BERNSTEIN OPERATOR AND IT'S VARIOUS MODIFICATIONS

Thesis Submitted
in Partial Fulfillment of the Requirement for the
Degree of

MASTER OF SCIENCE
IN
APPLIED MATHEMATICS

by

Amrita Mishra
(2K22/MSCMAT/04)

Shivansh Pratap Singh
(2K22/MSCMAT/39)

Under the supervision of
Prof. Naokant Deo
Department of Applied Mathematics
Delhi Technological University

DEPARTMENT OF APPLIED MATHEMATICS
DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College of Engineering)
Shahbad Daulatpur, Main Bawana Road, Delhi–110042,India

June, 2024

DELHI TECHNOLOGICAL UNIVERSITY

(Formerly Delhi College of Engineering)
Shahbad Daulatpur, MainBawana Road, Delhi-42

CANDIDATE'S DECLARATION

We, **Amrita Mishra**, Roll No. 2K22/MSCMAT/04, and **Shivansh Pratap Singh**, Roll No. 2K22/MSCMAT/39, hereby certify that the work which is being presented entitled *"Bernstein Operator and it's Various Modifications"*, in partial fulfillment of the requirements for the award of Degree of Master of Science, submitted in the Department of Applied Mathematics, Delhi Technological University is an authentic record of our own work carried out during the period from August 2023 to May 2024 under the supervision of Prof. Naokant Deo.
The matter presented in the thesis has not been submitted by us for the award of any other degree of this or any other Institute.

Candidate's Signature **Candidate's Signature**

This is to certify that the students have incorporated all the correction suggested by the examiners in the thesis and the statement made by the candidates is correct to the best of our knowledge.

Signature of Supervisor **Signature of External Examiner**

DELHI TECHNOLOGICAL UNIVERSITY

(Formerly Delhi College of Engineering)

Shahbad Daulatpur, MainBawana Road, Delhi-42

CERTIFICATE BY THE SUPERVISOR

Certified that **Amrita Mishra** (2K22/MSCMAT/04) and **Shivansh Pratap Singh** (2K22/MSCMAT/39) have carried their search work presented in this thesis entitled **"BERNSTEIN OPERATOR AND IT'S VARIOUS MODIFICATIONS"** for the award of **Master of Science** from Department of Applied Mathematics, Delhi Technological University, Delhi, under my supervision. The thesis embodies results of original work, and studies are carried out by the students themselves and the contents of the thesis do not form the basis for the award of any other degree to the candidate or to anybody else from this or any other University.

Signature

(Prof. Naokant Deo)

(Department of Applied Mthematics)

Date:

DELHI TECHNOLOGICAL UNIVERSITY

(Formerly Delhi College of Engineering)

Shahbad Daulatpur, MainBawana Road, Delhi-42

ACKNOWLEDGEMENT

We want to express our appreciation to Prof Naokant Deo, Department of Applied Mathematics, Delhi Technological University, New Delhi, for his careful and knowledgeable guidance, constructive criticism, patient hearing, and kind demeanour throughout our ordeal of the present report. We will always be appreciative of his kind, helpful and insightful advice, which served as a catalyst for the effective completion of our dissertation report.

We are grateful to our Department of Mathematics for their continuous motivation and involvement in this project work. We are also thankful to all those who, in any way, have helped us in this journey. I am indebted to all the PhD academics under Prof. Naokant Deo who healped us greatly in successful completeion of this thesis.

Finally, we are thankful to the efforts of our parents and family members for supporting us with this project.

Abstract

This thesis focuses on the convergence rate and error behaviour for different versions of the Bernstein operator, a crucial tool in approximation theory. The instrumental application of the first of these was made by Sergei Natanovich Bernstein in 1912 when he was proving Weierstrass's approximation theorem constructively. The result, which implies that on a closed interval any continous function can be approximated uniformly by polynomials, is one of the most important results in mathematical analysis and provides a basis for much of modern approximation theory.

In this regard we start our exploration with some basic findings, lemmas' etc. elucidating characteristics and properties of various versions of the Bernstein operator we will discussed below. Then we g0 more deeply into understanding basics. First, we thoroughly introduce what a classical Bernstein operator is all about; its birth, fundamental attributes and specific conditions where it can converge to an intended function as highlighted in the paper. Thus, we will include here the comprehensive examinations into graphic presentations of distinct degrees within Bernstein polynomials themselves. This makes it easier to see how these functions approach other given functions from visualization as their order grows bigger. To this, we examine the behavior of convergence for Bernstein polynomials with different degrees by considering the corresponding errors and how these errors diminish with an increase in polynomial degree.

Such preliminary information allows a detailed examination on several adjustments to be made on the Bernstein operator. These changes aim at making the Bernstein operator more useful in practical applications by either reducing approximation errors or increasing convergence rates. We also present graphical representations which illustrate their behavior and properties of error and convergence features of each case.

Next are some particular types of Bernstein operators like Bernstein-Kantorovich, α-Bernstein, Bernstein-Chlodovsky, Λ-Bernstein, and Bernstein-Durrmeyer variations and more, that we will look into later. We study individual characteristics of each adjustment highlighting subtle mathematical differences from classical Bernstein operator. Our aim is to determine whether amendments provided higher convergence rates or lesser approximation errors when compared to other operators for a given order polynomial. This comparison analysis must be done in order to choose the best operator for a certain application as well as ensuring its maximum efficiency.

This study also investigates the various real-life applications of the Bernstein operator and its variations apart from theoretical work. These operators are not limited to abstract mathematical creations but have many practical uses in a variety of areas. For instance, the exactness of these operators' approximation ability can be employed in reconstructing and accurately simulating human facial features during facial surgery. Also, the process helps in improvement of voice recognition systems through speech analysis and modeling. This makes it a more accurate system for recognizing various human voices as well as improving others. These illustrations indicate that use of Bernstein operator or any other form is highly critical since it is required by practicality. The basic idea behind this thesis is to provide detailed knowledge about various forms of Bernstein operator. For example, their convergence rates could be examined along with an error behavior as well as performance comparisons; thus would help us understand how best to apply them theoretically and practically speaking. It seeks to ad-

vance approximation theory while enhancing application of these powerful mathematical tools in solving real-world problems.

Contents

Chapter 1

Introduction

The area of mathematical analysis known as approximation theory enables us to approximate given real-valued continuous functions to more straightforward functions such as trigonometric and algebraic polynomials. The convergence of these sequences has been a large and practical area in approximation theory over the years. Approximation theory has two sides: the theoretical side, which is frequently concerned with existence and uniqueness issues as well as applications in other theoretical areas, and the pragmatic side, which is primarily concerned with computational practicalities, accurate estimations of error, and so forth. Mostly theoretical, we begin with a brief overview of approximation theory's past. One of the key areas of mathematical analysis is approximation theory. P. L. Chebyshev established the groundwork for approximation theory in 1853 by posing the following problem:

A continuous function g on a closed interval $[p, q]$ can be written by a polynomials $\phi(x) = \sum_{j=0}^{n} d_j x^j$ with at most degree $n(n \in \mathbb{N})$. In such condition, the maximum error can be minimized for any $x \in [p, q]$ by controlling error $\max_{x\in[p,q]} |g(x) - \phi(x)|$.

For better understanding, the fundamental development of Chebyshev problem was later introduced by , Karl Weierstrass(1815-1897) [14], in 1885, proved that the continuous real valued functions such as algebraic polynomials and trigonometric polynomials on compact interval can be uniformly approximated by polynomials. Further, S.N. Bernstein(1880-1968), in 1912 introduced the Bernstein polynomial also known as Bernstein operator and proved the lengthy and complex proof of Weierstrass theorem in one of the simplest manner. In this thesis, we start with illustrating the Weierstrass approximation theorem.

1.1 Weierstrass 1^{st} Theorem

Given, a continuous function $g : [p, q] \to \mathbb{R}$ and for any arbitrary $\epsilon > 0$, $\exists$ an algebraic polynomial ϕ such that

$$|g(x) - \phi(x)| \leq \epsilon,$$

for all x $\in [p, q]$

1.2 Positive Linear Operators

Let A and B be the two space of linear functions, then the mapping $F : A \to B$ is said to be positive linear operators if it satisfies following properties:

1. $F(af + bg) = aF(f) + bF(g)$
2. $F(f) \geq 0$, for all $f \geq 0$ for all $f, g \in A$ and $a, b \in \mathbb{R}$.

Proposition Let $F : A \to B$ be linear positive operators. Then the following irequalities holds:

1. If $f, g \in A$ with $f \leq g$, then $Ff \leq Fg$.
2. For all $f \in A$, we have $|Ff| \leq F|f|$.

The fundamental theorem ,i.e, Weierstrass Approximation theorem has multiple proofs, the first of which was provided by K. Weierstrass in 1885. However, the lengthy and intricate original proof of this approximation theorem encouraged other eminent mathematicians to produce shorter and more understandable versions. S.N. Bernstein in 1912 had given the simplest proof of Weierstrass theorem using Bernstein polynomial. First we will define Bernstein Theorem as follows:

1.3 Bernstein Theorem

Let $g : [0, 1] \to \mathbb{R}$ be continuous and bounded, then

$$\lim_{n\to\infty} B_n(g; x) = g(x),$$

for each $x \in [0, 1]$ [1]. Furthermore, if $g \in C([0, 1])$ then $B_n(g; x)$ converges to g uniformly.

1.4 Bernstein Polynomial

Let $g : [0, 1] \to \mathbb{R}$ and n be a non negative integer The Bernstein polynomial [1] of g of degree (n) is defined as

$$B_n(g, x) = \sum_{k=0}^{\infty} p_{n,k}(x) g\left(\frac{k}{n}\right),$$

$\forall x \in [0, 1]$, where

$$p_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k (1 - x)^{n-k}.$$

Proof:- Now we shall show that if $g \in C[0, 1]$ then $\lim_{n\to\infty} B_n(g; x) = g(x)$, uniformly in $x \in [0, 1]$

To prove it, let $\varepsilon \geq 0$ be arbitrary then by the uniform continuity of g over $[0, 1]$, we can find a $\delta > 0$, such that

$$|g(x_1) - g(x_2)| < \frac{\varepsilon}{2},$$

whenever $|x_1 - x_2| < \delta \; x_1, x_2 \in [0,1]$
we may write

$$B_n(g;x) - g(x) = \sum_{k=0}^{n} n_{c_k} x^k (1-x)^{n-k} g\left(\frac{k}{n}\right) - g(x), \quad x \in [0,1]$$
$$= \sum_{k=0}^{n} n_{c_k} x^k (1-x)^{n-k} \left\{ g\left(\frac{k}{n}\right) - g(x) \right\}$$

then $|B_n(g;x) - g(x)| \leq \sum_{k=0}^{m} nc_k x^k (1-x)^{n-k} \left| g\left(\frac{k}{n}\right) - g(x) \right|$. Let us define

$$\Gamma_n = \left\{ R : \left| \frac{k}{n} - x \right| < \delta, \quad k = 0, 1, 2, \ldots n \right\}$$
$$\tau_n = \left\{ R : \left| \frac{k}{n} - x \right| \geq \delta, \quad k = 0, 1, 2, \ldots n \right\}$$

then

$$| \; B_n(g;x) - g(x) \; | \leq \left(\sum_{R \in \Gamma_n} + \sum_{R \in J_n} \right) n_{C_k} x^k (1-x)^{n-k} \left| g\left(\frac{k}{h}\right) - g(x) \right|$$
$$= I_1 + I_2$$

$$I_1 = \sum_{k \in T_n} \binom{n}{k} x^k (1-x)^{n-k} \left| g\left(\frac{k}{n}\right) - g(x) \right|$$
$$\leq \frac{\varepsilon}{2} \sum_{k \in \sqrt{n}} \binom{n}{k} x^k (1-x)^{n-k}$$
$$\leq \frac{\varepsilon}{2} \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$$
$$= \frac{\varepsilon}{2}$$
$$I_2 = \sum_{k \in J_n} \binom{n}{k} x^k (1-x)^{n-k} \left| g\left(\frac{k}{n}\right) - g(x) \right|$$
$$\leq 2M \sum_{k \in J_n} \binom{n}{k} x^k (1-x)^{n-k}$$
$$\leq \frac{2M}{4n\delta^2}$$
$$= \frac{M}{2n\delta^2}$$

Collecting the estimates of I_1 & I_2, we get

$$|B_n(g;x) - g(x)| < \frac{\varepsilon}{2} + \frac{M}{2n\delta^2},$$

Now, let us choose n to be large that

$$\frac{M}{n\delta^2} < \varepsilon$$

then,

$$|B_n(g;x) - g(x)| < \varepsilon \qquad \forall n \geq N_\varepsilon$$

where $N_\varepsilon = \frac{M}{\varepsilon\delta^2+1}$.
Hence the result.

1.5 Weierstrass 2^{nd} Theorem

Let $g \in C_{2\pi}$ then for every $\varepsilon > 0 \quad \exists$ a trigonometric polynomial say $T(x)$ such that

$$|g(x) - T(x)| < \varepsilon \quad \forall x \in \mathbb{R}.$$

The theory of approximation by positive linear operators relies heavily on Bernstein polynomials due to their clear and practical nature. Numerous scientific endeavors have been dedicated to increasing the rate of convergence and reducing the approximation error throughout history. However, Bernstein polynomials have been transferred to a space of functions that are both Riemann and Lebesgue integrable. Now, we will discuss the moments of Bernstein operator in the next section.

1.6 Moments of Bernstein Polynomial

Case 1.

Put $g(x) = 1$, then eqn (1)

$$\begin{aligned} B_n(g,x) &= \sum_{k=0}^{n} 1 \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} P_{n,k}(x) \\ &= 1 \end{aligned}$$

Case 2.

When $g(x) = x$, then

$$
\begin{aligned}
B_n(g,x) &= \sum_{k=0}^{n} \left(\frac{k}{n}\right) \left(\begin{array}{c} n \\ k \end{array}\right) x^k (1-x)^{n-k} \\
&= \sum_{k=0}^{n} \left(\frac{k}{n}\right) \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \\
&= \sum_{k=0}^{n} \frac{k}{n} \frac{x(n-1)!}{k(k-1)!(n-k)!} x^k (1-x)^{n-k} \\
&= \sum_{k=0}^{n-1} \frac{(n-1)! x^k (1-x)^{n-k}}{(k-1)!(n-k)!} \\
&= \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-k-1)!k!} x^{k+1} (1-x)^{n-k-1} \\
&= \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-k-1)!k!} x^k \cdot x(1-x)^{n-1-k} \\
&= x \sum_{K=0}^{n-1} P_{n-1,k}(x) \\
&= x
\end{aligned}
$$

Case 3.

when $g\left(x\right)=x^2$

$$\begin{aligned}
B_n(g,x) &= \sum_{k=0}^{n}\left(\frac{k}{n}\right)^2\binom{n}{k}x^k(1-x)^{n-k}\\
&= \sum_{k=0}^{n}\frac{k^2}{n^2}\frac{n!}{(n-k)!k!}x^k(1-x)^{n-k}\\
&= \frac{1}{n^2}\sum_{k=0}^{n}[k(k-1)+k]\frac{n!}{k!(n-k)!}x^k(1-x)^{n-k}\\
&= \frac{1}{n^2}\sum_{k=0}^{n}k(k-1)\frac{n!}{(n-k)!k!}x^k(1-x)^{n-k}+\frac{k}{n^2}\frac{n!}{(n-k)!k!}x^k(1-x)^{n-k}\\
&= \frac{1}{n}\sum_{k=0}^{n}k(k-1)\frac{(n-1)(n-2)!}{(n-k)!k(k-1)(k-2)!}x^k(1-x)^{n-k}+\frac{1}{n^2}nx\\
&= \frac{n-1}{n}\left[\sum_{k=2}^{n}\frac{(n-2)!x^k(1-x)^{n-k}}{(n-k)!(k-2)!}\right]+\frac{x}{n}\\
&= \frac{(n-1)}{n}\left[\sum_{k=0}\frac{(n-2)!}{k!(n-k-2)!}x^{k+2}(1-x)^{n-k-2}\right]+\frac{x}{n}\\
&= x^2\frac{(n-1)}{n}\left[\sum_{k=0}^{n-2}\frac{(n-2)!}{k!(n-k-2)!}x^k(1-x)^{n-2-k}\right]+\frac{x}{n}\\
&= x^2\frac{(n-1)}{n}P_{n-2,k}(x)+\frac{x}{n}\\
&= \left(\frac{n-1}{n}x^2+\frac{x}{n}\right)\quad [\because P_{n-2,k}(x)=1]\\
&= x^2+\frac{x^2}{n}+\frac{x}{n}\\
&= x^2
\end{aligned}$$

1.7 Preliminaries

Defintion 1. In 1910, H. Lebesgue proposed the modulus of continuity for $g\in C\mid p,q]$ and $j\in\mathbb{N}$ as follows:

$$\omega_j(g;\delta)=\sup\left\{\left|\Delta_k' g(x)\right|;|k|\leq\delta,x\in[p,q]\right\},\delta\geq 0,$$

For $j=1$, it denotes the first modulus of continuity $\omega(.;\delta)$ and given by

$$\omega(g;\delta)=\max\nolimits_{|x-y|\leqslant b}|g(x)-g(y)|,x,y\in[p,q].$$

Proposition For $g\in C[p,q],\omega(g;.)$ holds following properties:

1. $\omega(g; .)$ is non-negative and non-decreasing uniformly continuous function in $[0, \infty)$.
2. When $\delta^{+} \to 0$ $\omega(g, \delta) = 0$.
3. For any $\delta \geq 0$ and $s \geq 0$, the following inequality holds:
$$\omega(g, s\delta) \leq (1 + s)\omega(g, \delta).$$
4. For any $\delta \geq 0$ and $x, y \in [0, \infty)$, it follows:
$$|g(x) - g(y)| \leq \omega(g, |x - y|) \leq \left(1 + \frac{|x - y|}{\delta}\right) \omega(g, \delta).$$

Definition 2. In 1963, Peetre introduced another tool to estimate the smoothness of function for $f \in C_B[p, q]$ termed as Peetre-K-functional and given by

$$K_2(g; \delta) = \inf_{h \in C_B^2[p,q]} \{\|g - h\| + \delta \|h''\|\}, \delta > 0,$$

where $C_B^2[p, q] = \{h \in C_B[p, q] : h', h'' \in C_B[p, q]\}$. There exists a positive constant C such that

$$K_2\left(g, \delta^2\right) \leq C\omega_2(g, \delta).$$

Definition 3. For $0 \leq \lambda \leq 1$, $\varphi(x) : (0, \infty) \to \mathbb{R}$ is an admissible weight functiom and $g \in C_B[0, \infty)$, the Ditzian-Totik modulus of continuity are defined as

$$\omega_{\varphi^\lambda}(g, \delta) = \sup_{0 \leq h \leq \delta} \sup_{x \pm \frac{h\varphi^\lambda(x)}{2} \in [0,\infty)} \left| g\left(x + \frac{h\phi^\lambda(x)}{2}\right) - g\left(x - \frac{h\phi^\lambda(x)}{2}\right) \right|,$$

$$\omega_{\varphi^\lambda}^2(g, \delta) = \sup_{0 < h \leq \delta} \sup_{x \pm h\varphi^\lambda(x) \in [0,\infty)} \left| \Delta_{h\varphi^\lambda}^2 g(x) \right|, \delta > 0,$$

where

$$\Delta_{h,x}^2 g(x) = g\left(x + h\varphi^\lambda(x)\right) - 2g(x) + g\left(x - h\varphi^\lambda(x)\right).$$

and their K-functionals are:

$$K_{\phi^\lambda}(g, \delta) = \inf_{h \in \mathbb{W}_\lambda} \left\{\|g - h\| - \delta \left\|\phi^\lambda h'\right\|\right\},$$
$$K_{\varphi^\lambda}^2\left(g, \delta^2\right) = \inf_{h \in D_\lambda^2} \left\{\|g - h\| + \delta^2 \left\|\varphi^{2\lambda} h''\right\|\right\},$$

where $W_\lambda = \left\{g \in C_B[0, \infty) : g' \in A \cdot C._{loc}[p, q], \left\|\varphi^\lambda g'\right\| < \infty\right\}$,
$D_\lambda^2 = \{g \in C_B[0, \infty) : g' \in \text{A.C.}_{\text{loc}}[0, \infty), \left\|\varphi^{2\lambda} g''\right\| < \infty\}$ and $A.C_{\text{loc}}$ means locally absolutely continuous functions g on $[0, \infty)$.
In Theorem, there exists a constant $C > 0$ such that

$$C^{-1}\omega_{\varphi^\lambda}(g, \lambda) \leq K_{\varphi^\lambda}(g, \delta) \leq C\omega_{\varphi^\lambda}(g, \delta)$$

and

$$C^{-1}\omega_{\varphi^\lambda}^2(g,\delta)\leq K_{\varphi^\lambda}^2\left(g,\delta^2\right)\leq C\omega_{\varphi^\lambda}^2(g,\delta).$$

Definifion 4. The Lipschitz class Lip $M(\alpha)$, $0<a\leq 1$. We say that $g\in C_B[0,\infty)$

$$|g(t)-g(x)|\leq M|t-x|^\alpha$$

holds.

Definition 5 We define Lipschitz-type space as:

$$\mathrm{Lip_M}(\beta)=\left\{g\in C_B[0,\infty):| g(t)-g(x) |\leq M\frac{|t-x|^\beta}{(t+x)^{\beta/2}}\right\},$$

where M is a positive constant for all $0<\beta\leq 1$ for fixed $\beta,\gamma>0$, we have

$$\mathrm{Lip}_M^{\beta,\gamma}(r):=\left\{g\in C_n[0,\infty):|g(t)-g(x)|\leq M\frac{|t-x|^r}{\left(t+\beta x^2+\gamma x\right)^{r/2}}\middle|\ x,t\in(0,\infty)\right\}.$$

Where: M is any positive constant and $r\in(0,1]$.

Definition 6. Let $B_2[0,\infty)$ be the space of all functions defined on $[0,\infty)$ given by

$$B_2[0,\infty)=\left\{g=|g|\leq M_j\left(1+x^2\right\}\right\},$$

where M_g is positive constant may be depends on g with the form

$$\|g\|_2=\sup_{x\geq 0}\frac{|g(x)|}{1+x^2}.$$

$C_2[0,\infty)$ denotes the space of all continuous functions in $B_2[0;\infty)$. For $g\in C_2[0,\infty)$, the weighted modulus of continuity $\Omega(g;\delta)$ is given by

$$\Omega(g;\delta)=\sup_{0\leq\beta\leq\delta}\frac{|g(x+\beta)-g(x)|}{\left(1+\beta^2\right)\left(1+x^2\right)}.$$

Let $C_2^*[0,\infty):=\Big\{g\in C_2[0,\infty):\lim_{x\to\infty}\frac{g(x)}{1+x^2}$ exists and therefore is finite. It is shown in that for every f $\to C_2^*[0,\alpha)$, $\Omega(.;\delta)$ has the properties

$$\lim_{\delta\to 0}\Omega(g;\delta)=0,$$

and

$$\Omega(g;d\delta)\leq 2(1+d)\left(1+\delta^2\right)\Omega(g;\delta)\quad d>0.$$

For $f\to C_2^*[0,\infty)$ the properties of equations (1.23) and (1.21), we can write

$$\begin{aligned}|g(t)-g(x)|&\leq\left(1+(t-x)^2\right)\left(1+x^2\right)\Omega(g;|t-x|)\\&\leq 2\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\Omega(g;\delta)\left(1+(t-x)^2\right)\left(1+x^2\right).\end{aligned}$$

Yuksal and lspir also introduced the another weighted modulus of continuity as follows:

$$\Omega(g;\delta) = \sup_{x\in[0,\infty),0<\beta<\delta} \frac{g(x+\beta)-g(x)}{1+(x+\beta)^2}.$$

Definition 7. For $g \in C_B(I_0)$, the Steklov-mean is given as:

$$g_\gamma(x) = \frac{4}{\gamma^2}\int_0^{\frac{\gamma}{2}}\int_0^{\frac{\gamma}{2}} [2g(x+v+w) - g(x+2(v+w))]dvdw.$$

The following inequalities satisfy:

1. $\|g_\gamma - f\| \le \omega_2(g,\gamma)$.
2. $g'_\gamma + g''_\gamma \in C_B(I_0)$, $\|g'_\gamma\| \le \frac{5}{7}\omega(g,\gamma)$, and $\|g''_\gamma\| \le \frac{9}{\gamma^2}\omega_2(g,\gamma)$.

1.8 Main results and inequalities

Lemma 1- Let $x \in [0,1]$ and $\delta > 0$ Also,

$$\delta_n(x) = \left\{k : \left|\frac{k}{n} - x\right| \geqslant \delta\right\},$$

n is any integer then we have

$$\sum_{k\in\delta_n(x)} p_{n,R}(x) \le \frac{1}{4n\delta^2}.$$

Lemma 2-

$$I_n = \int_0^{\pi/2} \cos^{2n} t dt = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}$$

where n is a positive integer.
Lemma 3- The product of two trignometric polynomials of order m and n respectively is a trigonometric polynomial of $m + n.n$.
Lemma 4- An even trigonometric polynomial is of the form

$$T(x) = a_0 + \sum_{k=1}^{n} a_k \cos k\hat{x}.$$

Basic Inequalities

1. $\sum_{k=0}^{n} c_k x^k (1-x)^{n-k} = 1$
2. $\sum_{k=0}^{n} nc_k x^k (1-x)^{n-k} = nx$
3. $\sum_{k=0}^{n} (k-nx)^2 \cdot {}^n c_k x^k (1-x)^{n-k} = nx(1-x)$

Chapter 2

Bernstein Polynomials

A Bernstein polynomial, given by Sergei Natanavich Bernstein, is a polynomial in the Bernstein form in mathematics, which is a linear combination of Bernstein basis polynomials. A numerically stable approach to evaluate polynomials in Bernstein form is de Casteljau's algorithm. Firstly, Bernstein used the Bernstein polynomial to give a constructive proof of the Stone-Weierstrass approximation theorem with the introduction of computer graphics Bernstein polynomials are restricted to the interval $x \in (0, 1)$.. In this chapter,we will examine the mathematical properties and forms of polynomials in this chapter.

2.1 The Bernstein Basis Function

The Bernstein basis functions of degree n are defined by

$$B_{i,n}(t) = \binom{n}{1} t'(1-t)^{n-1}$$

for $i = 0, 1, \ldots, n$ where $t \in \{0, 1\}$

$$\binom{n}{i} = \frac{n!}{r!(n-r)!}$$

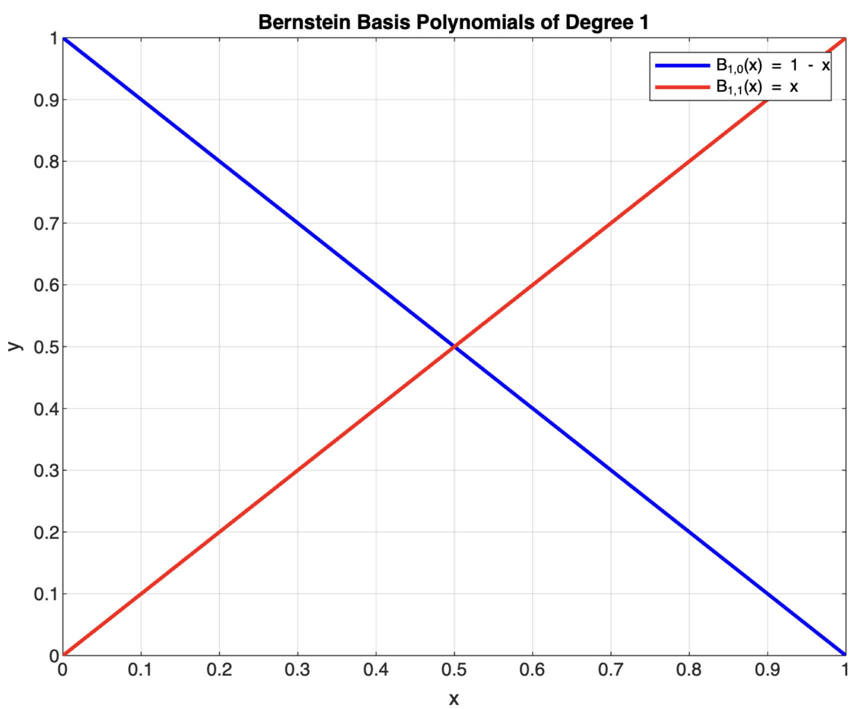
There are $(n+1)n^{\text{th}}$ degree polynomials. For mathematical convenience we use $B_(i, n) = 0$ for $i < 0$ or $i > n$. Let us have a look at few degrees of the polynomial:

2.2 Bernstein Polynomials of Degree 1

There are 2 polynomials since $n = 1$. These are mathematically given as :

$$B_{0,1}(t) = 1 - t$$
$$B_{0,1}(t) = t$$

for $t \in [0, 1]$.

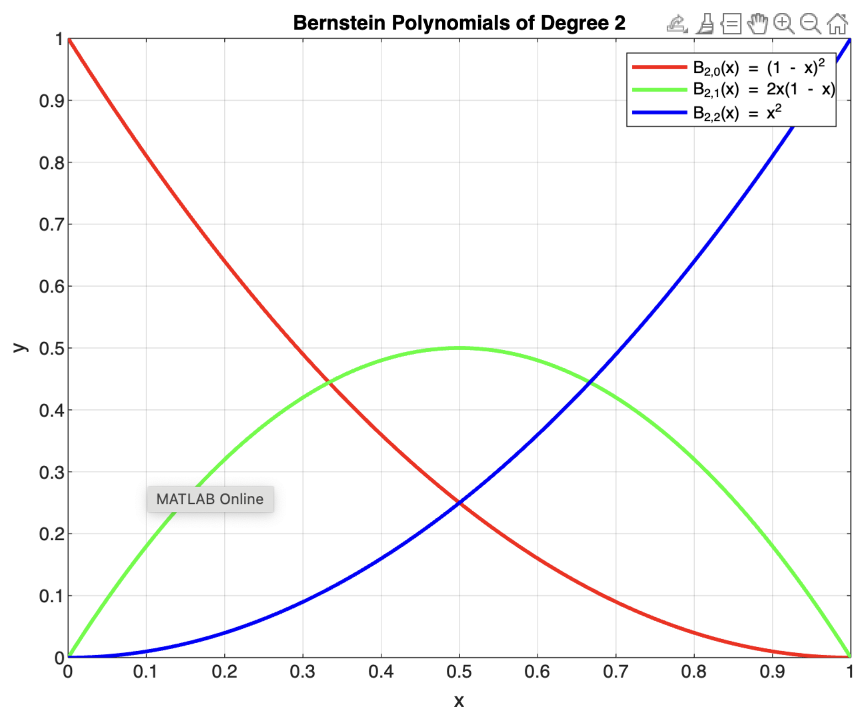


2.3 Bernstein Polynomials of Degree 2

There are 3 polynomials since $n = 2$. These are mathematically given as :

$$B_{0,2}(t) = (1-t)^2$$
$$B_{1,2}(t) = 2t(1-t)$$
$$B_{2,2}(t) = t^2$$

for $t \in [0, 1]$.



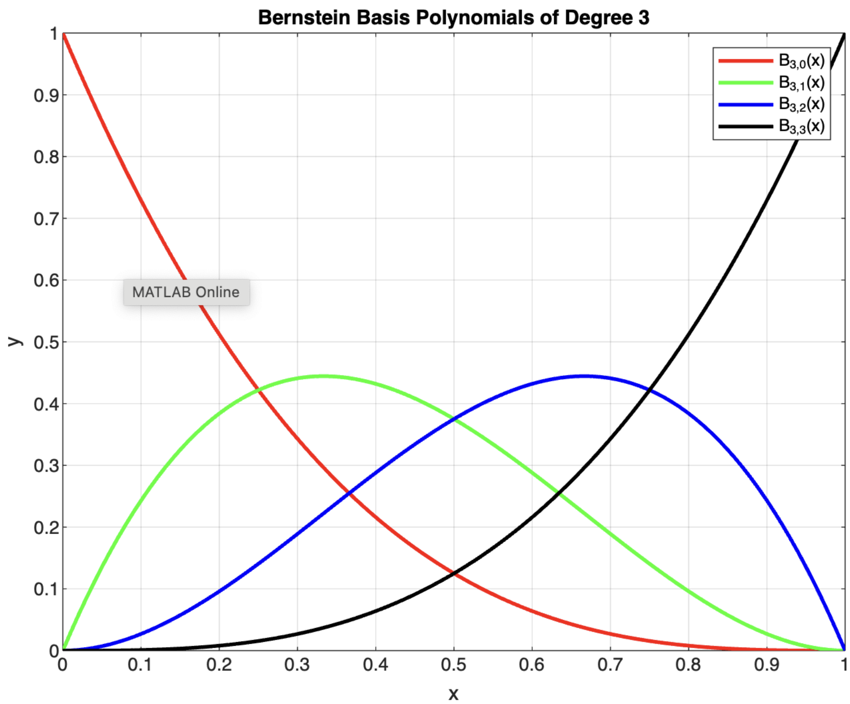
Above is the graphical representation of Bernstein polynomial of degree 2 in matlab as we have defined above.

2.4 Bernstein Polynomials of Degree 3

There are $n + 1 = 4$ polynomials since $n = 3$. These are mathematically given as:

$$B_{0,3}(t) = (1-t)^3$$
$$B_{1,3}(t) = [3t(1-t)]^2$$
$$B_{2,3}(t) = (3t)^2(1-t)$$
$$B_{3,3}(t) = t^3$$

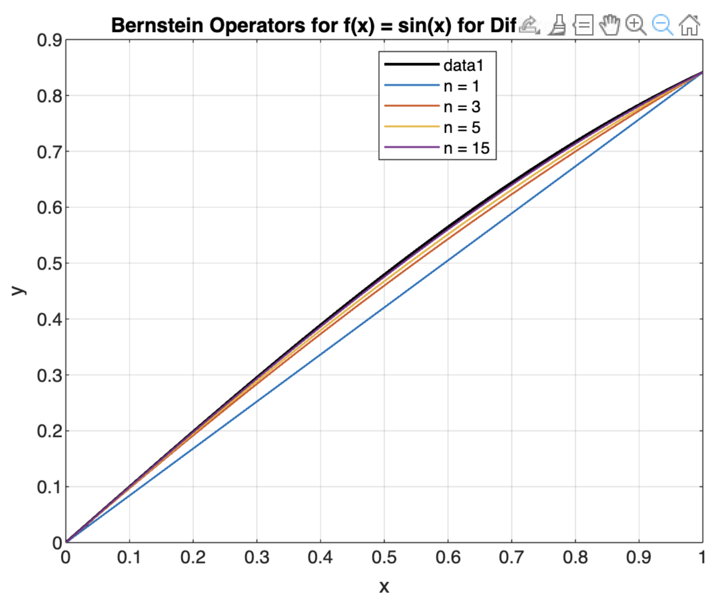
for $t \in [0, 1]$. These are plotted as:



Above is the graphical representation of Bernstein polynomial of degree 3 in matlab as we have defined above.

2.5 Convergence of Bernstein polynomial

In this section, we plot Bernstein polynomial for different degrees where we take $f(x) = sin(x)$ in the interval $[0, 1]$ and try to study the pattern of convergence of this polynomial for different degrees.



The above figure shows the bernstein polynomial plotted graphically for different degrees and the line in black color displays the actual curve of $sinx$.

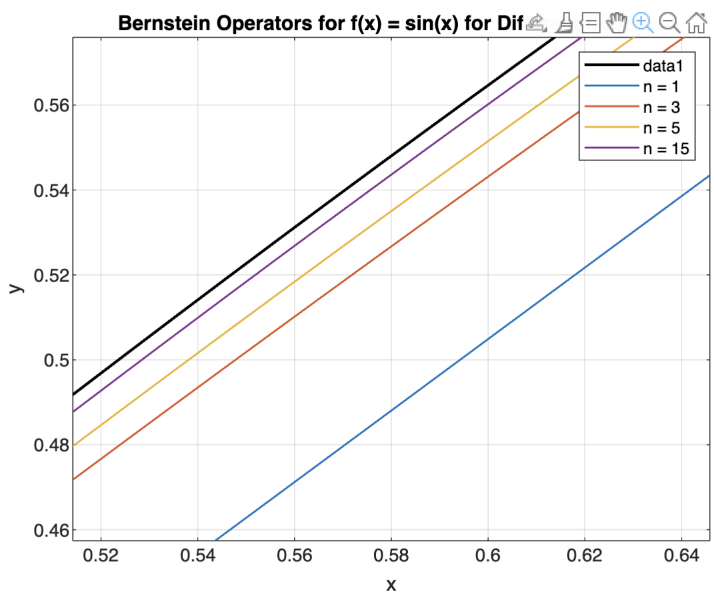
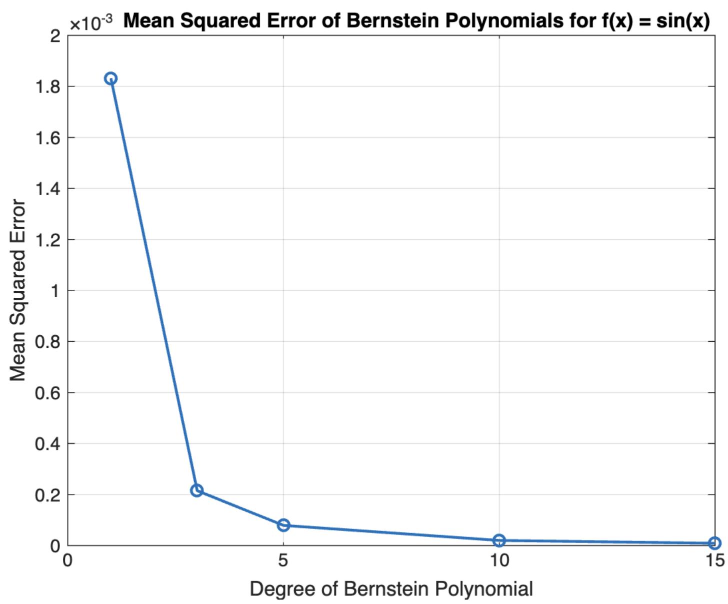


Figure 2.1: This is zoomed in image of the above plot.

Now, when we carefully observe the above plots we can see that the polynomial of the least degree is farthest from the actual graph of $sinx$ and as the degree increases the distance between the polynomial and actual graph decreases. Hence, we can conclude that, as the degree of the bernstein polynomial increases its convergence.

Error plot of the bernstein polynomial



The above plot shows that the difference between actual values and the values predicted by bernstein polynomial decreases as degree of polynomial increases.

Chapter 3

VARIOUS MODIFICATIONS OF BERNSTEIN OPERATOR

In this section, we study the various modifications of Bernstein polynomial defined by different mathematians for better approximation of polynomials and trigonometric functions, their graphs and error in approximating the given polynomial by various modifications of Bernstein. We will also study the convergence of different operators for a given function.

3.1 Bernstein-Chlodovsky Operator -1932

The polynomial $B_n(f(x))$ is called the M. S. Bernstein polynomial. Formula shows us that the coefficients of the polynomials of M. S. Bernstein are defined by means of the values of the function that we seek to approximate to the rational points of the segment [0, 1]. From this point of view the polynomials of M. S. Bernstein are bring Newton-Lagrange interpolation polynomials together. It should be noted that the polynomials of M. S. Bernstein have the advantage of being uniformly convergent whatever the continuous function f(x), while interpolation polynomials of Newton do not always converge. The aim of this work is to study the representation by a sequence of M. S. Bernstein polynomials of defined functions in an infinite interval. We can limit ourselves to the case of a function f(x) defined in $0 \leq x \leq \infty$.
It is easy to see that all results obtained can be extended to the case of the interval $\infty \leq x \leq \infty$.

In 1932, Chlodovsky [5] introduced the generalization of Bernstein polynomial known as classical Bernstein-Chlodovsky polynomials on an unbounded interval. These polynomials $C_n : C[0,\infty) \to C[0,\infty), n \in \mathbb{N}$, defined by

$$C_n(g,x) = \begin{cases} \sum_{\upsilon=0}^{n} g\left(\frac{\upsilon}{n} y_n\right) \begin{pmatrix} n \\ \upsilon \end{pmatrix} \left(\frac{x}{y_n}\right)^{\upsilon} \left(1 - \frac{x}{y_n}\right)^{n-\upsilon}, 0 \le x \le y_n \\ g(x), \quad x > y_n, \end{cases}$$

where $g \in C[0,\infty), x \in [0,\infty)$ and $0 \le x \le y_n$ and $\{y_n\}$ is a positive sequence with $\lim_{n\to\infty} y_n = \infty$, and $\lim_{n\to\infty} \frac{y_n}{n} = 0$.

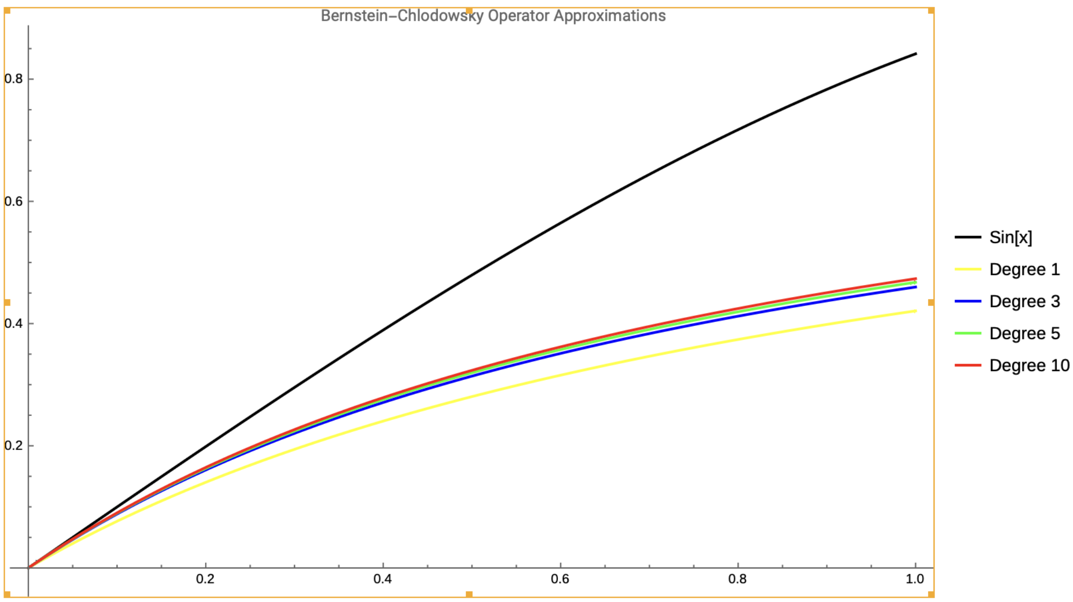


Figure 3.1: Graphical representation of Bernstein-Chlodovsky operator

3.2 Bernstein Operator of Kantorovich form-1939

In 1939, an integral analogue of Bernstein polynomial was introduced by Leonid Kantorovich using the following:

$$K_n(g,x) = \frac{d}{dx}\left(K_{n+1}G\right) \quad , \quad x \in [0,1]$$

where $G(x) = \int_0^x g(t)dt$

$$K_{n+1}(g,x) = \sum_{i=0}^{n+1} \begin{pmatrix} n+1 \\ i \end{pmatrix} x^i (1-x)^{n+i+1} g\left(\frac{i}{n+1}\right)$$

The Bernstein-Kantorovich operator incorporates an integral of the function over subinterval $[k/n, (k+1)/n]$. This can provide a smoother approximation compared to the standard Bernstein operator, especially for functions with less regular behavior. The Bernstein-Kantorovich operator is useful in various applications requiring function approximation, such as: ,numerical integration and interpolation, computer-aided geometric design, signal processing and data smoothing. By incorporating the integration step, the Bernstein-Kantorovich operator can better handle functions with discontinuities or rapid variations, providing a more robust approximation method.

3.2.1 Derivation of Bernstein Kantorovich

$$K_{n+1}(g,x) = \frac{d}{dx}\left(\sum_{k=0}^{n+1}\begin{pmatrix} n+1 \\ k \end{pmatrix} x^k(1-x)^{n-k+1}F\left(\frac{k}{n+1}\right)\right)$$

$$= \frac{d}{dx}\left(\sum_{k=0}^{n+1}\begin{pmatrix} n+1 \\ k \end{pmatrix} x^k(1-x)^{n-k+1}\int_0^{k/n+1} f(t)dt\right)$$

$$= \sum_{k=0}^{n+1}\begin{pmatrix} n+1 \\ k \end{pmatrix}\left[k\cdot x^{k-1}(1-x)^{n-k+1} - (n-k)x^k(1-x)^{n-k-1}\right]\int_0^{k/n+1} f(t)dt$$

$$= \sum_{k=0}^{n+1}\frac{(n+1)!}{k!(n-k+1)!}\cdot k\Bigg)^{k-1}(1-x)^{n-k+1} - \sum_{k=0}^{n+1}\frac{(n+1)!}{k!(n-k+1)!}(k-1)!\int_0 f(t)dt$$

$$= \sum_{k=1}^{n+1}\frac{(n+1)!}{(k-1)!(n-k+1)!}x^{k-1}(1-x)^{n-k+1}\int_0^{k/n+1} x^k - \sum_{k=0}^{n}\frac{(n+1)(n)!}{k!(n-k)!}x^k(1-x)^{n-k}\int_0 f(t)dt$$

$$= \sum_{k=0}^{n}\frac{(n+1)n!x^s(1-x)^{n-s}}{s!(n-s)!}\int_0^{s+1/n+1} f(t)dt - \sum_{k=0}^{n}(n+1)\binom{n}{k}x^k(1-x)^{n-k}\int_0^{k/n+1} f(t)dt$$

$$= (n+1)\sum_{R=0}^{n} c\begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}\left[\int_0^{k+1/n+1} f(t)dt - \int_0^{k/n+1} f(t)dt\right]$$

$$= (n+1)\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}\int_{(k)/n+1}^{(k+1)/n+1} f\left(\frac{k}{n}\right)$$

$$= (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{(k)/n+1}^{(k+1)/n+1} f(t)dt$$

Graphical representation of Bernstein-Kantorovich

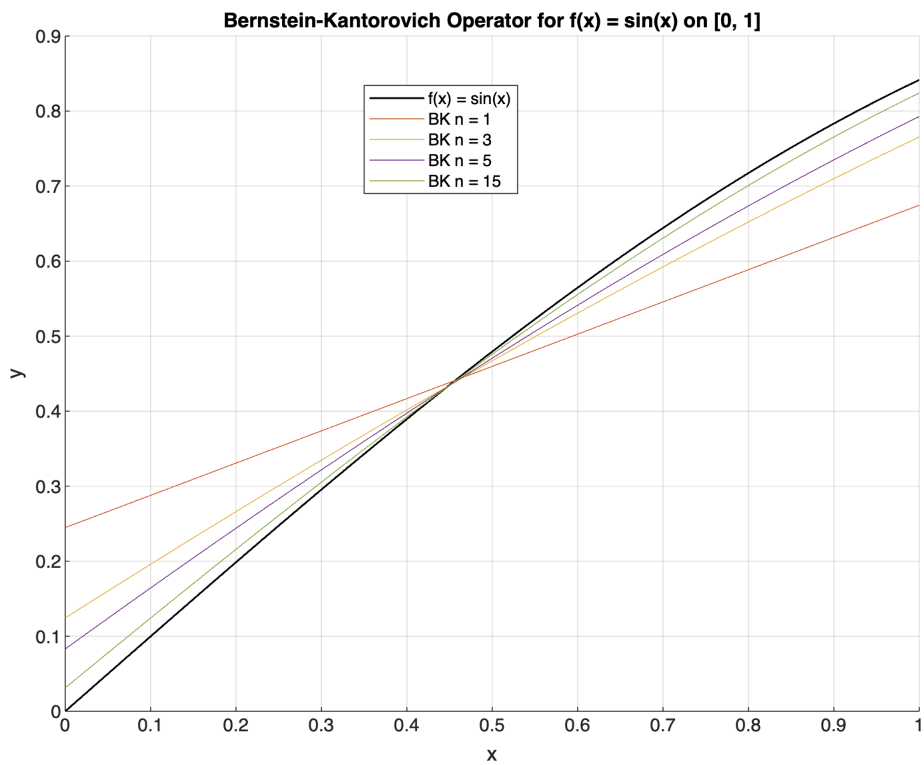


Figure 3.2: Bernstein Kantorovich for various degree

Above is the graph of Bernstein-kantorovich operator for different degrees. As we increase the degree of the polynomial we observe that the distance between the bernstein kantorovich polynomial and the plot of $sinx$ decreases and hence, rate of convergence increases.

3.2.2 Error graph of Bernstein-Kantrovich

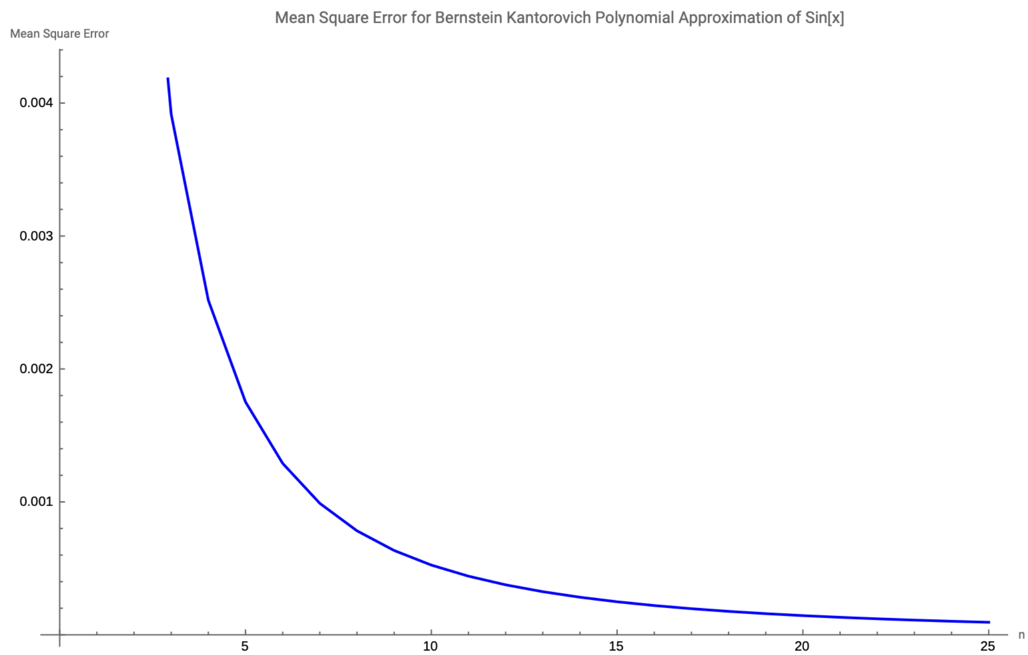


Figure 3.3: Error Representation of Bernstein-Kantorovich Operator

In the above graph, we observe that the error between the actual function and and bernstein-kantrovich operator decreases as we increases the degree of the operator, i.e, we get the better approximation with higher degree.

3.3 Cheney-Sharma Bernstein Operator-1964

Another prominent modification of Bernstein polynomial was given by Cheney-Sharma [4] in 1964 using the famous inequalities given by Niels Henrik Abel as follows:

$$(c+d)^n = \sum_{\upsilon=0}^{n} \left(\begin{array}{c} n \\ \upsilon \end{array} \right) c(c-\upsilon\beta)^{\nu-1}(y+\upsilon\beta)^{n-\upsilon}, \beta \geq 0.$$

$$(c+d+n\beta)^n = \sum_{\upsilon=0}^{n} \begin{pmatrix} n \\ \upsilon \end{pmatrix} c(c+\upsilon\beta)^{\upsilon-1}(d+(n-\upsilon)\beta)^{n-\upsilon},$$

$$(c+d+n\beta)^n = \sum_{\upsilon=0}^{n} \begin{pmatrix} n \\ \upsilon \end{pmatrix} (c+\upsilon\beta)^{\upsilon} d(d+(n-\upsilon)\beta)^{n-\upsilon-1},$$

$$(c+d+n\beta)^{n-1} = \sum_{\upsilon=0}^{n} \begin{pmatrix} n \\ \upsilon \end{pmatrix} c(c+\upsilon\beta)^{\upsilon-1} d[d+(n-\upsilon)\beta]^{n-\upsilon-1},$$

$$(c+d)(c+d+n\beta)^{n-1} = \sum_{\upsilon=0}^{n} \begin{pmatrix} n \\ \upsilon \end{pmatrix} c(c+\upsilon\beta)^{\upsilon-1} d[d+(n-\upsilon)\beta]^{n-\upsilon-1},$$

where $c, d \in \mathbb{R}$. Using above inequalities the generalization Bernstein polynomial is as follows:

$$\phi_n(g,x) = (1+n\beta)^{1-n} \sum_{\upsilon=0}^{n} g\left(\frac{\upsilon}{n}\right) \begin{pmatrix} n \\ \upsilon \end{pmatrix} x(x+\upsilon\beta)^{\upsilon-1}(1-x)[1-x+(n-\upsilon)\beta]^{n-\upsilon-1}.$$

above operator reduce to the classical Bernstein operator when $\beta = 0$. reduce to the classical Bernstein operators. Cheney-Sharma proved that if $n\beta_n \to \infty$ as $n \to \infty$, then for $g \in C[0,1]$, these operators uniformly convergence to f on $[0,1]$. Cheney-Sharma operators preserve the Lipschitz constant and order of a Lipschitz continuous function as well as the properties of the function of modulus of continuity. They also gave a result of $G_n(f;x)$ under the convexity of f.

3.4 Bernstein-Durrmeyer Operator-1967

In 1967, an integral modification of Bernstein polynomials was proposed by Durrmeyer [7].These operators combine the properties of the Bernstein polynomials with those of the Durrmeyer operators, which are integral modification of the Bernstein operators. They are used to approximate functions and have applications in various areas such as numerical analysis, computer-aided geometric design, and solution of differential equations, the bernstein Durrmeyer operator is defined by :

$$\bar{C}_n(g,x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) g(t) dt \quad \forall x \in [0,1].$$

where $p_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k}$ is the Bernstein basis function.

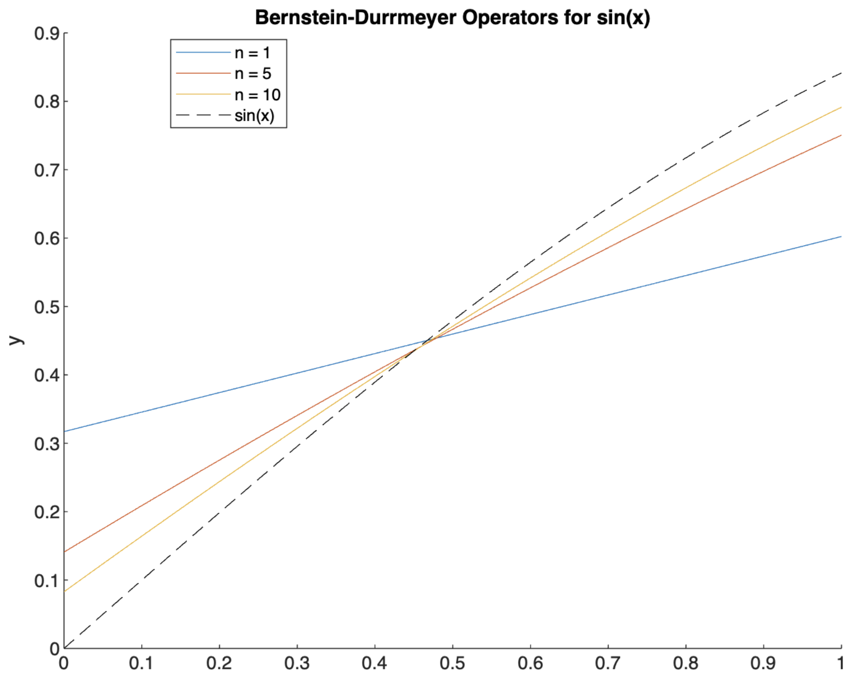


Figure 3.4: Graphical Representation of Bernstein Durrmeyer for different degrees

3.5 D. D. Stancu Bernstein Operator-1968

In 1984, D.D Stancu defined a new positive linear polynomial operator [12] : $C_m^{[a]}(g;x) = C_m^{[a]}(g(t);x)$, corresponding to a function $g = g(x)$, defined on the interval $[0,1]$, and to a parameter $\alpha \geqq 0$, which may depend only on the natural number m is defined by

$$C_m^{[\alpha]}(g;x) = \sum_{k=0}^{m} w_{m,k}(x;\alpha) g\left(\frac{k}{m}\right),$$

where

$$w_{m,k}(x;\alpha) \\ = \binom{m}{k} \frac{x(x+\alpha)\cdots(x+\overline{k-1}\alpha)(1-x)(1-x+\alpha)\cdots(1-x+\overline{m-k-1}\alpha)}{(1+\alpha)(1+2\alpha)\cdots(1+\overline{m-1}\alpha)}.$$

One observes that it represents a polynomial of degree m. Because $\alpha \geqq 0$, we have $w_{m,k}(x;\alpha) \geqq 0$ for $x \in [0,1]$. Therefore the linear (additive and homogeneous) operator is positive on the interval $[0,1]$. In fact, we have here a class of operators depending on the parameter α.

3.6 q-analogue of Bernstein Operator-1987

The remarkable evolution of q-calculus[9] has resulted in the identification of new generalizations of Bernstein polynomials that involve q-integers. The very first q-analogue of Bernstein polynomial for

$q \geq 0$ was introduced by Lupas A. in 1987 which is as follows:

$$C_n(g, q; x) = \sum_{k=0}^{n} g\left(\frac{[k]_q}{[n]_q}\right) c_{n,k}(x; q),$$

where

$$c_{n,k}(x; q) = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{n(n-1)}{2}} x^k (1-x)^{n-k}}{\prod_{j=0}^{n-1} \{(1-x) + q^j x\}}.$$

3.7 A new Bernstein type Operator by Deo-2008

A Bernstein type special operator V_n defined as

$$(V_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),$$

where,

$$p_{n,k}(x) = \left(1 + \frac{1}{n}\right)^n \begin{pmatrix} n \\ k \end{pmatrix} x^k \left(\frac{n}{n+1} - x\right)^{n-k}, \quad \frac{n}{n+1} \geqslant x \text{ and } x \in \left[0, 1 - \frac{1}{n+1}\right].$$

If n is sufficiently large then above operator becomes generalized form of Bernstein operator[6]. Lebesgue integrable function defined in the interval $[0, 1]$ can also be approximated by taking into account the integral modification of Bernstein operators. Driven by previous research on Bernstein operators, a certain integral was proposed as modification of the operators $V_n(f(x))$, defined as

$$(L_n f)(x) = n\left(1 + \frac{1}{n}\right)^2 \sum_{k=0}^{n} p_{n,k}(x) \int_0^{\frac{\pi}{n+1}} p_{n,k}(t) f(t) \mathrm{d}t.$$

3.8 λ-Bernstein-2010

A new Bezier bases was introduced by Ye et al [15] using shape parameter λ in 2010 which is known as Lambda Bernstein operator, defined by,

$$\begin{cases} \tilde{c}_{n,0}(\lambda; x) = c_{n,0}(x) - \frac{\lambda}{n+1} c_{n+1,1}(x), \\ \tilde{c}_{n,i}(\lambda; x) = c_{n,i}(x) + \lambda\left(\frac{n-2i+1}{n^2-1} c_{n+1,i}(x) - \frac{n-2i-1}{n^2-1} c_{n+1,i+1}(x)\right) \quad (1 \leq i \leq n+31), \\ \tilde{c}_{n,n}(\lambda; x) = c_{n,n}(x) - \frac{\lambda}{n+1} c_{n+1,n}(x), \end{cases}$$

where $\lambda \in [-1, 1]$. put $\lambda = 0$, the above will reduce to Bernstein basis function $P_{n,k}(x)$ where $x \in [0, 1]$ and $\quad n = 0, 1, 2, \ldots$

In 2018, Cai et al. [2] introduce the λ-Bernstein operators,

$$C_{n,\lambda}(g;x) = \sum_{i=0}^{n} \tilde{c}_{n,i}(\lambda;x) g\left(\frac{i}{n}\right),$$

where $\tilde{c}_{n,i}(\lambda;x)(i = 0, 1, \ldots, n)$ are defined above and λ lies in the closed interval$[-1, 1]$.

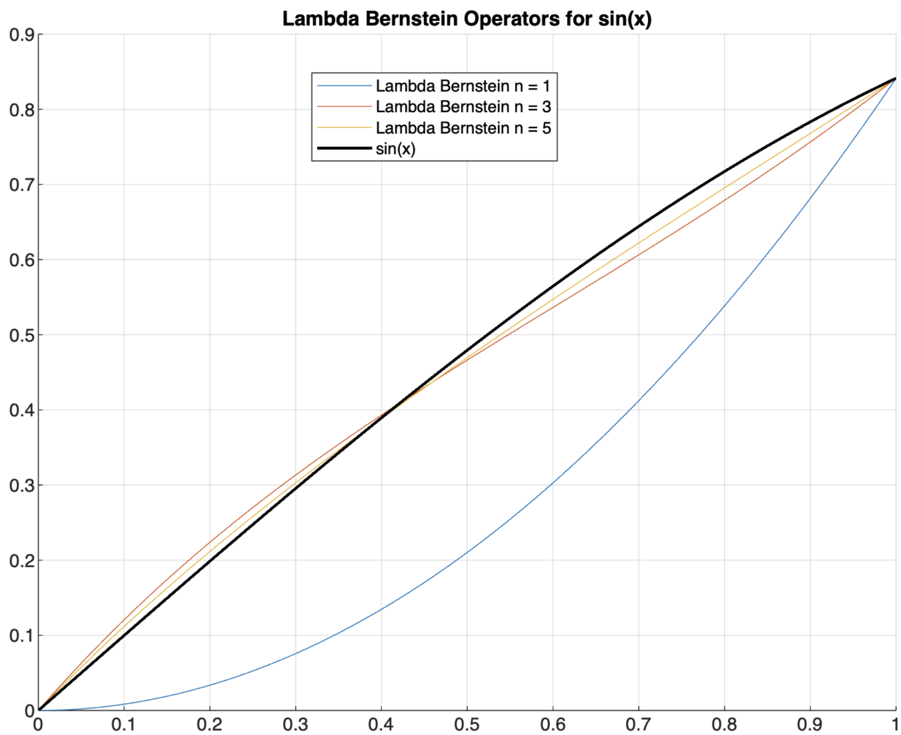


Figure 3.5: Graphical representation of Lambda-Bernstein

3.9 α-Bernstein-2017

Chen et al [3], in 2017, defined a function $g(x)$ on $[0, 1]$, for each positive integer n and any fixed real α, we define α-Bernstein operator for $g(x)$ as

$$C_{n,\alpha}(g;x) = \sum_{j=0}^{n} g_j p_{n,j}^{(\alpha)}(x),$$

where $g_j = g\left(\frac{j}{n}\right)$. The α-Bernstein polynomial,
for $i = 0$, $p_{n,j}^{(\alpha)}(x)$ of degree n is defined by $p_{1,0}^{(\alpha)}(x) = 1 - x$, for $i = 1$, $p_{1,1}^{(\alpha)}(x) = x$ and

$$p_{n,i}^{(\alpha)}(x) = \left[\begin{pmatrix} n-2 \\ i \end{pmatrix} (1-\alpha)x + \begin{pmatrix} n-2 \\ i-2 \end{pmatrix} (1-\alpha)(1-x) + \begin{pmatrix} n \\ i \end{pmatrix} \alpha x(1-x) \right] x^{i-1}(1-x)^{n-i-1},$$

where $x \in [0,1]$ and $n \geq 2$ and the binomial coefficients $\binom{i}{l}$ are defined as

$$\binom{i}{l} = \begin{cases} \frac{i!}{l!(i-l)!}, & \text{if } 0 \leq l \leq k \\ 0, & \text{else.} \end{cases}$$

Corresponding to each function $g(x)$ there is a sequence of α-Bernstein operator which maps a function, defined on $[0,1]$, to $C_{n,\alpha}(g)$, where the function is defined by $C_{n,\alpha}(g;x)$, i.e, function $C_{n,\alpha}(g)$ evaluated at x.

The α-Bernstein polynomial reduces to the classical Bernstein polynomial when $\alpha = 1$, i.e,

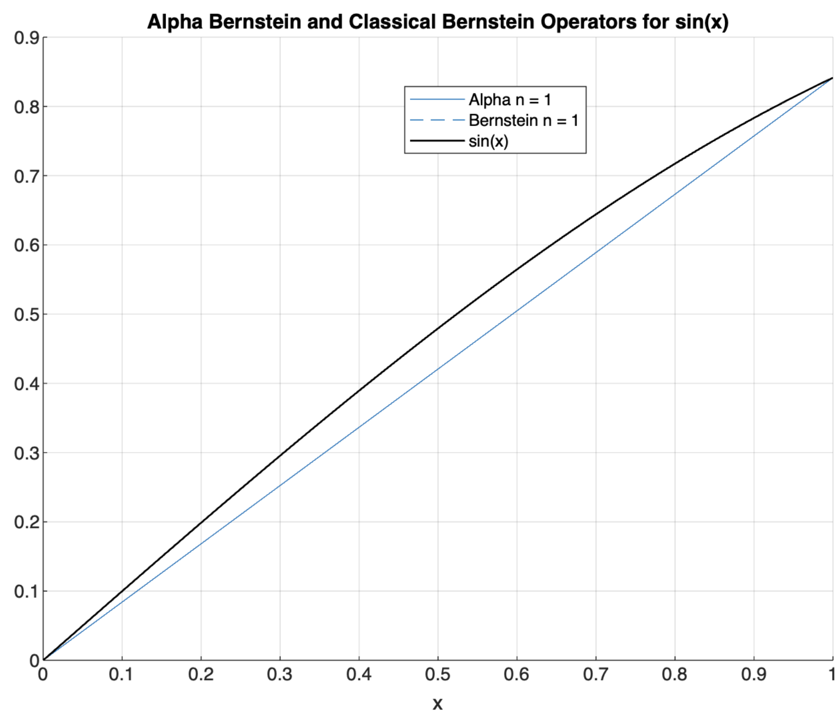
$$p_{n,i}^{(1)}(x) = \binom{n}{i} x^i (1-x)^{n-i},$$

So the α-Bernstein operator has following identity

$$C_{n,1}(g;x) = \sum_{i=0}^{n} g_i \binom{n}{i} x^i (1-x)^{n-i} = B_n(g;x),$$

this shows that the class of α-Bernstein operators contains the classical Bernstein.

3.9.1 Graphical representation of α-Bernstein when $\alpha = 1$



We have studied that for α= 1 the α Bernstein is transformed to classical Bernstein operator. In the above graph we have plotted the opeator for α= 1, and notice that graph for classical Bernstein Operator and α- Bernstein Operator overlap each other.

Chapter 4

Convergence and Error Analysis of Different Operators

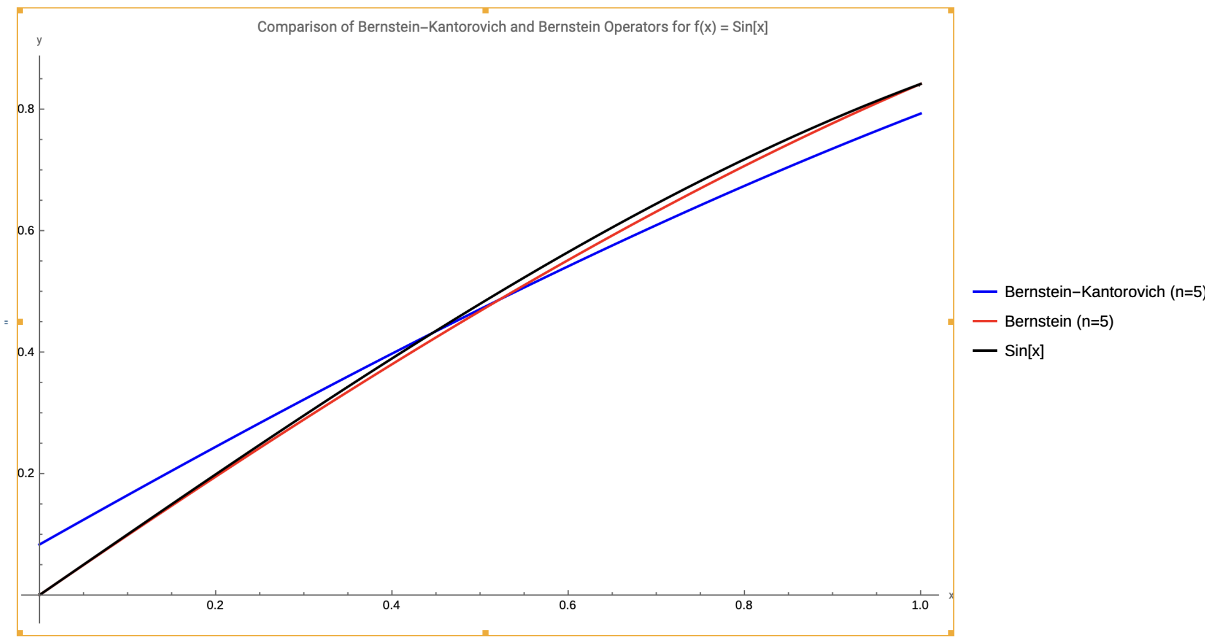
This chapter will provide an in-depth discussion of the rate at which approximations are made and the error characteristics of various operators. For an easier understanding and visual depiction of these qualities, we shall plot them graphically. As a result, we can compare different operators' convergence rates by displaying them together in one graph. Also, via mean squared error analysis, we shall determine the errors associated with each operator using their predicted values to match their real ones.

We will plot these operators for $sin(x)$ function in the interval $[0, 1]$.

The aim is thus to have a clear contrast in behavior between these operators as they converge at various degrees. This way we can establish which of these operators approximates a function best for a given degree.

4.1 Comparing Bernstein and Bernstein-Kantorovich Operator

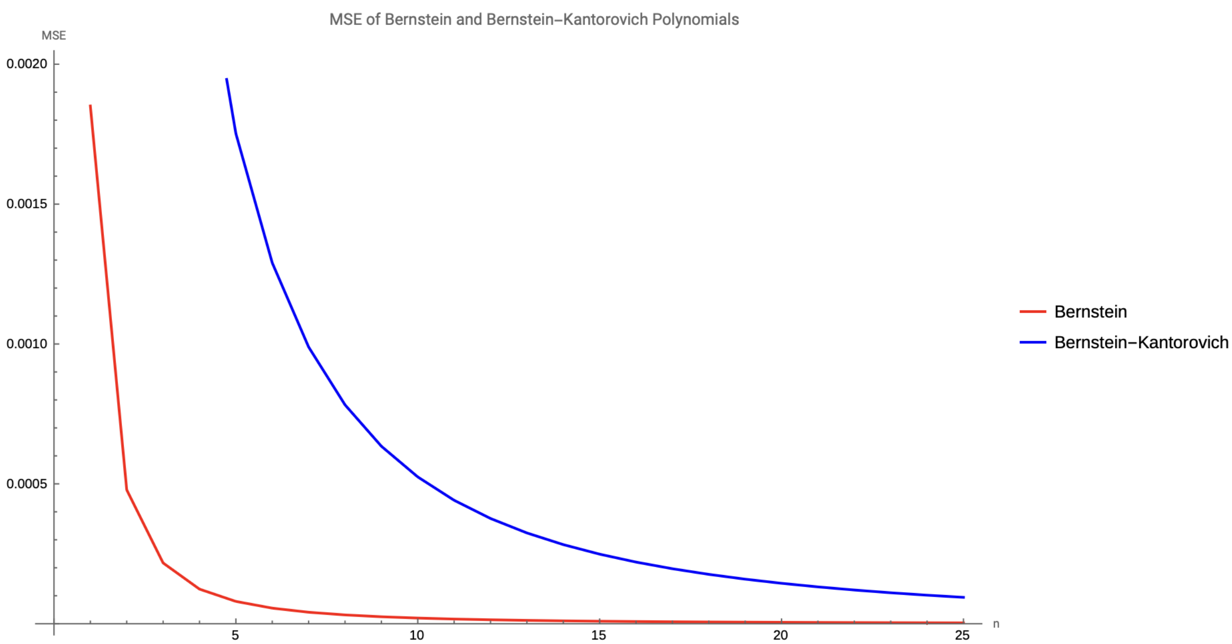
We start off by plotting Bernstein operator and Bernstein-Kantorovich operator for $f(x) = sin(x)$ in the interval $[0, 1]$ for degree five and observe:



We see that plot of bernstein polynomial is closer to the graph of $sin(x)$ as compared to Bernstein-kantorovich polynomial. Hence, we conclude that Bernstein operator approximates a function better as compared to Bernstein-Kantorovich operator for degree 5. Also, the same trend follows for other degrees also.

4.1.1 Error comparison of Bernstein and Bernstein-Kantorovich

Now we know that Bernstein operator has better convergence property but to understand it more efficiently, we calculate the error for both the functions and plot it graphically to understand the difference between the convergence of two operators.



From the above graph we see that initially, for smaller degrees like 1, 3 and 5 there is a significant difference between the error of two operators but as the degree increases, this difference also decreases.

4.2 Comparing Bernstein and Bernstein-Durrmeyer

We start off by plotting Bernstein operator and Bernstein-Durrmeyer operator for $f(x) = sin(x)$ in the interval $[0, 1]$ for degree five and observe:

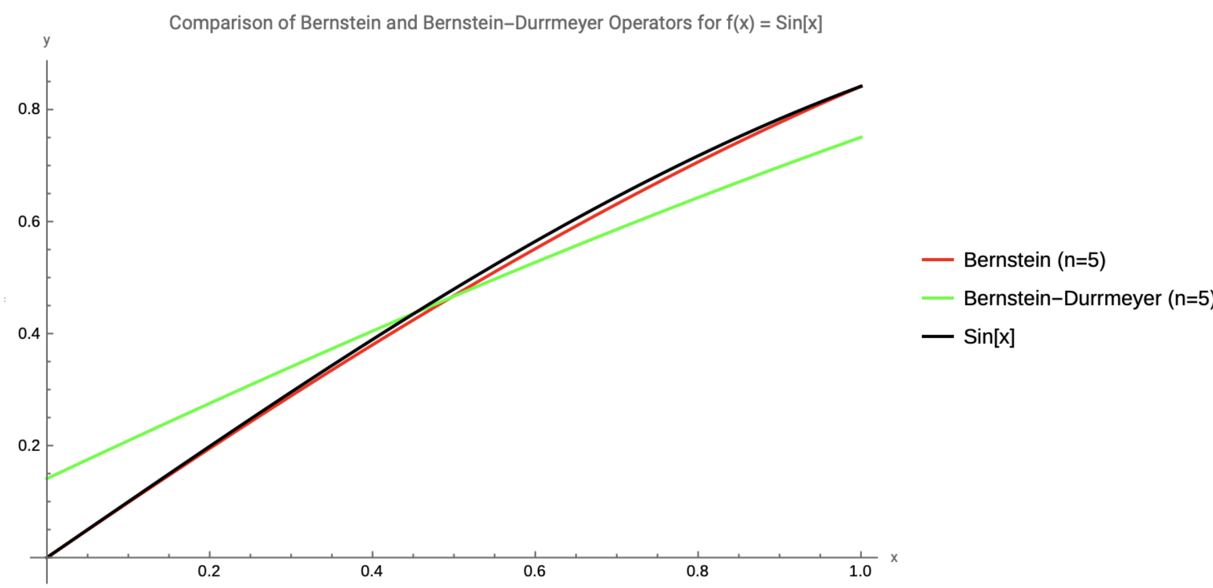
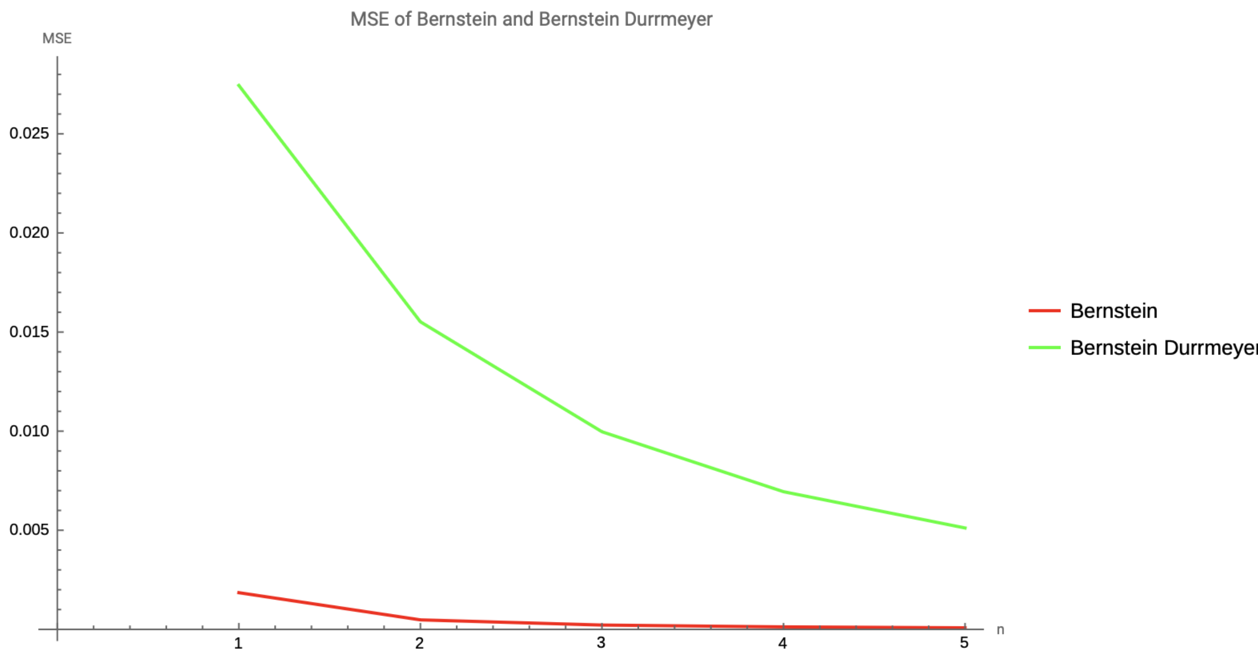


Figure 4.1: Bernstein and Bernstein-Durrmeyer Operator

We see that plot of bernstein polynomial is closer to the graph of $sin(x)$ as compared to Bernstein-Durrmeyer polynomial. Hence, we conclude that Bernstein operator approximates a function better as compared to Bernstein-Durrmeyer operator for degree 5. Also, the same trend is followed as we alter the degrees i.e., Bernstein Operator has better convergence for any given degree.

4.2.1 Error comparison of Bernstein and Bernstein-Durrmeyer

Now we try to plot the error for Bernstein and Bernstein durrmeyer operator for various degrees for $sin(x)$ function.



From the above given graph we can notice that error for Bernstein operator in significantly less as compared to Bernstein Durrmeyer Operator intially for degrees 1, 2, 3. Also the error is decreasing for both the operators as the degree is increasing. Hence convergence increases as degree increases.

4.3 Comparing all three Operators

In this part we will combine both the above graphs togethere to compare the convergence of all the three operator to analyse their performance. So we plot the graph for Beernstein Operator and both it's modification for degree 5 along with the graph of fuction $sin(x)$

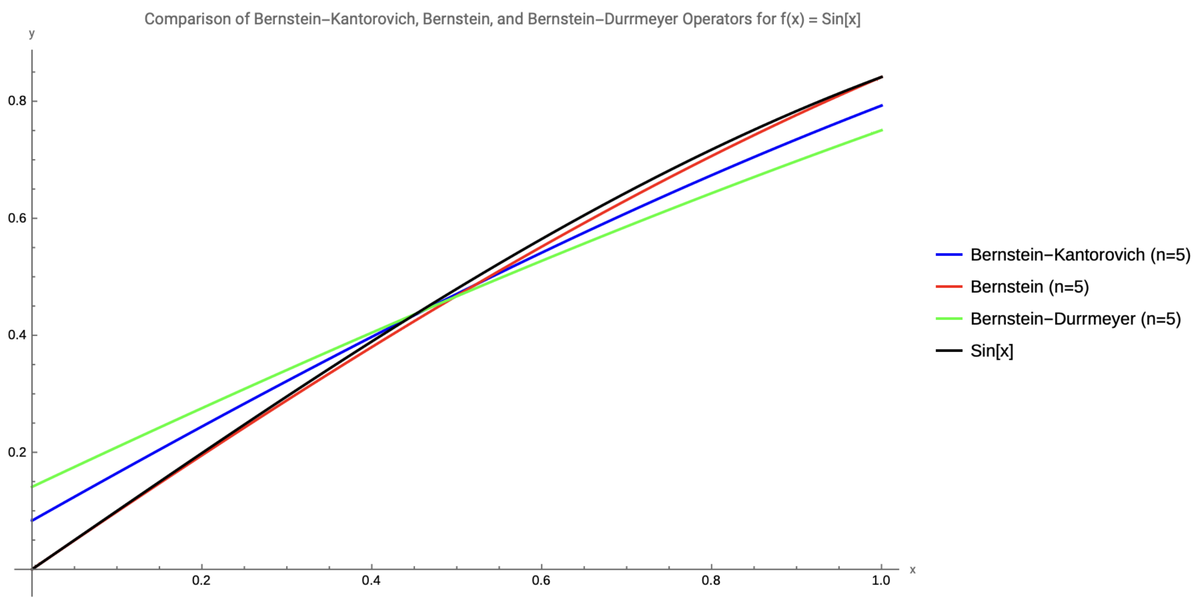


Figure 4.2: Operators Plot

In the above plot we see that the graph of Bernstein Operator is closest to the graph of function $sin(x)$. Also the graph of Bernstein Kantorovich Operator is closer the graph of $sin(x)$ compared to the plot of Durrmeyer variant. Hence Bernstein Operator has the best convergence compared to other two, followed by Bernstein Kantorovich Operator and Durrmeyer variant has the least convergence.

4.3.1 Error Comparison for Bernstein, Bernstein Kantorovich and Bernstein Durrmeyer Operator

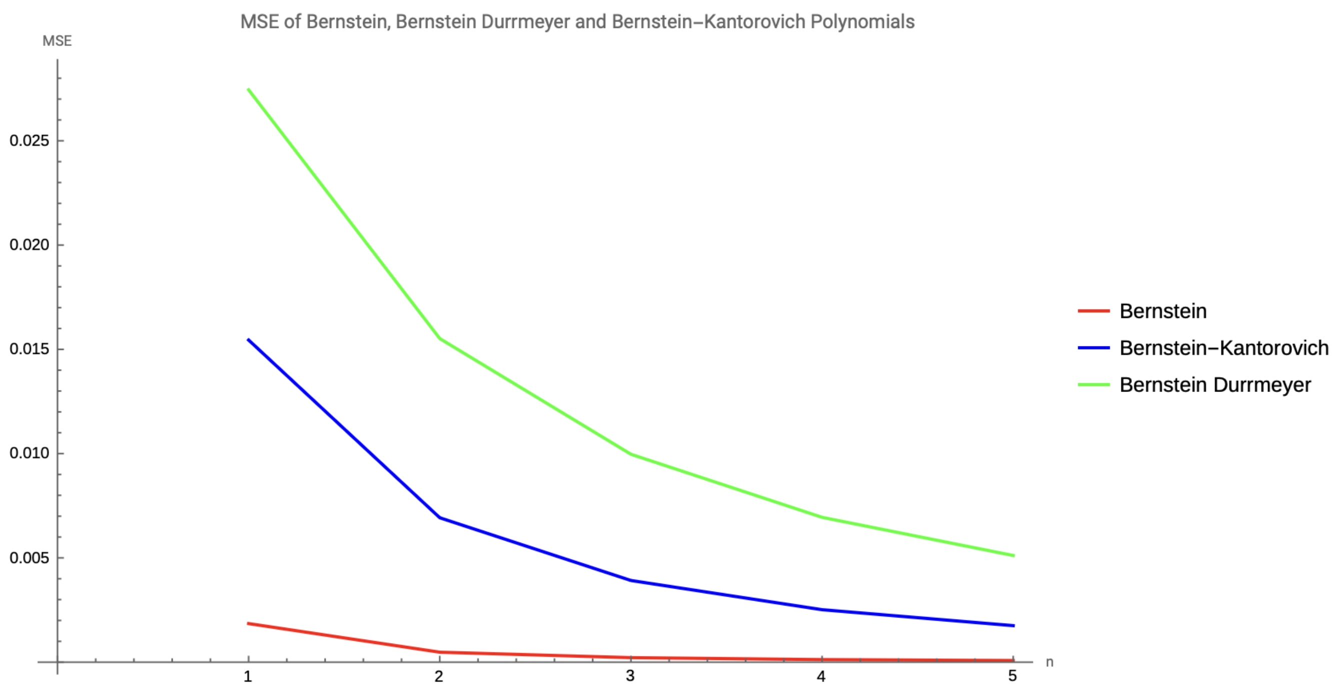


Figure 4.3: Error Plot

Above is the graph for error of Bernstein , Bernstein-Kantorovich and Bernstein Durrmeyer for $sin(x)$ in the interval $[0, 1]$ for different degrees. We can clearly see the difference in the error for the three operator, Bernstein Operator is the most accurate one followed by Bernstein-Kantorovich and the Durrmeyer variant has the highest error. Also, as the degree is increasing the difference between the error of the operators is decreasing.

4.4 Comparing Bernstein and Bernstein-Chlodovsky

First, we will plot the graph of Bernstein operator along with graphs of Bernstein-Chlodwsky operator and $sin(x)$ in $[0, 1]$ to visualize their convergence property.

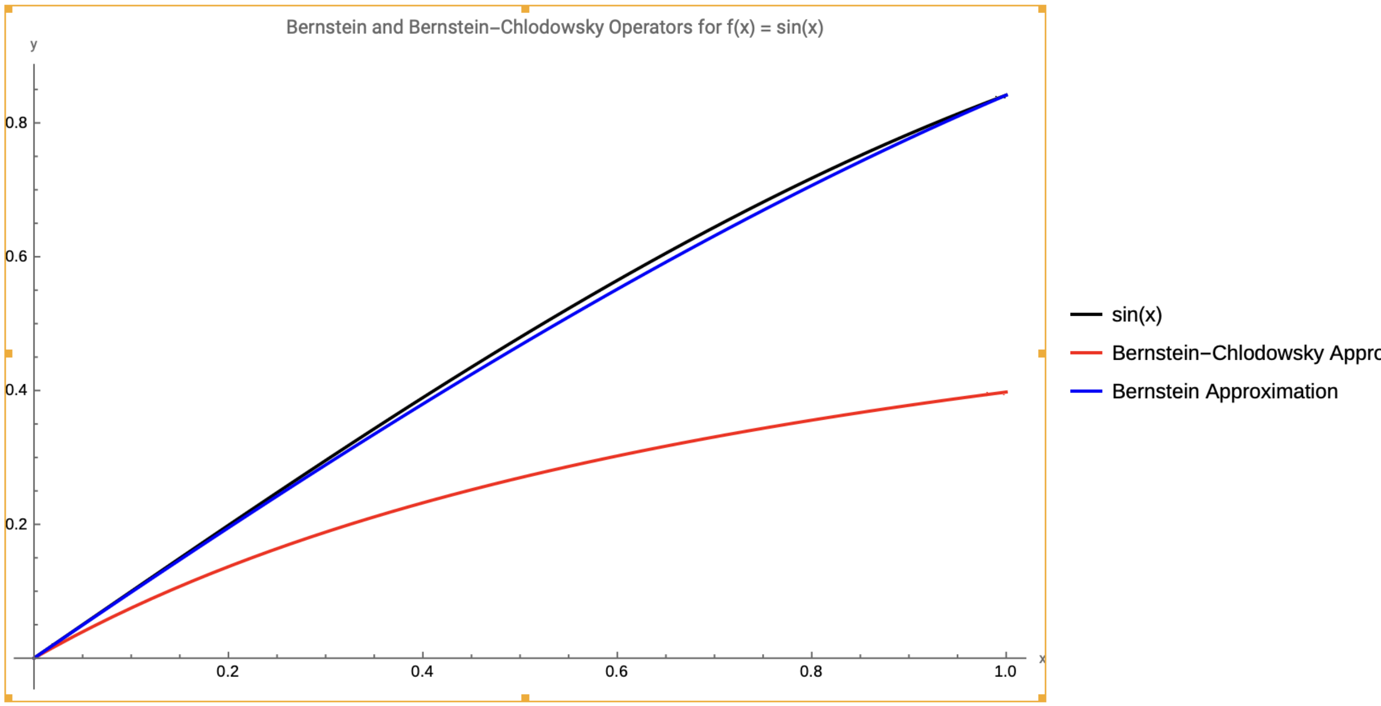


Figure 4.4: Bernstein and Bernstein-Chlodwsky operators

From the above plotted graph, we observe that Bernstein Operator is closest to the graph of function $sin(x)$. Also, the graph of Bernstein-Chlodovsky operator is far away from the graph of $sin(x)$. It clearly shows that Bernstein operator has far better convergence than Chlodovsky variant. Also the results does not improve much of Bernstein-Chlodovsky upon increasing the degree of the operator.

4.4.1 Error graph for Bernstein and Bernstein-Chlodovsky Operator for f(x) = $sin(x)$ in the interval $[0, 1]$

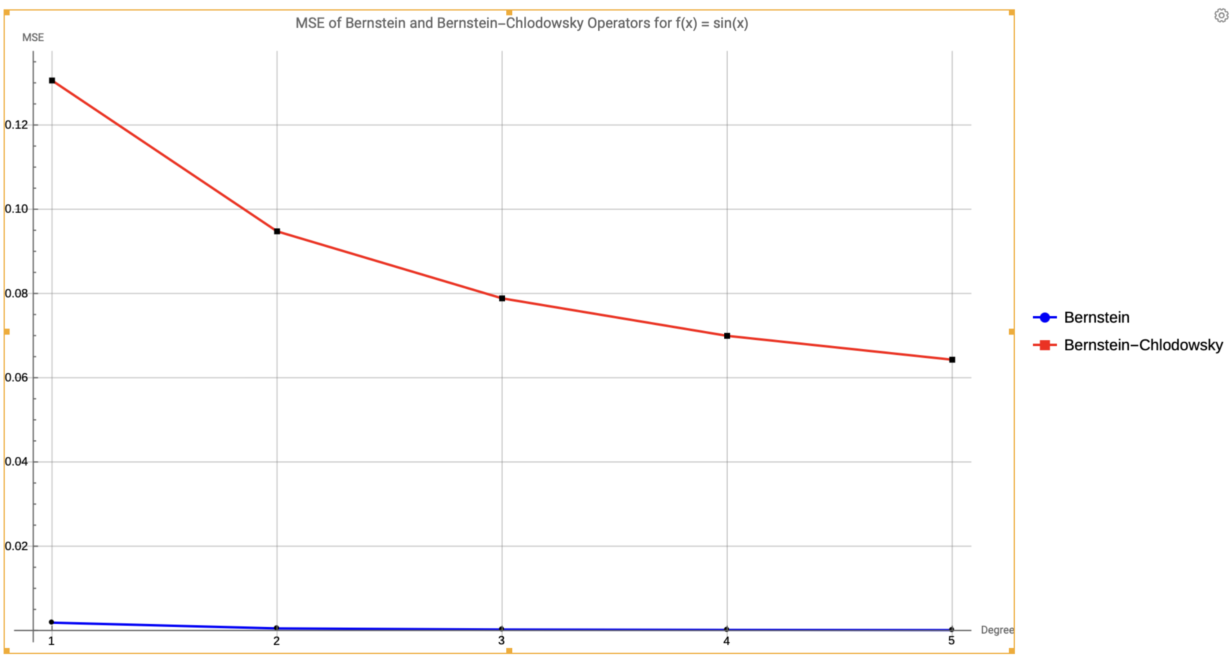
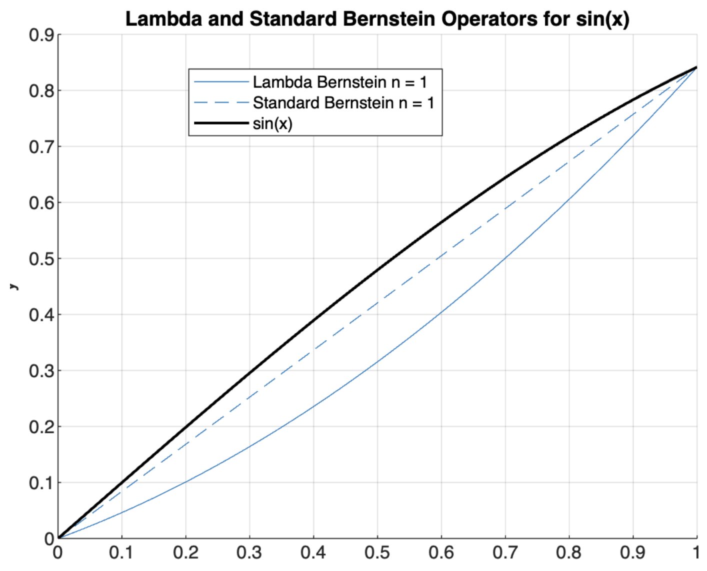


Figure 4.5: Error Plot

In the above we can clearly see that the error for Bernstein-Chlodovsky operator is significant higher as compared to the Bernstein operator. Also as compared to other modifications where the error reduces greatly on increasing the degree, here there is no such trend here and upon increasing the degree there is significant difference between error of the two operators.

4.5 Comparing Bernstein and λ-Bernstein Operator

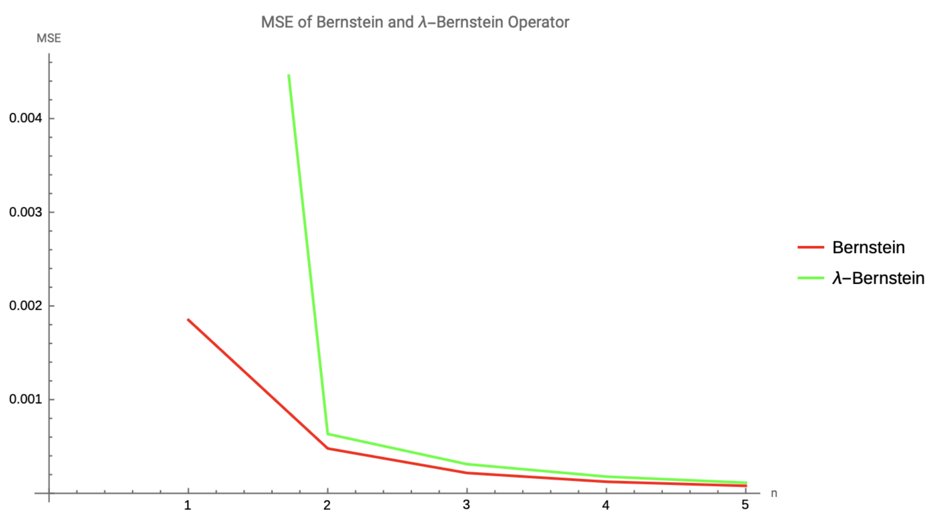
We will now plot the graph the graph for Bernstein and λ-Bernstein operator to understand which one approximates a function better. We plot both the operators for f(x) = $sin(x)$ and in interval $[0, 1]$.



In the above graph we can see that plot of Bernstein operator is significantly closer to the graph of function for degree 1 as compared to λ-Bernstein. But this difference reduces greatly as we increase the degree of the operator. Both the operators have almost identical convergence as we increase the degree.

4.5.1 Error Plot of Bernstein and λ-Bernstein

We will now plot the Mean Squared Error for both the operators for degrees one to five to help us better understand the convergence property of these operators and analyse the error in the approximation of function.



From the above figure, we notice that the error for λ-Bernstein operator is very high as compared to Bernstein operator. From degree 2 the error for both the operator is almost similar for every degress, the error for Bernstein is slightly less. Hence, both the operator have almost same rate of approximation with Bernstein operator being slightly better.

4.6 Comparing Bernstein and α-Bernstein

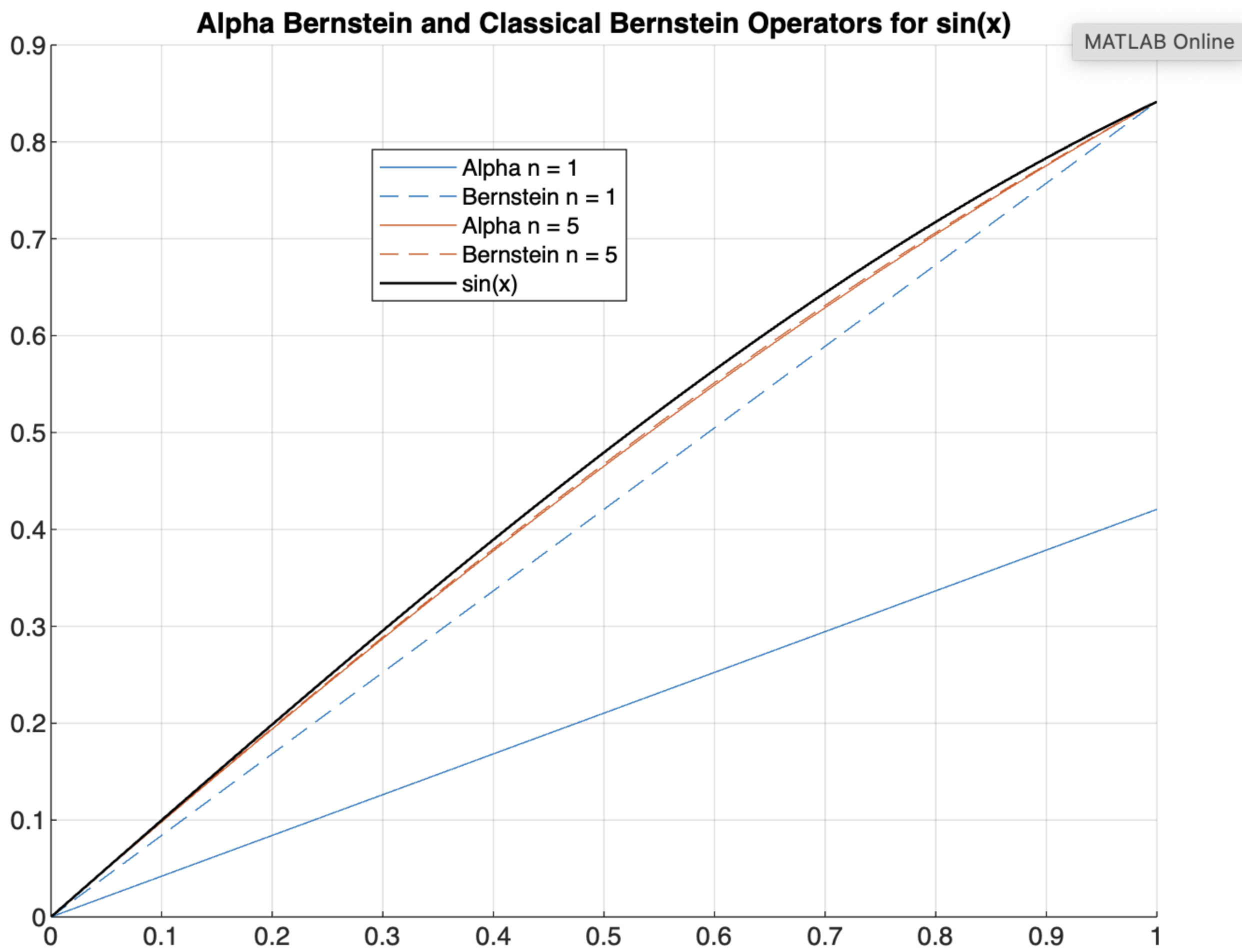


Figure 4.6: Bernstein and α-Bernstein

In the above plot we notice that there is notable difference between the graph of the two operator for degree one with Bernstein operator being significantly closer to the graph of $sin(x)$. But when we increase the degree of the operators from one to five, we notice that the graph of both the operators are really close to each other, with Bernstein operator being slightly better than the other one. Also the same trend is followed on further increasing the degree of the operator.

4.7 Comparing Convergence of Different Operators

Now we try to analyse the convergence property of these operators to understand which one approximates a function with least error. To do so we will plot all the operators togethere on a single graph. Here we use the function $sin(x)$ in the interval $[0, 1]$, each operator is plotted for degree 5.

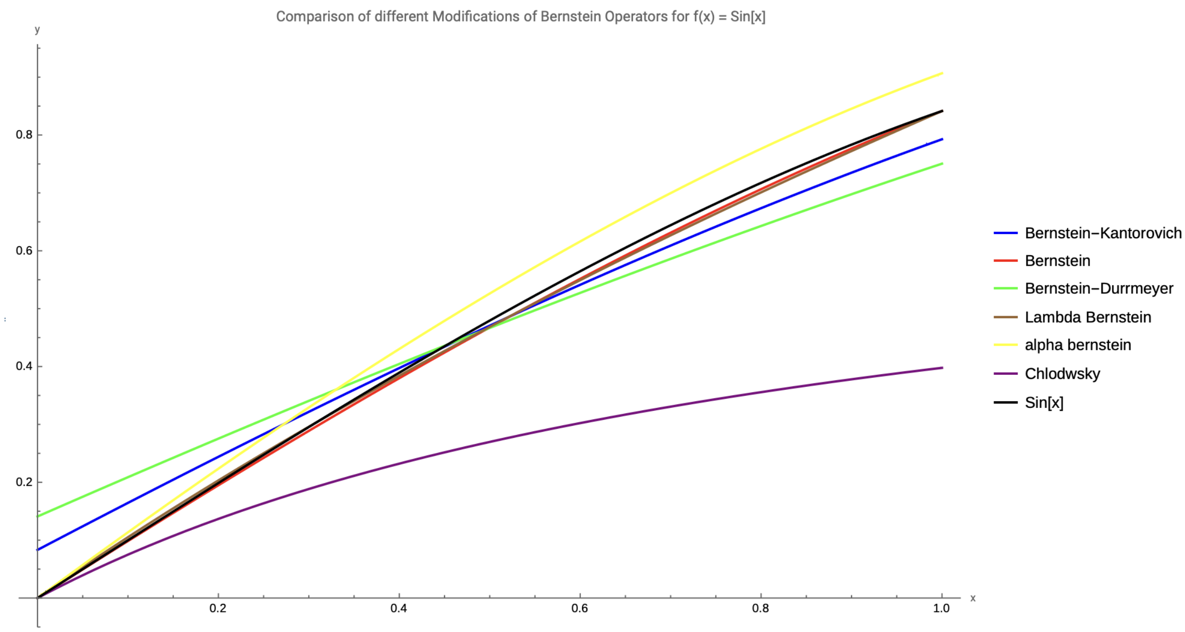


Figure 4.7: Bernstein Operator and it's modification

From the above graph we see that the plot of Bernstein Operator is closest to plot of function $sin(x)$ and the graph of Bernstein-Chlodovsky Operator is farthest from the function. Hence we conclude that Bernstein Operator approximates a function better than any of it's modification that we discussed. Following Bernstein Operator is the Lambda Bernstein operator which also has better convergence than the remaining other modifications. Bernstein-Kantorovich Operator is the third best Operator and the Chlodovsky Variant has the highest error in approximating a function for a given degree.

Chapter 5

Some applications of Bernstein Polynomial

(a) One of the primary uses of the Bernstein polynomial is that, in contrast to the previous proof that was available to us, it offers a straightforward and simplified version of the Weierstrass theorem. When solving differential equations of higher order, Bernstein polynomials are frequently utilised.

(b) The Hadamard transform, Fredholm integral equations (FIE), and the Hilbert transform are approximated using the generalised Bernstein polynomial. Since they may be used to describe a wide range of elasticity, fluid dynamics, and other problems. Fredholm integral equations are widely used in applied sciences [10]. The Hilbert transform is widely used in many areas, such as wave propagation equations, electromagnetic and quantum mechanics, and signal processing.

(c) Furthermore, the Hilbert and Hadamard transformations have a wide range of uses in mathematical physics, including the formulation of boundary-value problems in potential theory, fracture mechanics, aerodynamics, elasticity, and other fields.

(d) Computer aided geometric design (CAGD) can be thought of as having its roots in the study of parametric curves and surfaces for use in CAD (computer aided design)[11]. Bézier patch theory was a key development in CAGD. The expression for Bézier curves is given by Bernstein polynomials. A degree n Bézier curve is an expansion of the Bernstein operator B (t), is defined by

$$b^n(t) = \sum_{k=0}^{n} b_i B^n(t)$$

with the control points b.

(e) In Facial Surgery

The context of finite element analysis has been applied to Bernstein-Bézier representations. Raising the degree of finite element interpolation had the opposite impact as mesh refinement. Both strategies enable for progressively more precise simulation outcomes. Increasing the degree of continuity of the finite element solution has been the focus of additional research in an effort to address the massive increase in the issue dimension brought on by mesh refinement and higher order interpolation. Consequently, a theoretical n-dimensional interpolant has been used to create a tetrahedral Bernstein–Bézier finite element. Despite the fact that linear elasticity is widely acknowledged to be an issue, the element was used to this particular case. Neither the locality nor the approximation qualities of the Bernstein finite set are invalidated by the tetrahedral construction approach.

The prototype simulator developed for post-simulation of actual surgery aims to compare real surgical outcomes with simulated ones, striving for equivalence. It operates with minimal user interaction, focusing on automatic model building and jaw movement determination to reduce inconsistencies between simulation and reality. Key to its functionality are cutting-edge registration techniques for data set alignment, crucial for accurate bone displacement field computation [11].

Successfully applied in correcting short face syndrome, the simulator optimized finite element and mesh size by comparing real surgery results with various simulations. It determined that while higher-order interpolation methods enhance approximation quality, they are computationally expensive. Instead, linear and quadratic simulations on a finer grid proved superior in terms of both approximation quality and computation time. Consequently, mesh refinement was favored over higher-order interpolation, with quadratic elements identified as the optimal balance between accuracy and efficiency. Current meshing neglects tetrahedral mesh benefits, opting for a simplistic cylindrical projection onto the skull, resulting in a low-quality prism mesh. The Clough-Tocher split worsens issues by creating problematic tetrahedra with challenging angles.

Triangular Bézier clipping, integrated into a raytracer for triangular Bézier patches, accurately computes ray intersections. With a bounding volume hierarchy, it offers efficient computation and memory use. Its accuracy matches tensor product equivalents, surpassing the original algorithm in reliability.

(f) Recognition of human speech using q-Bernstein polynomial

Speech command recognition offers significant advantages for human-computer interaction. Acquiring speech command data is straightforward and does not require any special skills, unlike keyboard proficiency needed for other data entry methods. Using speech commands allows for the rapid conversion of spoken text into electronic form. This method grants users freedom of movement and the ability to use their hands for other tasks.

q-Bernstein polynomials, which are mathematical operators, have been applied to pattern recognition, leading to the development of a new method. This method has been implemented using software developed with the Delphi 7 programming language. With this software, the method has been utilized to process verbal expressions for recognition. Additionally, the developed program computerized 16,000 samples of 8-bit stereo images, conducted speech recognition tests for six words, and obtained results [8].

As mathematical operators, have been utilized in pattern recognition, resulting in the development of a new method. This method was applied to six words using the software created in this research. The data revealed that the human speech recognition rate was higher with this method compared to the traditional pattern recognition method without q-Bernstein polynomials. The potential application of this method for voice and fingerprint recognition technologies is currently being considered.

(g) Bernstein Polynomial as a machine learning technique

The application of Bernstein Polynomial in machine learning regression has been studied for various combination methods and over various datasets. The variation of 'a' and 'b' parameter and their effect over the output has also been studied. Bernstein's polynomials can be used to create a fuzzy regression model and perform better than existing conventional models on fuzzy real-life data. They applied the Bernstein regression model to a real life dataset and further improved this idea by combining its features with other models. This method can be used for much better performance by using Bernstein's polynomials approach to expand the data into 'n' dimensions and using that expansion in different models. Lian Fang et al. described the concept of parametric curve fitting and implemented it using the Hermite Basis function. Using unorganized data, the author tries to depict the model performance on messy real-life data collected in an unorganized manner.

Bernstein Polynomial are approximate curves used for curve fitting [13]. Bernstein Polynomial is used to create this formula, which is given by

$$G(t) = \sum_{i=0}^{n} P_i B_i^n$$

Where, $G(t)$ is the generator function used to create the parametric equation for plotting in each dimension. P_i is i_{th} input to the generator function. B_i^n is the Bernstein coefficient -

$$B_i^n(t) = \left(\frac{\binom{n}{i}}{(b-a)^n} \right) (t-a)^i (b-t)^{n-1}$$

Hence,

$$G(t) = P_0 B_0^n + P_1 B_1^n + P_2 B_2^n + \cdots + P_n B_n^n$$

$$G(t) = P_0 \binom{n}{0} \frac{(t-a)^0 (b-t)^n}{(b-a)^n} + P_0 \binom{n}{1} \frac{(t-a)^1 (b-t)^{n-1}}{(b-a)^n} + \cdots + P_0 \binom{n}{n} \frac{(t-a)^n (b-t)^0}{(b-a)^n}$$

This generator function is used to plot each feature of the dataset independently against the output. Hence for each feature of the data, a generator function equation is created

Chapter 6

Conclusion

This research paper thoroughly examines the convergence properties and errors of the Bernstein operator and its various modifications. We begin by comparing the Bernstein operator with its Kantorovich and Durrmeyer variants. Our findings indicate that the Bernstein operator approximates functions better than the other two within the interval $[0, 1]$ yielding the least error for any degree of the operator. Next, we compare the Bernstein operator with the Bernstein-Chlodowsky, Lambda-Bernstein, and Alpha-Bernstein operators. Again, the Bernstein operator outperforms the others.

Using the function $sin(x)$ defined in the interval $[0, 1]$, we systematically compare all these modifications to determine which operator converges to a function more effectively. Our results show that the Bernstein operator provides the best approximation, followed by the Lambda-Bernstein operator. The Bernstein-Kantorovich operator ranks third in accuracy, while the Bernstein-Chlodowsky operator produces the least accurate results among all the operators studied.

The research paper also deals with Bernstein operator and it's variations including applications in real life. The study delves into its mathematical properties and practical implications from the basic Bernoulli polynomial to various types of Chlodovsky operators, Kantorovich forms, q-analogs and others.Moreover, it highlights a few uses for Bernstein operators in mathematical physics, computer-aided geometric design, differential equations, and integral equations, highlighting their versatility.

In conclusion, it should be noted that Bernstein polynomials are essential to both theoretical and applied mathematics. They help solve higher-order differential equations and offer a condensed version of the Weierstrass theorem. Approximating the Hadamard transform, Fredholm integral equations, and the Hilbert transform—all of which are widely utilized in domains including elasticity, fluid dynamics, wave propagation, electromagnetics, and quantum mechanics—requires the generalized Bernstein polynomial. In addition, mathematical physics relies heavily on the Hilbert and Hadamard transformations to solve boundary-value issues in potential theory, fracture mechanics, aerodynamics, and elasticity. Moreover, Bézier patch theory, which forms the basis for parametric curves and surfaces in computer-aided design (CAD), is derived from Bernstein polynomials in computer-aided geometric design (CAGD).

Bernstein-Bézier representations have been used in finite element analysis to increase simulation accuracy. Although higher-order interpolation requires a lot of computing power, research indicates that mesh refining and increasing the degree of interpolation both improve precision. For linear elasticity, a tetrahedral Bernstein-Bézier finite element was created while preserving approximation properties and locality. By balancing quality and computing time, a prototype simulator for post-surgical analysis obtained great accuracy through mesh refinement with linear and quadratic elements. Triangle Bézier clipping in raytracing shows efficient, precise computation, outperforming earlier approaches, even in the face of meshing problems. This method made finite element analysis more useful for real-world uses, such as treating short face syndrome.

Bibliography

[1] Serge Bernstein. Démo istration du th'eorème de weierstrass fondée sur le calcul des probabilités. *i* , 13(1):1–2, 1912.

[2] Qing-Bo Cai, Bo-Yong Lian, and Guorong Zhou. Approximation properties of λ-bernstein operators. *Journal of inequalities and applications*, 2018:1–11, 2018.

[3] Xiaoyan Chen, Jieqing Tan, Zhi Liu, and Jin Xie. Approximation of functions by a new family of generalized bernstein operators. *Journal of Mathematical Analysis and Applications*, 450(1):244–261, 2017.

[4] EW Cheney and A Sharma. On a generalization of bernstein polynomials. *Riv. Mat. Univ. Parma*, 5(2):77–84, 1964.

[5] I Chlodovsky. Sur le développement des fonctions définies dans un intervalle infini en séries de polynomes de ms bernstein. *Compositio Mathematica*, 4:380–393, 1937.

[6] Naokant Deo, Muhammad Aslam Noor, and Mohammad Arif Siddiqui. On approximation by a class of new bernstein type operators. *Applied mathematics and computation*, 201(1-2):604–612, 2008.

[7] Jean Loup Durrmeyer. *Une formule d'inversion de la transformée de Laplace: Applications à la théorie des moments*. PhD thesis, 1967.

[8] Abdulkadir Karacı, İbrahim Büyükyazıcı, and Muharrem Aktümen. Recognition of human speech using q-bernstein polynomials. *International Journal of Computer Applications*, 975:8887.

[9] A Lupas. A q-analogue of the bernstein operator. In *Seminar on numerical and statistical calculus, University of Cluj-Napoca*, volume 9, 1987.

[10] Donatella Occorsıo, Maria Grazia Russo, and Woula Themıstoclakıs. Some numerical applications of generalized bernstein operators. *Constructive Mathematical Analysis*, 4(2):186–214, 2021.

[11] Samuel Hans Martin Roth. *Bernstein-Bézier representations for facial surgery simulation*, volume 16. ETH Zurich, 2002.

[12] Dimitrie D Stancu. On a new positive linear polynomial operator. *Proceedings of the Japan Academy*, 44(4):221–224, 1968.

[13] Deekshant Wadhwa, Nitesh Kumar Singh, and Nishtha Jatana. Evaluation of bernstein polynomial as a machine learning technique. In *IOP Conference Series: Materials Science and Engineering*, volume 1022, page 012076. IOP Publishing, 2021.

[14] Karl Weierstrass. Über die analytische darstellbarkeit sogenannter willkürlicher functionen einer reellen veränderlichen. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*, 2(633-639):364, 1885.

[15] Zhangxiang Ye, Xiang Long, and Xiao-Ming Zeng. Adjustment algorithms for bézier curve and surface. In *2010 5th International conference on computer science & education*, pages 1712–1716. IEEE, 2010.