}^- \leq \tilde{a}_{ij^*}$$

for any $i = 1, 2, \dots, n, j = 1, 2, \dots, m$.

Moreover, taking $j = j^*$ on left and $i = i^*$ on the right, we get $\tilde{a}_{i^*j^*} = \tilde{v}^+ = \tilde{v}^-$. This suffices (i^*, j^*) to be the saddle point. $\square$

Now, we illustrate the above concept using an example.

Example 1 Suppose a builder tries to sell its two buildings I and II to a buyer. Three criteria namely Price, Architecture, and Gentry are selected for the sale. The payoffs are in terms of proportions chosen from the LT set

$$U = \{u_0 : \text{Very Low}, u_1 : \text{Low}, u_2 : \text{Average}, u_3 : \text{High}, u_4 : \text{Very High}\}.$$

The payoffs correspond to what the builder gets based on the above criterion. Let the payoff matrix of building I be:

Builder/Buyer	Price	Architecture	Gentry
Price	(0.2u ₂ , 0.8u ₃)	(0.2u ₂ , 0.8u ₃)	(0.2u ₂ , 0.8u ₃)
Architecture	(0.4u ₀ , 0.6u ₁)	(0.7u ₁ , 0.3u ₂)	(0.2u ₂ , 0.8u ₃)
Gentry	(0.4u ₀ , 0.6u ₁)	(0.1u ₃ , 0.9u ₄)	(0.1u ₃ , 0.9u ₄)

This means that if the buyer purchases building I by seeing its price and building II based on its architecture, respectively, then the builder is assured to receive a payoff which is 20% Average and 80% High of the market share. Moreover,

$neg(\tilde{A})$

Builder/Buyer	Price	Architecture	Gentry
Price	(0.8u ₁ , 0.2u ₂)	(0.8u ₁ , 0.2u ₂)	(0.8u ₁ , 0.2u ₂)
Architecture	(0.6u ₃ , 0.4u ₄)	(0.3u ₂ , 0.7u ₃)	(0.8u ₁ , 0.2u ₂)
Gentry	(0.6u ₃ , 0.4u ₄)	(0.9u ₀ , 0.1u ₁)	(0.9u ₀ , 0.1u ₁)

is the payoff matrix for building II .

The payoffs are interpreted in a way that if the buyer purchases building I based on Price and building II based

on Architecture, respectively, then the builder is assured to receive a payoff which is 80% Low and 20% Average of the market share.

The lower value for building I is

$$\begin{aligned}\tilde{v}^{1-} &= \max_{i=1,2,3} \min_{j=1,2,3} (\tilde{a}_{ij}) \\ &= \max [(0.2u_2, 0.8u_3), \\ &\quad (0.4u_0, 0.6u_1), (0.4u_0, 0.6u_1)] \\ &= (0.2u_2, 0.8u_3)\end{aligned}$$

Hence, the builder is assured of achieving at least a market share which is 20% Average and 80% High by selling building I .

In the similar manner, the lower value of building II is

$$\begin{aligned}\tilde{v}^{2-} &= \max_{i=1,2,3} \min_{j=1,2,3} (neg(\tilde{a}_{ij})) \\ &= \max [(0.8u_1, 0.2u_2), (0.8u_1, 0.2u_2), \\ &\quad (0.9u_0, 0.1u_1)] \\ &= (0.8u_1, 0.2u_2)\end{aligned}$$

Hence, the builder will surely receive a payoff of at least 80% Low and 20% Average by selling building II .

Also, the lower value of the game is

$$\tilde{v}^- = \tilde{v}^{1-} = (0.2u_2, 0.8u_3)$$

and the upper value is

$$\tilde{v}^+ = neg(\tilde{v}^{2-}) = (0.2u_2, 0.8u_3)$$

$\Rightarrow (1, 1)$ is the saddle point and $(0.2u_2, 0.8u_3)$ is the value of the game.

Next, it is worth noting that it is not always possible to apply pure strategy in every case, or a saddle point always exists. For example, if we interchange $\tilde{a}_{21}$ with $(0.9u_3, 0.1u_4)$ in the above matrix $\tilde{A}$, then $\tilde{v}^- = (0.2u_2, 0.8u_3)$ and $\tilde{v}^+ = (0.1u_3, 0.9u_4)$. So we have to apply the case where mixed strategies are involved. Hence, if the players propose mixed strategies to play the game, then their payoffs can be calculated by finding what is “expected” by the player to receive.

In the next section, we step forward to propose a Proportional Linguistic Linear Programming (PLLP) approach for solving the Proportional Matrix Game wherein players undergo mixed strategies (as well as pure strategies) to play the game.

4 Proportional linguistic linear programming (PLLP) formulation of a proportional matrix game

In the following section, we formulate Proportional Linguistic Linear Programming (PLLP) to solve a proportional matrix game (involving games using mixed strategies) by making use of the expected payoff to evaluate the payoffs of the players.

Let $U = \{u_0, u_1, \dots, u_g\}$ be the LT set and $\bar{U} = \{(\delta u_i, (1 - \delta)u_{i+1}) : \delta \in [0, 1] \text{ and } i = 0, 1, \dots, g - 1\}$ be the set of proportions, g being a positive integer. Consider the proportional linguistic matrix $\tilde{A} = (\tilde{a}_{ij})_{n \times m}$, where each $\tilde{a}_{ij} \in \bar{U}$ represents the payoff to player I received after opting the strategy 'i' while player II opts for strategy 'j'. Let $\Phi = (\phi_1, \phi_2, \dots, \phi_n) \in S^n$ and $\Psi = (\psi_1, \psi_2, \dots, \psi_m) \in S^m$ be the mixed strategies of player I & II, respectively. Then, in a proportional matrix game having uncertain and unbalanced payoff entries, the proportional expected payoff of players may be defined as

$$\tilde{E}_{\tilde{A}}(\Phi, \Psi) = \bigoplus_{i=1}^n \left(\phi_i \left(\bigoplus_{j=1}^m \tilde{a}_{ij} \psi_j \right) \right). \quad (1)$$

Thus, the expected payoff of player I corresponding to the j^{th} strategy chosen by player II is given by $\tilde{a}_{1j}\phi_1 \oplus \tilde{a}_{2j}\phi_2 \oplus \dots \oplus \tilde{a}_{nj}\phi_n$. This expression is the weighted average of all $\tilde{a}_{ij}$ which is defined as below:

Let $\tilde{a}_{ij} = (\delta u_k, (1 - \delta)u_{k+1}) \in \bar{U}$, $k = 0, 1, \dots, g - 1$ be a proportional 2-tuple variable and $w = (w_1, w_2, \dots, w_n)$ be the associated weight vector satisfying $0 \leq w_i \leq 1$, $i = 1, \dots, n$, $\sum_{i=1}^n w_i = 1$.

Then, the proportional weighted average operator may be defined in the following manner:

$$\begin{aligned} PWA[\tilde{a}_{ij}, i = 1, \dots, n] &= \tilde{a}_{1j}w_1 \oplus \tilde{a}_{2j}w_2 \oplus \dots \oplus \tilde{a}_{nj}w_n \\ &= \pi^{-1} \left(\sum_{i=1}^n w_i \pi(\tilde{a}_{ij}) \right), \\ &\quad j = 1, \dots, m \end{aligned}$$

Consequently,

$$\pi \left(\bigoplus_{i=1}^n \tilde{a}_{ij} w_i \right) = \sum_{i=1}^n w_i \pi(\tilde{a}_{ij}) \quad (2)$$

Note: Several operators are used to aggregate PLV's, e.g., CCV operator as defined in Wang and Hao (2006) by Wang and Hao. But usage of the CCV operator draws our attention toward the defuzzification of variables which is not our objective because of the information loss involved in the method.

So we use the π function for such purpose because here we deal with the position indices of the variables.

Hence, we need to evaluate the mixed strategy $\Phi \in S^n$ of player I such that the value of player I, i.e., $\tilde{v}^{1-}$ is maximized. So the required PLLP for player I is:

$$\begin{aligned} \max \quad & \tilde{v}^{1-} \\ \text{s.t.} \quad & \begin{cases} \bigoplus_{i=1}^n \tilde{a}_{ij} \phi_i \geq \tilde{v}^{1-} \\ \text{where} \\ \phi_i \geq 0, \sum_{i=1}^n \phi_i = 1. \\ \tilde{a}_{ij} \in \bar{U}. \end{cases} \end{aligned} \quad (P1)$$

As π is a monotonically increasing function, we apply π to each constraint of P1 and then use Equation 2.

Thus, the strategies and minimum gain of player I can be obtained by solving the following optimization problem:

$$\begin{aligned} \max \quad & \tilde{v}^{1-} \\ \text{s.t.} \quad & \begin{cases} \sum_{i=1}^n \pi(\tilde{a}_{ij}) \phi_i \geq \pi(\tilde{v}^{1-}) \\ \text{where} \\ \phi_i \geq 0, \sum_{i=1}^n \phi_i = 1. \\ \tilde{a}_{ij} \in \bar{U}. \end{cases} \end{aligned}$$

Substituting $\pi(\tilde{a}_{ij}) = a_{ij}$, and $\pi(\tilde{v}^{1-}) = v$. Here, it is clear that $\pi(\tilde{v}^{1-}) = v > 0$.

So, constraints of P1 become

$$\begin{aligned} \sum_{i=1}^n a_{ij} \phi_i &\geq v \\ \text{s.t. } \phi_i &\geq 0, \sum_{i=1}^n \phi_i = 1. \end{aligned}$$

Set $X_i = \frac{\phi_i}{v}$; $i = 1, 2, \dots, n$ and $\frac{1}{v} = V$. This is summarized as a linear program:

$$\begin{aligned} \min \quad & V = \sum_{i=1}^n X_i \\ \text{s.t.} \quad & \begin{cases} \sum_{i=1}^n a_{ij} X_i \geq 1 \\ \text{where} \\ X_i \geq 0. \end{cases} \end{aligned} \quad (P2)$$

and then, we can solve P2 as a usual linear programming problem.

It must be noted that in all the above constraints and expressions, the variable i varies as $i = 1, 2, \dots, n$ and the variable j varies as $j = 1, 2, \dots, m$.

Similarly, the PLLP for player II would be

$$\begin{aligned} \max \quad & \tilde{v}^{2-} \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \bigoplus_{j=1}^m \text{neg}(\tilde{a}_{ij}) \psi_j \geq \tilde{v}^{2-} \\ \text{where} \\ \psi_j \geq 0, \sum_{j=1}^m \psi_j = 1. \\ \tilde{a}_{ij} \in \bar{U}. \end{array} \right. \end{aligned} \quad (\text{P3})$$

We have to find the mixed strategy $\Psi \in S^m$ for player II. Let $\text{neg}(\tilde{v}^{2-}) = \tilde{v}^+$.

$\Rightarrow$ From the properties of negation operator, the problem becomes

$$\begin{aligned} \min \quad & \tilde{v}^+ \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \bigoplus_{j=1}^m \tilde{a}_{ij} \psi_j \leq \tilde{v}^+ \\ \text{where} \\ \psi_j \geq 0, \sum_{j=1}^m \psi_j = 1. \\ \tilde{a}_{ij} \in \bar{U}. \end{array} \right. \end{aligned} \quad (\text{P4})$$

Now, we apply π operator in P4 and use the expression in Equation 2.

Since earlier $\pi(\tilde{v}^{1-}) = v$ and if v is the value of the game, then $\pi(\tilde{v}^+) = v$.

So the problem for player II becomes

$$\begin{aligned} \min \quad & \tilde{v}^+ \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \sum_{j=1}^m \pi(\tilde{a}_{ij}) \psi_j \leq \pi(\tilde{v}^+) \\ \text{where} \\ \psi_j \geq 0, \sum_{j=1}^m \psi_j = 1. \\ \tilde{a}_{ij} \in \bar{U}. \end{array} \right. \end{aligned}$$

Put $Y_j = \frac{\psi_j}{v}$; $j = 1, 2, \dots, m$ and $\pi(\tilde{a}_{ij}) = a_{ij}$, the problem for player II reduces to the problem

$$\begin{aligned} \max \quad & V = \sum_{j=1}^m Y_j \\ \text{s.t.} \quad & \left\{ \begin{array}{l} \sum_{j=1}^m a_{ij} Y_j \leq 1 \\ \text{where} \\ Y_j \geq 0. \end{array} \right. \end{aligned} \quad (\text{P5})$$

So, by solving P2 and P5, we get the optimal solution $X = (X_1, X_2, \dots, X_n)$ and $Y = (Y_1, Y_2, \dots, Y_m)$ and the optimal value V . Hence, unwinding the formulation back to our original variables, the optimal strategies Φ and Ψ of player I and player II, respectively, could be achieved and the value v of the game follows. Also, the linguistic proportional

values for player I and player II would be

$$\tilde{v}^- = \tilde{v}^{1-} = \pi^{-1}\left(\frac{1}{V}\right)$$

and,

$$\tilde{v}^{2-} = \text{neg}\left(\pi^{-1}\left(\frac{1}{V}\right)\right)$$

respectively.

Again, the variables i & j vary as $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$ in all of the above constraints.

Note: It is to be noted that applying the π function directly in the payoff matrix to convert payoffs into numbers must not be a good idea because the value of α is different for each cell. Instead, to do so, we apply the function π on both sides of the PLLP (π being a monotonic function, this is justifiable).

Remark 3 Note that the linear programming problems P2 and P5 obtained by using a transformation function π are particular crisp problems corresponding to given proportional linguistic 2-tuples and form primal-dual pair of each other. Since it is envisaged that the translation function π is an isomorphism, henceforth, there exists a similarity relation between the numerical and symbolic proportional linguistic spaces, i.e., $\bar{U} \cong [0, g]$ and both are structurally similar sets. As the duality result holds for the crisp LPP, therefore, it follows directly for PLLP as well.

Example 2 Consider a 3×3 proportional payoff matrix for a constant sum game whose elements belong to the set

$$U = \{u_0 : \text{Very Poor}, u_1 : \text{Poor}, u_2 : \text{Average}, u_3 : \text{Good}, u_4 : \text{Very Good}\}$$

given as

$$\tilde{A} = \begin{bmatrix} (0.2u_0, 0.8u_1) & (0.3u_1, 0.7u_2) & (0.3u_1, 0.7u_2) \\ (0.3u_1, 0.7u_2) & (0.4u_2, 0.6u_3) & (0.9u_3, 0.1u_4) \\ (0.2u_0, 0.8u_1) & (0.2u_0, 0.8u_1) & (0.4u_2, 0.6u_3) \end{bmatrix}$$

Let $\Phi = (\phi_1, \phi_2, \phi_3)$, $\phi_i \geq 0$ for $i = 1, 2, 3$ such that $\sum_{i=1}^3 \phi_i = 1$, and $\Psi = (\psi_1, \psi_2, \psi_3)$, $\psi_j \geq 0$, $j = 1, 2, 3$ such that $\sum_{j=1}^3 \psi_j = 1$ be the mixed strategies for player I and player II, respectively. We formulate the PLLP for player I as in P1 in following manner:

$$\max \quad \tilde{v}^-$$

$$\text{s.t.} \begin{cases} (0.2u_0, 0.8u_1)\phi_1 \oplus (0.3u_1, 0.7u_2)\phi_2 \oplus (0.2u_0, 0.8u_1)\phi_3 \geq \tilde{v}^- \\ (0.3u_1, 0.7u_2)\phi_1 \oplus (0.4u_2, 0.6u_3)\phi_2 \oplus (0.2u_0, 0.8u_1)\phi_3 \geq \tilde{v}^- \\ (0.3u_1, 0.7u_2)\phi_1 \oplus (0.9u_3, 0.1u_4)\phi_2 \oplus (0.4u_2, 0.6u_3)\phi_3 \geq \tilde{v}^- \\ \phi_1 + \phi_2 + \phi_3 = 1 \\ \phi_1, \phi_2, \phi_3 \geq 0. \end{cases}$$

Applying π function in the constraints, we get

$$\begin{aligned} 0.8\phi_1 + 1.7\phi_2 + 0.8\phi_3 &\geq \pi(\tilde{v}^-) \\ 1.7\phi_1 + 2.6\phi_2 + 0.8\phi_3 &\geq \pi(\tilde{v}^-) \\ 1.7\phi_1 + 3.1\phi_2 + 2.6\phi_3 &\geq \pi(\tilde{v}^-) \\ \phi_1 + \phi_2 + \phi_3 &= 1 \\ \phi_1, \phi_2, \phi_3 &\geq 0. \end{aligned}$$

If $\pi(\tilde{v}^-) = v$, substituting $X_i = \frac{\phi_i}{v}$ and $V = \frac{1}{v}$, the problem reduces in the similar manner as P2 like

$$\begin{aligned} \min \quad & V = X_1 + X_2 + X_3 \\ \text{s.t.} \quad & \begin{cases} 0.8X_1 + 1.7X_2 + 0.8X_3 \geq 1 \\ 1.7X_1 + 2.6X_2 + 0.8X_3 \geq 1 \\ 1.7X_1 + 3.1X_2 + 2.6X_3 \geq 1 \\ X_1, X_2, X_3 \geq 0. \end{cases} \end{aligned}$$

The optimal solution of this problem is $(X_1, X_2, X_3) = (0, 0.59, 0)$ and $V = 0.59 \implies (\phi_1, \phi_2, \phi_3) = (0, 1, 0)$ and $v = 1.6949$. So, $\tilde{v}^- = \pi^{-1}(v) = \pi^{-1}(1.6949) = (0.3051u_1, 0.6949u_2)$. Hence, the optimal strategy for player I is $\Phi = (0, 1, 0)$ and the value is $\tilde{v}^- = (0.3051u_1, 0.6949u_2)$.

Similarly, the PLLP for player II as in P4 is

$$\begin{aligned} \min \quad & \tilde{v}^+ \\ \text{s.t.} \quad & \begin{cases} (0.2u_0, 0.8u_1)\psi_1 \oplus (0.3u_1, 0.7u_2)\psi_2 \oplus (0.3u_1, 0.7u_2)\psi_3 \leq \tilde{v}^+ \\ (0.3u_1, 0.7u_2)\psi_1 \oplus (0.4u_2, 0.6u_3)\psi_2 \oplus (0.9u_3, 0.1u_4)\psi_3 \leq \tilde{v}^+ \\ (0.2u_0, 0.8u_1)\psi_1 \oplus (0.2u_0, 0.8u_1)\psi_2 \oplus (0.4u_2, 0.6u_3)\psi_3 \leq \tilde{v}^+ \\ \psi_1 + \psi_2 + \psi_3 = 1 \\ \psi_1, \psi_2, \psi_3 \geq 0. \end{cases} \end{aligned}$$

By substitution as given in P5, the problem becomes

$$\begin{aligned} \max \quad & V = Y_1 + Y_2 + Y_3 \\ \text{s.t.} \quad & \begin{cases} 0.8Y_1 + 1.7Y_2 + 1.7Y_3 \leq 1 \\ 1.7Y_1 + 2.6Y_2 + 3.1Y_3 \leq 1 \\ 0.8Y_1 + 0.8Y_2 + 2.6Y_3 \leq 1 \\ Y_1, Y_2, Y_3 \geq 0. \end{cases} \end{aligned}$$

The optimal solution is $(Y_1, Y_2, Y_3) = (0.59, 0, 0)$ and $V = 0.59 \implies (\psi_1, \psi_2, \psi_3) = (1, 0, 0)$ and $v = 1.6949$. So, $\tilde{v}^+ = \pi^{-1}(v) = \pi^{-1}(1.6949) = (0.3051u_1, 0.6949u_2)$.

$\therefore$ the optimal strategy used by player II is $\Psi = (1, 0, 0)$ and optimal value is $\tilde{v}^+ = (0.3051u_1, 0.6949u_2) = \tilde{v}$.

It is noteworthy that $\tilde{v}^- = \tilde{v}^+ = (0.3051u_1, 0.6949u_2) = \tilde{v}$ is the value of the game.

For a better understanding of the process of evaluating a proportional linguistic matrix game, one can refer to Fig. 3.

In the subsequent section, we apply the proposed proportional linguistic game-theoretic approach in solving a linguistic MCDM problem to validate the proposed methodology in a physical realm.

5 Proportional matrix game for MCDM problems

Decision-making is a necessary process to achieve the best possible solution for a problem as per the information provided by the decision-makers. For example, in an MCDM problem, there are different alternatives (A_i) which are to be chosen based on some criterion (C_j) using the previously defined linguistic terms set LT. As explained by Chen and Larbani (2006), the aim of the decision-maker (DM) is to pick the best alternative from the available data in the matrix with fuzzy payoffs. The decision maker (DM) and the Nature act as two players of the game called player I and player II, respectively, having strategies $\Phi = (\phi_1, \phi_2, \dots, \phi_n)$ and $\Psi = (\psi_1, \psi_2, \dots, \psi_m)$, where $\sum_{i=1}^n \phi_i = 1$, $\sum_{j=1}^m \psi_j = 1$ and $\phi_i, \psi_j \geq 0$. Hence, a two-player zero-sum game is obtained in which $\tilde{a}_{ij}$ is treated as the payoff obtained by the DM after choosing the i^{th} strategy having probability ϕ_i and the Nature chooses the j^{th} strategy having probability ψ_j . This game problem is then solved using some models, and the strategies of both the players are evaluated along with the value of the game.

In the sequence, Singh et al. (2018) illustrated the applications of the linguistic matrix game approach in the MCDM problem. They aim to select the most appropriate alternative out of all based on some attributes. Analogously, we present the following example to solve an MCDM problem based on the proposed proportional linguistic matrix game approach.

Example 3 The pharmaceutical company Abbott is willing to understand the perspective of doctors in the “first-line treatment” of constipation. The study is conducted on different laxatives, namely

- A_1 : Milk of Magnesia (MoM)
- A_2 : Lactulose
- A_3 : Isphagula
- A_4 : PEG

The decision of doctors is based on the following attributes:

- C_1 : Safest Laxative

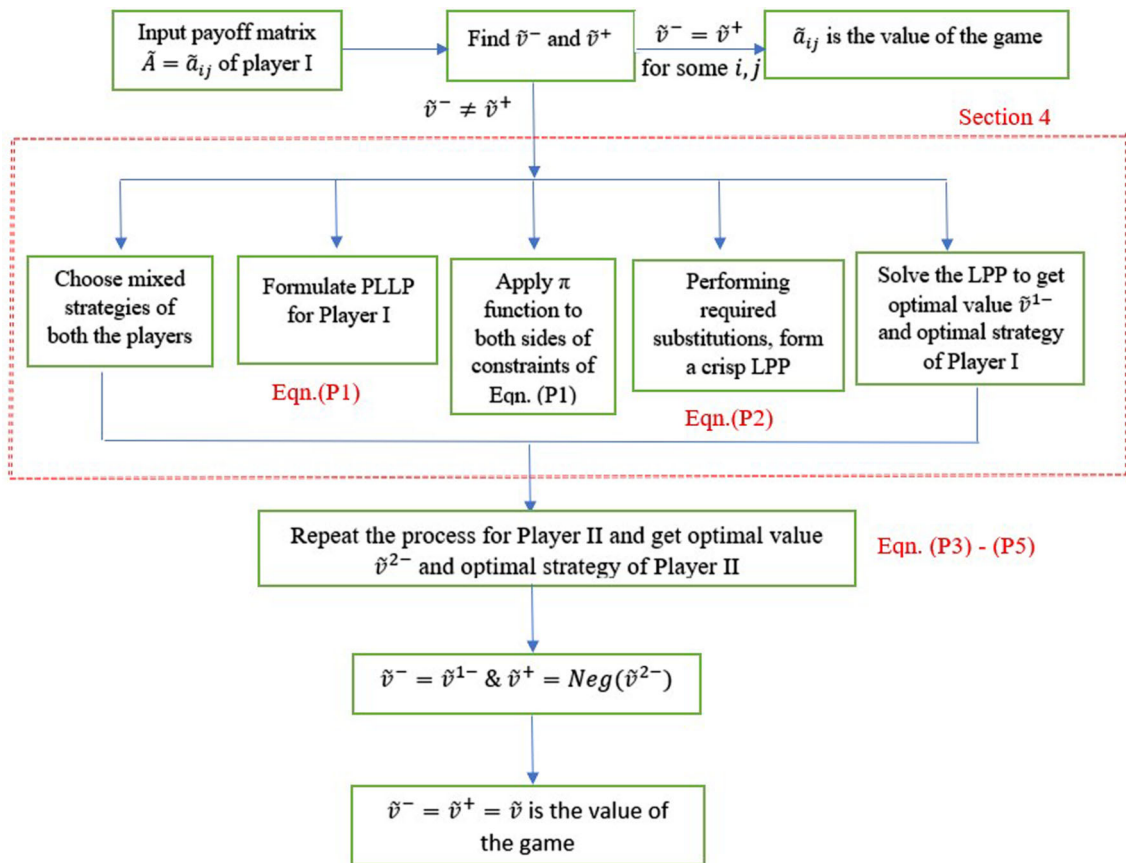


Fig. 3 The methodology to evaluate Proportional Linguistic Matrix Game problem

- C_2 : Fast Acting
- C_3 : Provides Overnight Relief
- C_4 : Provides Long-lasting Relief
- C_5 : Cost Effective
- C_6 : Is a mild laxative
- C_7 : Suitable for Long Term Usage
- C_8 : Backed by Strong Clinical Evidence
- C_9 : Non-habit forming

The doctors provide the response to each alternative based on given criteria using a predefined linguistic term set:

$U = \{u_0 : \text{Not Effective}, u_1 : \text{Less Effective}, u_2 : \text{Average Effective}, u_3 : \text{Very Effective}, u_4 : \text{Most Effective}\}$

The following matrix having proportional linguistic pay-offs is for the game problem between alternatives and criterion:

$$\tilde{A} = \begin{matrix} & \begin{matrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \begin{bmatrix} (u_0, 0) & (0.35u_0, 0.65u_1) & (0.71u_0, 0.29u_1) & (u_0, 0) & (0.76u_0, 0.24u_1) & (0.76u_0, 0.24u_1) & (u_0, 0) & (0.79u_0, 0.21u_1) & (u_0, 0) \\ (0.44u_1, 0.56u_2) & (0.61u_2, 0.39u_3) & (0.43u_1, 0.57u_2) & (0.35u_2, 0.65u_3) & (0.42u_1, 0.58u_2) & (0.77u_0, 0.23u_1) & (0.56u_1, 0.44u_2) & (0.66u_2, 0.34u_3) & (0.58u_1, 0.42u_2) \\ (0, u_4) & (u_0, 0) & (0.33u_0, 0.67u_1) & (0.56u_1, 0.44u_2) & (0.24u_2, 0.76u_3) & (0.22u_2, 0.78u_3) & (0.22u_2, 0.78u_3) & (0.54u_1, 0.46u_2) & (0.25u_2, 0.75u_3) \\ (u_0, 0) & (0.53u_1, 0.47u_2) & (0.37u_0, 0.63u_1) & (0.52u_1, 0.48u_2) & (0.8u_0, 0.2u_1) & (u_0, 0) & (u_0, 0) & (0.42u_1, 0.58u_2) & (u_0, 0) \end{bmatrix} \end{matrix}$$

The payoffs are described so that the payoff $(0.35u_0, 0.65u_1)$ depicts that 35% of the doctors consider the laxative as non-effective while 65% doctors find it effective, but some less.

In accordance with P2, the corresponding linear programming problem for alternatives would be

$$\begin{aligned} \min \quad & V = X_1 + X_2 + X_3 + X_4 \\ \text{s.t.} \quad & \begin{cases} 0X_1 + 1.56X_2 + 4X_3 + 0X_4 \geq 1 \\ 0.65X_1 + 2.39X_2 + 0X_3 + 1.47X_4 \geq 1 \\ 0.29X_1 + 1.57X_2 + 0.67X_3 + 0.63X_4 \geq 1 \\ 0X_1 + 2.65X_2 + 1.44X_3 + 1.48X_4 \geq 1 \\ 0.24X_1 + 1.58X_2 + 2.76X_3 + 0.2X_4 \geq 1 \\ 0.24X_1 + 0.23X_2 + 2.78X_3 + 0X_4 \geq 1 \\ 0X_1 + 1.44X_2 + 2.78X_3 + 0X_4 \geq 1 \\ 0.21X_1 + 2.34X_2 + 1.46X_3 + 1.58X_4 \geq 1 \\ 0X_1 + 1.42X_2 + 2.75X_3 + 0X_4 \geq 1 \\ X_1, X_2, X_3, X_4 \geq 0. \end{cases} \end{aligned}$$

The optimal solution of above LPP is $(0, 0.5, 0.32, 0)$ & $V = 0.82$, and hence by implementing the required computations, the optimal strategy of player I comes out to be $(\phi_1, \phi_2, \phi_3, \phi_4) = (0, 0.6095, 0.39008, 0) \approx (0, 0.6, 0.4, 0)$. Moreover, $v = 1.219$ and hence $\tilde{v}^- = (0.781u_1, 0.219u_2)$.

Similarly, for attributes, the LPP formulated using P5 is

$$\begin{aligned} \max \quad & V = Y_1 + Y_2 + \dots + Y_9 \\ \text{s.t.} \quad & \begin{cases} 0Y_1 + 0.65Y_2 + 0.29Y_3 + 0Y_4 + 0.24Y_5 + 0.24Y_6 + 0Y_7 + 0.21Y_8 + 0Y_9 \leq 1 \\ 1.56Y_1 + 2.39Y_2 + 1.57Y_3 + 2.65Y_4 + 1.58Y_5 + 0.23Y_6 + 1.44Y_7 + 2.34Y_8 + 1.42Y_9 \leq 1 \\ 4Y_1 + 0Y_2 + 0.67Y_3 + 1.44Y_4 + 2.76Y_5 + 2.78Y_6 + 2.78Y_7 + 1.46Y_8 + 2.75Y_9 \leq 1 \\ 0Y_1 + 1.47Y_2 + 0.63Y_3 + 1.48Y_4 + 0.2Y_5 + 0Y_6 + 0Y_7 + 1.58Y_8 + 0Y_9 \leq 1 \\ Y_1, Y_2, \dots, Y_9 \geq 0. \end{cases} \end{aligned}$$

The solution obtained from this LPP is $(0, 0, 0.61, 0, 0, 0.21, 0, 0, 0)$ & $V = 0.82$.

$\therefore$ The optimal strategy used by player II comes out to be $(\psi_1, \psi_2, \dots, \psi_9) = (0, 0, 0.7439, 0, 0, 0.2559, 0, 0, 0)$. Also, $v = 1.219$. So, $\tilde{v}^+ = (0.781u_1, 0.219u_2)$.

$\Rightarrow \tilde{v}^- = \tilde{v}^+ = \tilde{v} = (0.781u_1, 0.219u_2)$ is the value of the game.

This means that based on overnight relief and the power of laxative, 78.1% doctors find the laxative “Lactulose” followed by “Isphagula” as effective (but less) and only 21.9%

doctors find it not effective out of all other laxatives in the first line treatment of constipation.

6 Analysis of the work done

This section presents the analysis of the work done in the paper. Section 6.1 offers the analysis of the results obtained in the previous sections, and Sect. 6.2 compares the defined methodology with the existing methodologies. In Sect. 6.3, we discuss the computation complexity of the proposed methodology.

6.1 Result analysis

In this subsection, we discuss the results of the game's value and the players' strategies obtained in earlier sections.

Lemma 1 in Sect. 3 describes the case of having saddle point if the players in the game apply the pure strategies. Example 1 depicts the existence of saddle point in case pure strategies are played as $\tilde{v}^-$ comes out to be equal to $\tilde{v}^+$. Hence, the value of the game may be obtained as $\tilde{v} = \tilde{v}^- = \tilde{v}^+$ and the row (column) for which value of the game is obtained is called the optimal pure strategy of player I (player II). The same may be evaluated for player II by applying this procedure on the negation of the game matrix corresponding to player I.

In case the pure strategies cannot be applied, the players use mixed strategies to evaluate the value of the game. This

occurs when $\tilde{v}^- \neq \tilde{v}^+$, which is depicted in Example 2 of Sect. 4. For such a case, a PLLP is formulated for player I in equation P1 which consists of proportional linguistic payoffs. These proportional linguistic payoffs maybe then converted into numerical terms by doing required substitutions. By doing so, the original PLLP may be converted into a usual crisp LPP that can be solved directly. The solution of the LPP maybe then re-converted into linguistic form by reverting the substitutions. This gives the values of both the players and the value of the game, and the players' strategies

may be found. The same process is then repeated for player II whose PLLP is as given in equation P3. Equal values of both the players provide the value of the game.

6.2 Comparative analysis

This subsection compares the proposed study with the existing theories of matrix games having payoffs from an uncertain domain.

Many theories deal with matrix games having fuzzy payoffs, e.g., Bector et al. (2004a,b); Bector and Chandra (2005); Campos (1989); Dengfeng (1999); Vijay et al. (2005a); Zaichenko (2020). These theories discuss matrix games whose payoffs belong to a fuzzy set. Nevertheless, these theories have an associated limitation that the payoffs may take the form of linguistic variables as well. Hence, matrix games are important to study in the realm of linguistic term sets also. Researchers have worked in the field of linguistic matrix games and have found significant results Arfi (2006a,b); Sahoo (2017). However, in these methods, the linguistic payoffs are defuzzified to convert them into numeric LPPs. Defuzzification of payoffs causes the loss of information. The loss of information of payoffs is handled well by Singh et al. (2018); Singh and Gupta (2018) and Malhotra et al. (2019) where the payoffs belong to a balanced LT set, and these payoffs are converted into crisp values using a bijective function. But these theories are limited to a balanced LT set only. The theory defined in this paper is believed to be the first study considering matrix games having payoffs from an unbalanced LT set. The payoffs belong to a proportional LT set whose proportions belong to the interval $[0, 1]$. Moreover, the payoffs belonging to balanced LT sets may also be converted to their corresponding proportional ones by using the inverse of the h function defined in Definition 5. Note that this can only be done for a particular δ because δ is different for each cell. Hence, the matrix games with 2-tuple linguistic payoffs as defined by Singh et al. (2018) may also be solved by considering it as a particular case of the methodology defined in this paper.

6.3 Computation complexity

This subsection discusses the computation complexity involved in the defined methodology of unbalanced linguistic matrix games.

If we consider a game problem as a decision-making problem, then rather than picking random weights, the weights in this method are calculated within the game problem itself (considering that the mixed strategies play the role of weights). This makes the computation of this approach more appropriate than the existing decision-making methodologies. Moreover, the PLLP of Players I and II is first converted into usual crisp LPPs that are easy and fast to compute (man-

ually or by using any optimization software), and the result is then re-translated into the linguistic one using bijective function. Thus, the defined methodology considers the complexity in solving game problems and computes them without making any “loss of information.”

7 Concluding remarks

The payoffs based on symmetrical linguistic term sets have been successfully applied in many game problems to date. However, in a more realistic viewpoint, the payoffs of a matrix game may not always belong to an LT set where the terms are equidistant. Overcoming this limitation, we consider matrix games having payoffs belonging to an asymmetrical LT set in our paper. These payoffs are expressed in ordinal linguistic information via 2-tuples, comprising of two proportional linguistic terms with coefficients in $[0, 1]$. This representation of payoffs is more advantageous as it assesses the payoffs consisting of linguistic terms that belong to a continuous scale. To compute such game problems, first, the case of pure strategies is discussed where the existence of saddle points is shown. Further, when pure strategies cannot be applied, the application of mixed strategies is discussed. For this, a Proportional Linguistic Linear Programming (PLLP) approach is introduced to obtain the optimal strategies for players I and II. The value of such a game, where the conventional requirement of linguistic terms being equidistant may not be required, is also discussed. This approach is then illustrated with the help of some examples.

Lastly, we develop an application of our approach for solving the linguistic MCDM problem to demonstrate its validity in a practical sense. The suggested approach provides an integrated structure as it can be accepted to solve unbalanced linguistic matrix games and PLLP problems, and MCDM problems.

Although our approach is advantageous in solving matrix games where the data are asymmetrical or unbalanced, giving rise to unbalanced linguistic matrix games, it has a limitation associated with it. Concerning the proportional linguistic term set, the symbolic proportions are allocated only to two consecutive terms of a linguistic term set. So, in the future, we look forward to extending our methodology to solve unbalanced linguistic matrix games overcoming the current limitation and enhancing the applicability of linguistic matrix games in this ambiguous world. Further, we anticipate that this methodology may be extended to solve n -person games and can be applied to evaluate the results in cooperative game theory.

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Declarations

Conflict of Interest Authors declare that they have no conflict of interest.

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Curriculum Vitae

I am Parul Chauhan, born and brought-up in Delhi. I come from a rural background where I belong to a joint family. I have completed my secondary school from Brahma Shakti Public School, 83.4% (CBSE board) and higher secondary school from Jain Bharati Mrigavati Vidyalaya, 83.2% (CBSE Board). I have completed B.Sc. (Honours) Mathematics from Indraprastha College for Women, University of Delhi with 73% and M.Sc. Mathematics from Lady Shri Ram College for Women, University of Delhi with 69%. I have a teaching experience of 3 years (2015-2018) from HMR Institute of Technology and Management, Guru Gobind Sing Indraprastha University, Delhi. Currently I am pursuing Ph.D. from Delhi Technological University. My research area is Optimization techniques.

I got admission in Ph.D. in Delhi Technological University in 2019. I have published 3 research papers in reputed international journals (SCI/ SCIE indexing) and one in a Scopus-indexed conference proceeding. Besides that, two more research papers have been communicated. Two research papers are presented in the international conferences. I have attended a one-week short-term training programme on “Graph Theory and Its Applications” during November 25-29, 2019.

I have been awarded the Commendable Research award for Excellence in Research by Delhi Technological University during the year 2021. I have been an organizing member in the short-term training program entitled “Graph Theory and its Applications” sponsored by TEQIP-III. I have the knowledge of languages like C, C++, Maple, MATLAB and some software including SPSS, Latex.

My aim for life is to be an excellent researcher and a professor in the field of Applied Mathematics. My attitude towards life has been optimistic and embracing like embracing new challenges and opportunities to a better version of myself at every stage.