IMAGE ENCRYPTION USING RABINOVICH-FABRICANT SYSTEM AND STUDY ITS CHAOTIC BEHAVIOUR

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SUBMITTED BY:

Dhawal Kumar Dhruv
(2K22/MSCMAT/08)

Prince Sahni
(2K22/MSCMAT/32)

UNDER THE SUPERVISION OF
DR. DHIRENDRA KUMAR

DEPARTMENT OF APPLIED MATHEMATICS
DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College of Engineering)
Bawana Road, Delhi – 110042

Delhi Technological University
(Formerly Delhi College of Engineering)
Bawana Road, Delhi-110042

CANDIDATE'S DECLARATION

We, **Dhawal Kumar Dhruv** (Roll No.2K22/MSCMAT/08) and **Prince Sahni**, (Roll No.2K22 /MSCMAT/32), students of Master of Science in Applied Mathematics, hereby declare that the dissertation titled"IMAGE ENCRYPTION USING RABINOVICH-FABRICANT SYSTEM AND STUDY ITS CHAOTIC BEHAVIOUR", submitted to the Department of Applied Mathematics, Delhi Technological University, Delhi, in partial fulfillment of the requirements for the degree of Master of Science, is our original work. Proper citations have been given wherever necessary, and this work has not been submitted previously for any degree, diploma, associateship, or any other similar title or recognition.

Place: Delhi **Dhawal Kumar Dhruv**

Date: **Prince Sahni**

DELHI TECHNOLOGICAL UNIVERSITY

(Formerly Delhi College of Engineering)

Bawana Road, Delhi-110042

CERTIFICATE

We hereby certify that the Project Dissertation titled "**IMAGE ENCRYPTION USING RABINOVICH-FABRICANT SYSTEM AND STUDY ITS CHAOTIC BEHAVIOUR**" which is submitted by **Dhawal Kumar Dhruv** (2K22/MSCMAT/08) and **Prince Sahni** (2K22/MSCMAT/32) [Department of Applied Mathematics], Delhi Technological University, Delhi in partial fulfillment of the requirement for the award of the degree of **Master of Science**, is a record of the project work carried out by the students under my supervision. To the best of my knowledge, this work has not been submitted in part or full for any degree or diploma to this university or elsewhere.

Place: Delhi
Date:

Dr. Dhirendra Kumar
SUPERVISOR

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We would especially want to thank Dr.Dhirendra Kumar, our mentor, for his time and efforts throughout the course of the year. We greatly appreciated your helpful recommendations and advise as we finished the project. We shall always be appreciative of you in this regard. We would like to state that we, and no one else, worked on this project from start to finish.

Signature	Signature
Dhawal Kumar Dhruv	Prince Sahni

Abstract

This paper explores the application of the Rabinovich-Fabrikant chaotic system in image encryption. By leveraging the system's inherent chaotic properties and sensitivity to initial conditions, we develop a method for generating encryption keys and encrypting images. The proposed encryption process involves permuting and substituting pixel values using sequences derived from the Rabinovich-Fabrikant system, ensuring the transformed image is unrecognizable. Decryption is achieved by regenerating the same chaotic sequences and applying inverse operations. The complexity and unpredictability of the chaotic system enhance security, providing robust protection against various attacks. This method demonstrates the potential of chaotic systems in enhancing the security of digital image encryption. This paper also investigates the master-slave synchronization of the Rabinovich-Fabrikant system, a well-known model in nonlinear dynamics that exhibits chaotic behavior. The Rabinovich-Fabrikant system is governed by a set of differential equations, designated as the master and the slave, with the objective of achieving synchronization despite the chaotic nature of their dynamics. The dynamical characteristics and properties are studied as well like fixed points, behaviour at fixed point, Lyapunov exponent, system dissipativity and conservative, poincare section and bifurcation diagrams.

Keywords: Rabinovich-Fabrikant system, nonlinear dynamics, chaotic behaviour, fixed point, trajectories, sensitive to initial conditions.

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Chapter 1

Introduction

In 1979, theoretical physicists Mikhail Rabinovich and Anatoly Fabrikant introduced a novel chaotic system known as the Rabinovich-Fabrikant system. This system is continuous in the time domain and real in the space domain, encompassing three spatial dimensions. Unlike other well-known chaotic systems, such as the Lorenz system which includes only second-order nonlinearities, the Rabinovich-Fabrikant system features third-order nonlinearities. These higher-order nonlinearities give rise to unique and complex dynamics that were seldom observed in earlier systems. The resulting waveforms can resemble virtual "saddles" or even "double vortex tornadoes," depending on the parameters used in the Rabinovich-Fabrikant equations. This system's distinct behaviors highlight its significance in the study of chaos and nonlinear dynamics.

The Rabinovich-Fabrikant system, renowned for its nonlinear dynamical properties, has generated a great deal of interest because of its possible applications, particularly in the realm of secure communication. This paper aims to investigate the implementation and analyze the characteristics of the Rabinovich-Fabrikant system, with a specific focus on its stability and its capability to conceal digital signals for secure communication purposes.

Our research reveals notable insights into the stability of the Rabinovich-Fabrikant system. Notably, for specific parameter values, such as α=0.14 and γ=0.1, we observed stable behavior within the system. This stability is of paramount importance as it ensures the reliability and resilience of the system, particularly in scenarios where data integrity and confidentiality are critical, such as secure communication environments.

Synchronization of chaotic systems has garnered significant interest in research due to its wide range of potential applications across various industrial sectors . One of the most prominent areas leveraging chaos synchronization is communication security. In this context, message signals are embedded into a chaotic carrier within the transmitter, leading to their masking or encryption. These encrypted signals are then transmitted over a public channel to the receiver [7]. The receiver can only retrieve the original message through synchronization of the chaotic systems at both the transmitter and receiver ends.

The concept of synchronizing chaotic systems with different initial conditions was first introduced by Pecora and Carroll in 1990 [8]. Since then, numerous synchronization techniques have been explored and developed, including coupling control, adaptive control, feedback control, fuzzy control, and observer-based control [7]. These methodologies have significantly advanced the field and opened up new avenues for secure communication technologies.

The mathematical notation of the Rabinovich-Fabrikant system is given by the following set of differential equations:

$$\begin{aligned} \dot{x} &= y(z - 1 + x^2) + \gamma x \\ \dot{y} &= x(3z + 1 - x^2) + \gamma y \\ \dot{z} &= -2z(\alpha + xy) \end{aligned} \tag{1.1}$$

Here, α and γ are constants. It is noted that the dynamics of the system are highly sensitive to changes in α, while γ has a lesser effect. Consequently, α is often treated as the bifurcation parameter.

1.1 Synchronization of Chaotic Systems using Master-Slave Configuration

The synchronization of chaotic systems using a master-slave configuration is a powerful technique in chaos theory. One notable approach, proposed by M.S. Rulkov, I. Sushchik, L. Tsimring, and H.D.I. Abarbanel, involves utilizing the Rabinovich-Fabrikant system [4], which exhibits complex chaotic behavior.

1.1.1 Rabinovich-Fabrikant System Overview

The Rabinovich-Fabrikant system is described by the following set of differential equations: (1.1)
where α and γ are system parameters. [6]

1.1.2 Master-Slave Synchronization Using Rabinovich-Fabrikant System

1. Master System:

$$\begin{cases} \dot{x}_m = y_m(z_m - 1 + x_m^2) + \gamma x_m \\ \dot{y}_m = x_m(3z_m + 1 - x_m^2) + \gamma y_m \\ \dot{z}_m = -2z_m(\alpha + x_m y_m) \end{cases}$$

2. Slave System:

$$\begin{cases} \dot{x}_s = y_s(z_s - 1 + x_s^2) + \gamma x_s + k(x_m - x_s) \\ \dot{y}_s = x_s(3z_s + 1 - x_s^2) + \gamma y_s + k(y_m - y_s) \\ \dot{z}_s = -2z_s(\alpha + x_s y_s) + k(z_m - z_s) \end{cases}$$

The coupling terms $k(x_m - x_s)$, $k(y_m - y_s)$ and $k(z_m - z_s)$ in the slave system drives synchronization with the master system.

1.1.3 Master-Slave Configuration

1. Master System:

- The master system is an autonomous chaotic system that generates a chaotic signal. It can be outlined by a set of differential equations:

$$\begin{cases} \dot{x}_m = F(x_m, p) \end{cases}$$

where x_m represents the state vector of the master system, and p denotes the parameters of the system.

2. Slave System:

- The slave system which receives signals from the master system and adjusts to synchronize its behavior with the master. Its dynamics are influenced by the master's output:

$$\begin{cases} \dot{x}_s = G(x_s, x_m, q) \end{cases}$$

Here, x_s is the state vector of the slave system, x_m is the input from the master system, and q denotes the parameters of the slave system.

1.2 Process of Synchronization

1. **Initialization:** Both master and slave systems start with different initial conditions. 2. **Coupling:** The slave system is driven by the coupling term from the master system. 3. **Synchronization:** Over time, the slave system synchronizes its chaotic behavior with the master system.

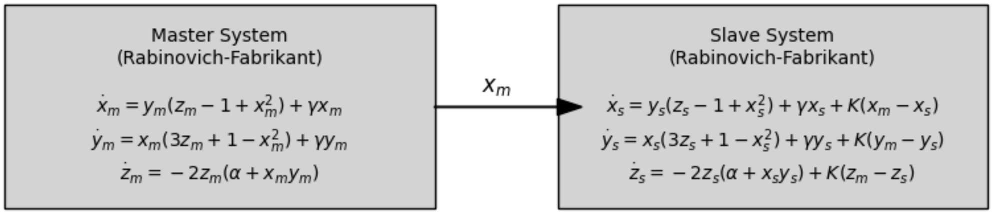


Figure 1.1: Illustration of master-slave system.

1.2.1 Methods for Synchronizing Chaotic Systems

1. **Subsystem Decomposition:** This method involves dividing the chaotic system into smaller subsystems to enable synchronization.

2. **Linear Mutual Coupling:** It entails creating linear connections between chaotic systems to exchange information and achieve synchronization.

3. **Linear Feedback:** This technique employs feedback mechanisms to modify system parameters or inputs, facilitating synchronization.

4. **Inverse System:** It involves developing an inverse model to attain desired dynamics and synchronize chaotic systems accordingly.

5. **Observer Based Design:** This approach focuses on designing an observer system to estimate the state of a chaotic system and achieve synchronization through feedback mechanisms.

Chapter 2

Chaotic Dynamics in Deterministic Systems

In this chapter, we explore a particular class of dynamical systems characterized by behavior that can be classified as chaotic under certain conditions [2]. Drawing on the work of Strogatz and other foundational texts, we define chaos as follows:

Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions [5].

Key Aspects of Chaotic Systems

Several important points arise from this definition:

Aperiodicity

This condition implies that within the system, there are trajectories that do not converge to fixed points or periodic orbits [2]. It is crucial to note that not all trajectories need to be aperiodic; it suffices that some exhibit this property. Additionally, aperiodicity excludes trajectories that tend towards infinity for large parameter values, as being at infinity can be seen as a type of periodicity. This distinction helps differentiate between chaotic behavior and instability. For example, the system described by

$$\dot{x}(t) = x(t)$$

is unstable since its trajectories,

$$x(t) = e^{tx_0},$$

grow unbounded as $t \to \infty$, [2] but this is not considered chaotic.

Determinism

The systems under consideration must be deterministic, meaning they do not involve any stochastic or random elements. This requirement ensures that the chaotic nature is inherent to the system itself, rather than being influenced by external, uncontrolled factors.

Sensitive Dependence on Initial Conditions

This property means that trajectories starting very close to each other will diverge exponentially fast over time. Even if they start infinitesimally close, they will eventually follow distinctly different paths, appearing unrelated in the long run.

By understanding these characteristics, we can better identify and study chaotic systems. These systems exhibit complex and unpredictable behavior despite being governed by deterministic rules, making them a fascinating subject in the field of dynamical systems.

2.1 Chaos in One Dimension

To analyze the long-term aperiodic behavior of a dynamical system, it is crucial to understand its fixed points (or equilibrium solutions). These are the constant solutions $x(t) = x^*$, where x^* satisfies the equation [2]:

$$f(x^*) = 0.$$

In the context of one-dimensional systems (i.e., for a system of the form $\dot{x}(t) = f(x(t))$, where $x : I \subset \mathbb{R} \to \mathbb{R}$ is defined on an open interval I and f is at least continuous) [2], the geometry is so constrained that chaotic behavior cannot occur. Suppose x_1 and x_2 are fixed points of the system, with $x_1 < x_2$. Thus, $f(x_1) = 0$ and $f(x_2) = 0$. Due to continuity, f maintains the same sign in the entire interval (x_1, x_2). Assume $f(x) > 0$ within this interval. Consequently, any solution will satisfy $\dot{x}(t) = f(x(t)) > 0$ for all t where $x(t) \in (x_1, x_2)$, leading to a monotonically increasing behavior [2], which is bounded above by the equilibrium $x(t) = x_2$.

According to the theorem on the existence and uniqueness of solutions to differential equations [2], distinct trajectories cannot intersect. Hence, a solution $x(t)$ that, at some time t_0, is in (x_1, x_2), must asymptotically approach the equilibrium $x(t) = x_2$. The same logic applies if $f(x) < 0$, resulting in monotonically decreasing solutions that asymptotically approach $x(t) = x_1$.

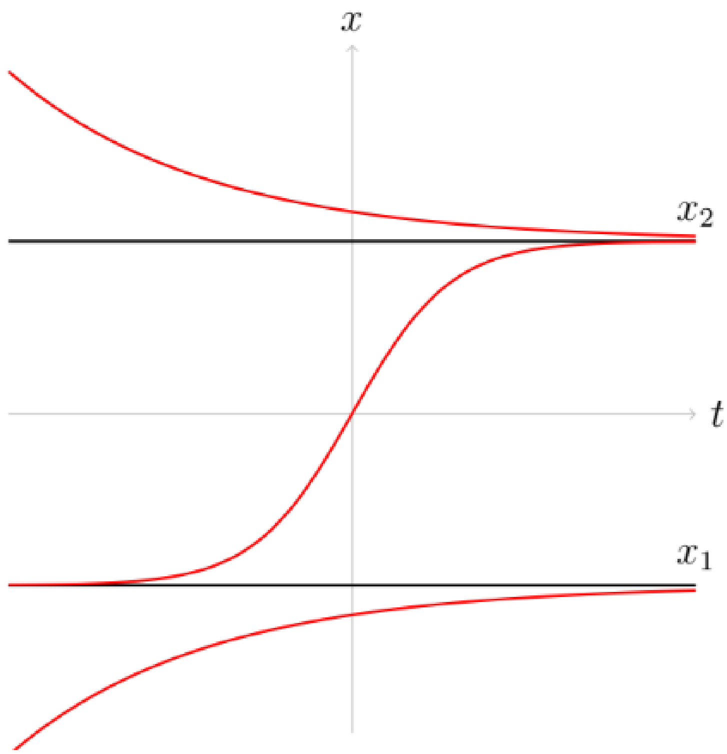


Figure 2.1: Chaos in one Dimension

2.2 Chaos in Two Dimension

It is a fundamental result in dynamical systems theory that chaos cannot arise in planar systems, owing to the Poincaré-Bendixson theorem. This theorem, discussed extensively in [2], states that for a plane system $\dot{x}(t) = f(x(t))$ governed by a smooth vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$, after evolving for a sufficient duration, the system will eventually reach one of the following configurations:

1. A fixed point.
2. A periodic orbit.
3. A connected set comprising a finite number of fixed points and trajectories connecting them [2]. These trajectories, connecting fixed points among themselves, are termed *homoclinic* if they start and finish at the same fixed point, and *heteroclinic* otherwise.

This theorem highlights the deterministic and ordered nature of planar dynamical systems, precluding the possibility of chaotic behavior. Instead, trajectories evolve towards stable configurations such as fixed points or periodic orbits, or they exhibit structured behavior in the form of connected sets involving fixed points and trajectories. This inherent predictability and orderliness make planar systems a valuable area of study in dynamical systems theory.

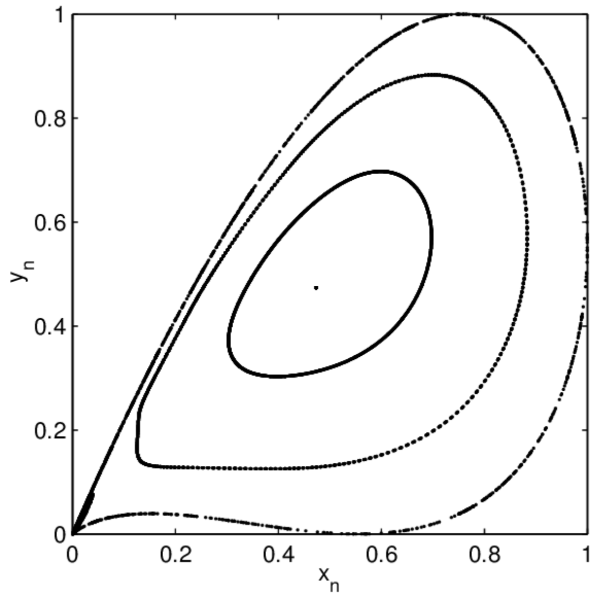


Figure 2.2: Chaos in two Dimension

2.3 Chaos in Three Dimension

Chaos is possible in a nonlinear 3D system. The Rabinovich-Fabrikant (RF) system is a notable example of a three-dimensional system that can demonstrate chaotic behavior. This system, introduced by Mikhail Rabinovich and Anatoly Fabrikant in 1979, models various physical phenomena, such as the dynamics of plasma and fluid flows. It is described by a set of coupled nonlinear differential equations.

The Rabinovich-Fabrikant (RF) system is described by equations: (1.1)

where:

- x, y, and z are the state variables,
- α and γ are parameters that control the dynamics of the system.

Dynamics and Chaos

The dynamics of the RF system depend critically on the values of the parameters α and γ. Let's explore the behavior of the RF system for specific parameter values:

1. Stable Periodic Orbits:
   For certain values of α and γ, the system exhibits stable periodic orbits. For instance, if $\alpha = 0.1$ and $\gamma = 0.87$, the system might display periodic behavior with trajectories settling into closed loops in phase space.

2. Onset of Chaos:

   As γ is increased while keeping α fixed, the system can transition to chaotic behavior. For $\alpha = 0.1$ and $\gamma = 0.98$, the system may exhibit chaotic dynamics with trajectories that appear random and cover a complex attractor.

3. Chaotic Regime:

   At certain parameter values, the system displays fully developed chaos. For example, $\alpha = 0.14$ and $\gamma = 0.1$ might result in a chaotic attractor, where trajectories are highly sensitive to initial conditions and exhibit no periodicity.

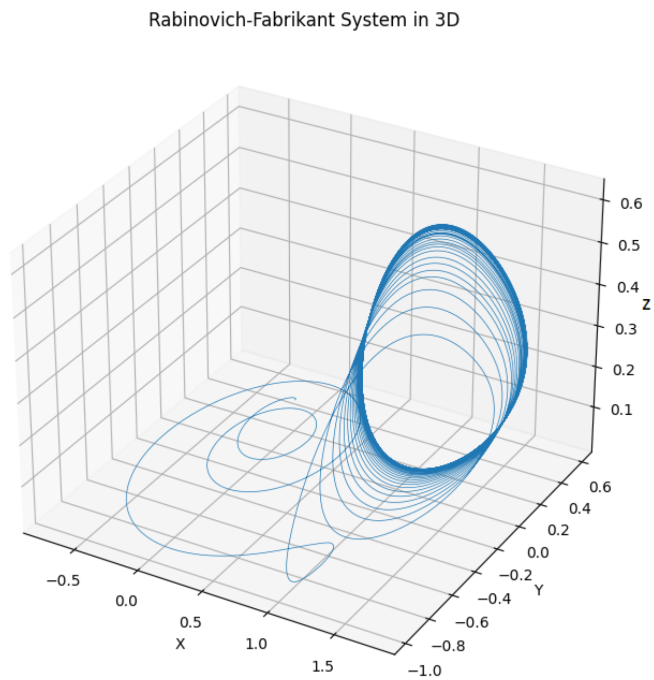


Figure 2.3: Rabinovich-Fabrikant System in 3D

Chapter 3

Methodology

3.1 Rabinovich-Fabrikant System

The system of ordinary differential equations representing the Rabinovich-Fabrikant system describes a set of chaotic oscillations. Each equation describes the rate of change of a state variable with respect to time. The equations are as follows: (1.1)

$$\dot{x} = y(z - 1 + x^2) + \gamma x, \tag{3.1}$$

$$\dot{y} = x(3z + 1 - x^2) + \gamma y, \tag{3.2}$$

$$\dot{z} = -2z(\alpha + xy). \tag{3.3}$$

In these equations:

- $\dot{x}$ describes the rate of change of the variable x over time. It is influenced by the variables y and z, and the parameter γ. It represents the dynamics of the chaotic system in the x-direction.
- $\dot{y}$ describes the rate of change of the variable y over time. It is influenced by the variables x and z, and the parameter γ. It represents the dynamics of the chaotic system in the y-direction [1].
- $\dot{z}$ describes the rate of change of the variable z over time. It is influenced by the variables x and y, and the parameter α. It represents the dynamics of the chaotic system in the z-direction.

Here, $\dot{x}$, $\dot{y}$, and $\dot{z}$ represent the derivatives of the variables x, y, and z with respect to time, respectively. α and γ are parameters that control the behavior of the system. The terms involving x, y, and z in each equation contribute to the chaotic behavior of the system.

These equations are commonly used in the study of chaos theory and have applications in various fields such as physics, biology, and cryptography. They exhibit complex, unpredictable behavior, making them valuable for modeling nonlinear phenomena.

3.1.1 Rabinovich-Fabrikant Phase Diagram

Rabinovich-Fabrikant System exhibits chaotic behaviour in (X,Y) plane as shown in Figure (2) .

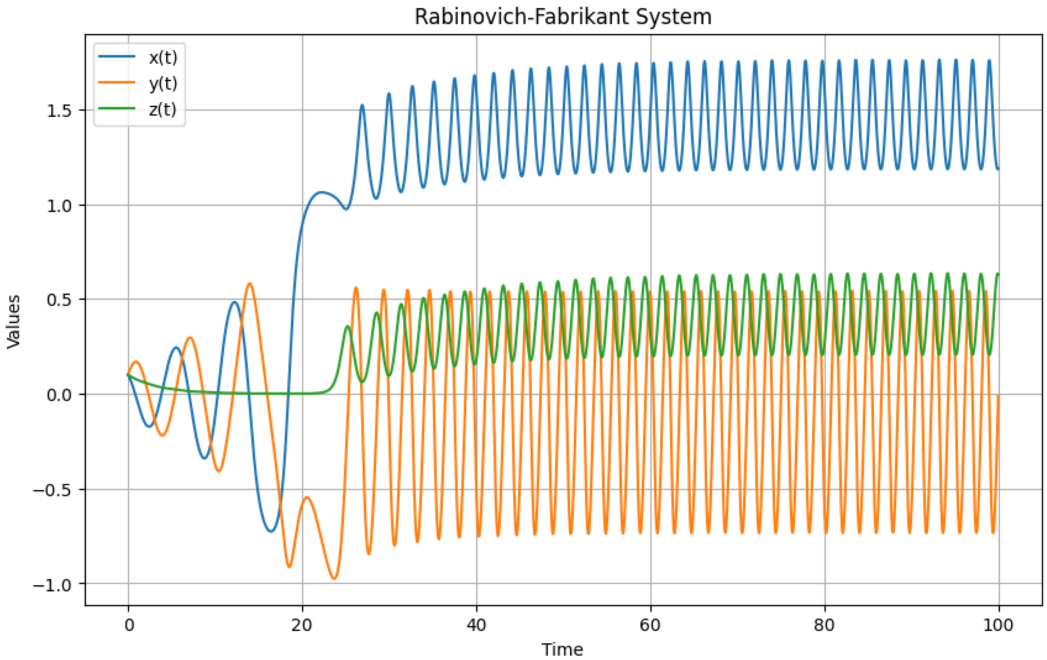


Figure 3.1: Illustration of Rabinovich-Fabrikant System 2D

3.1.2 Phase Diagram (X vs Y , X vs Z and Y vs Z) In 2D

The phase picture of the Rabinovich-Fabrikant System in Figure (3) in (x vs y , x vs z and y vs z) shows strange attractors in 2D.

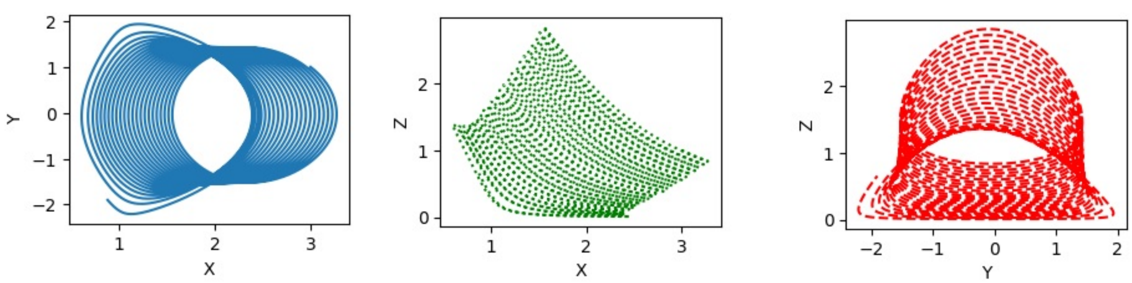


Figure 3.2: phase diagram in 2D

3.1.3 Rabinovich-Fabrikant Master-Slave System Phase Portrait In 2D

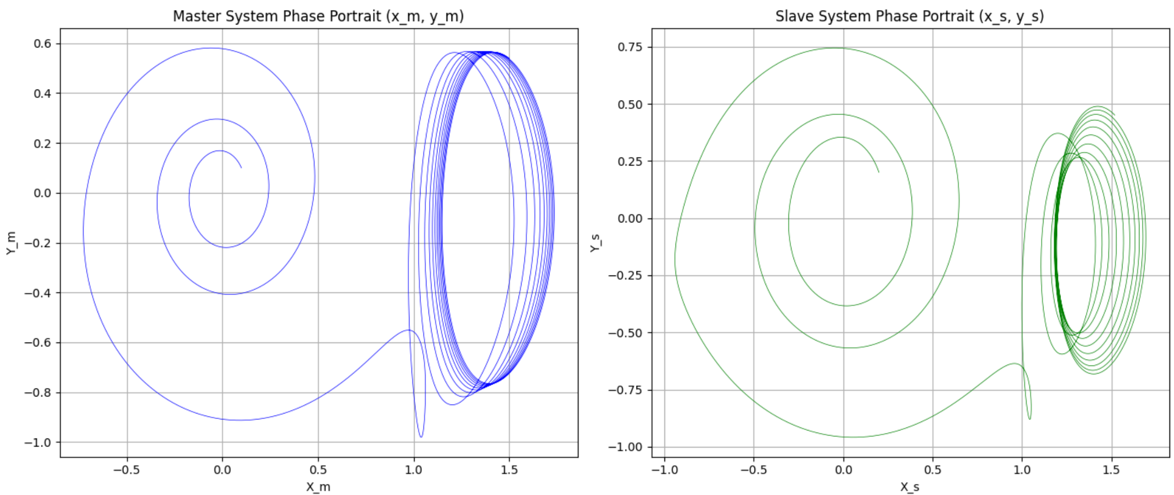


Figure 3.3: Master-Slave System Phase portrait in 2D.

Figure (3.3) illustrate the chaotic dynamics of Rabinovich-Fabrikant system with parameter values α=0.14 and γ=0.1. Two phase portrait for Rabinovich-Fabrikant system starting from two different initial conditions (x_m,y_m,z_m)=(0.1,0.1,0.1) for master system and (x_s,y_s,z_s)=(0.2,0.2,0.2) for slave system.

3.1.4 Rabinovich-Fabrikant Model in 3D

```
# Plotting in 3D
fig = figure()
ax = fig.add_subplot(111, projection='3d')
ax.plot(x, y, z, lw=0.5)
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')
show()
```

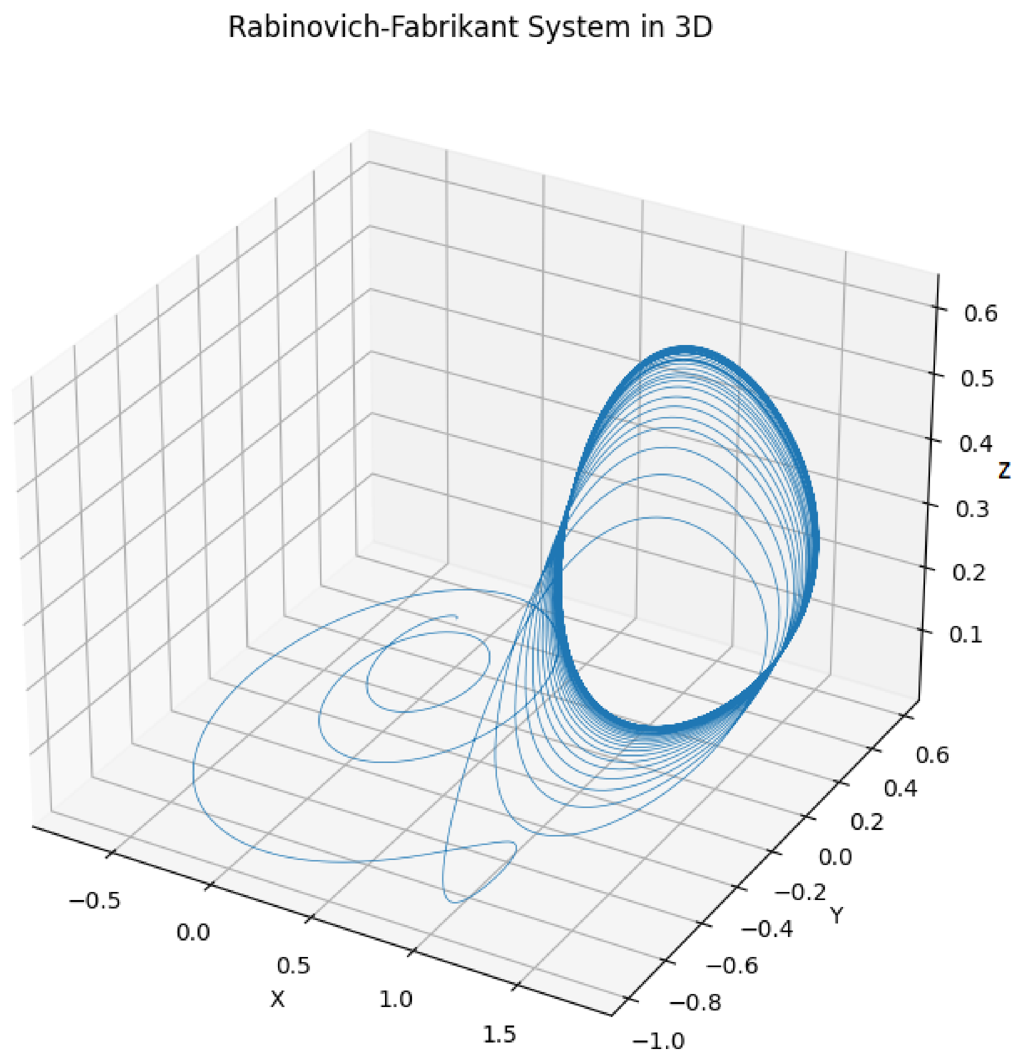


Figure 3.4: Rabinovich-Fabrikant System in 3D

3.1.5 Rabinovich-Fabrikant Master-Slave System Phase Portrait In 3D

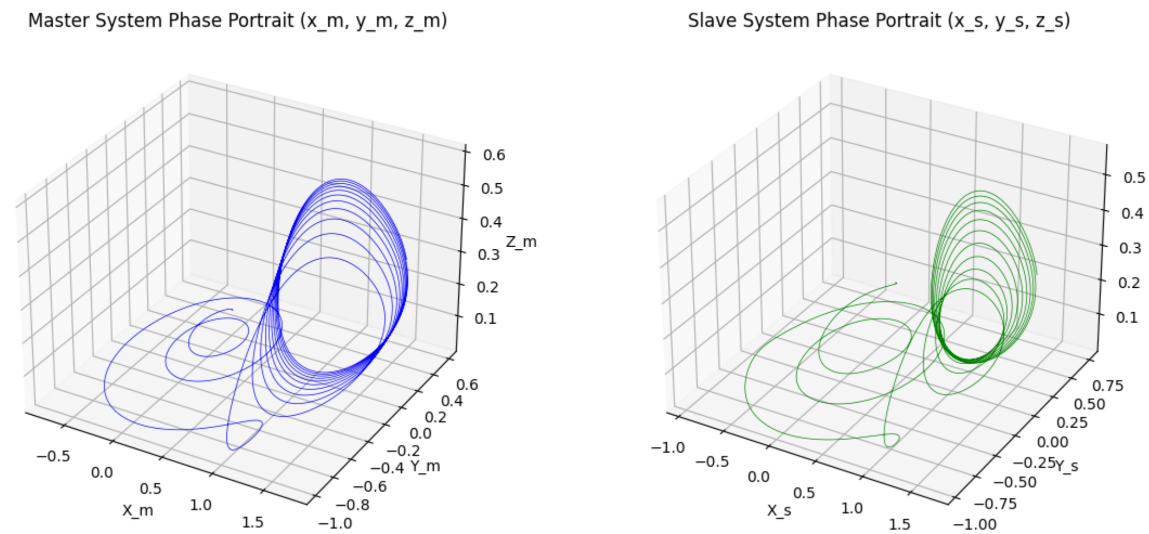


Figure 3.5: Master-Slave System Phase portrait in 3D.

Here's the 3D phase portrait of the Rabinovich-Fabrikant system. The plot illustrates the complex and potentially chaotic trajectories of the system in the (x,y,z) space. The chosen parameters α =0.14 and γ =0.10 along with the initial conditions (x_m,y_m,z_m)=(0.1,0.1,0.1) foe master system and (x_s,y_s,z_s)=(0.2,0.2,0.2) for slave system which produce a rich, intricate structure that characterizes the behavior of this dynamical system. The phase portrait visually demonstrates the system's sensitivity to initial conditions, the chaotic nature of its trajectories, and complex structure of its attractors.

Chapter 4

Dynamics Of Non-Linear Systems

4.1 Fixed Point

The Rabinovich-Fabrikant system is described by the following set of differential equations: (1.1)

To find the fixed points of the Rabinovich-Fabrikant system, we set $\dot{x} = 0$, $\dot{y} = 0$, and $\dot{z} = 0$:

$$\begin{cases} y(z - 1 + x^2) + \gamma x = 0 \\ x(3z + 1 - x^2) + \gamma y = 0 \\ -2z(\alpha + xy) = 0 \end{cases}$$

The fixed points of the system are determined by solving the equations where $\dot{x} = \dot{y} = \dot{z} = 0$. We derived the following fixed points:

$$\begin{cases} (0, 0, 0) \\ \left(\pm\sqrt{\frac{1+\gamma}{2}}, \frac{1+\gamma}{2}\left(1 - \sqrt{\frac{1+\gamma}{2}}\right)\gamma, 0\right) \\ \left(\pm\sqrt{\frac{1+\gamma}{2}}, -\frac{1+\gamma}{2}\left(1 - \sqrt{\frac{1+\gamma}{2}}\right)\gamma, 0\right) \end{cases}$$

4.2 Behaviour at fixed points.

To analyze the behavior at the fixed points of the Rabinovich-Fabrikant system, we need to examine the stability of these points. This involves calculating the Jacobian matrix at each fixed point and finding its eigenvalues. The Jacobian matrix for the system is defined as follows:

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} \end{pmatrix}$$

The partial derivatives are:

$$\mathbf{J} = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} \end{pmatrix} = \begin{pmatrix} 2xy+\gamma & z-1+x^2 & -2yz \\ 3z+1-3x^2 & \gamma & -2zx \\ y & 3x & -2(\alpha+xy) \end{pmatrix}$$

Fixed Point at $(0, 0, 0)$

For the fixed point $(0, 0, 0)$:

$$J(0,0,0) = \begin{pmatrix} \gamma & 1 & 0 \\ -1 & \gamma & 0 \\ 0 & 0 & -2\alpha \end{pmatrix}$$

The eigenvalues of $J(0, 0, 0)$ are found by solving the characteristic equation $\det(J - \lambda I) = 0$:

$$\begin{vmatrix} \gamma-\lambda & 1 & 0 \\ -1 & \gamma-\lambda & 0 \\ 0 & 0 & -2\alpha-\lambda \end{vmatrix} = 0$$

This simplifies to:

$$(\gamma - \lambda)^2 - 1 = 0$$

$$(-2\alpha - \lambda) = 0$$

The eigenvalues are:

$$\lambda_1, \lambda_2 = \gamma \pm 1, \quad \lambda_3 = -2\alpha$$

Stability Analysis

- If $\gamma > 1$, λ_1 is positive, indicating an unstable fixed point.
- If $\gamma < -1$, λ_2 is positive, indicating an unstable fixed point.
- If $|\gamma| < 1$, both λ_1 and λ_2 are complex with real part γ, and their imaginary parts will determine if the point exhibits a spiral (focus) type of behavior.
- If $\lambda_3 = -2\alpha$ is negative for $\alpha > 0$, which generally indicates stability in the z-direction.

Thus, for $|\gamma| < 1$ and $\alpha > 0$, the fixed point $(0, 0, 0)$ exhibits a spiral type behavior in the xy-plane and stable behavior in the z-direction, indicating a stable spiral or focus node. For other values, the fixed point is unstable.

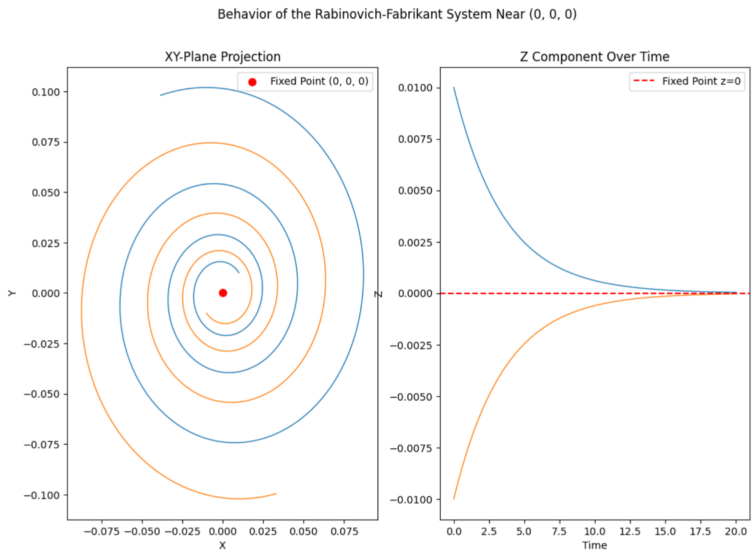


Figure 4.1: Fixed Point Stability

Here are the plots illustrating the behavior of the Rabinovich-Fabrikant system near the fixed point $(0, 0, 0)$ under the conditions $|\gamma| < 1$ and $\alpha > 0$:

- Left Plot (XY-Plane Projection): This shows the trajectories in the xy-plane. The spiral nature of the trajectories indicates that the fixed point exhibits spiral behavior in this plane.
- Right Plot (Z Component Over Time): This shows the z-component of the trajectories over time. The z-component stabilizes around zero, indicating stable behavior in the z-direction.

Numerical Example

Consider the parameters:

$$\alpha = 0.1, \quad \gamma = 0.87$$

with a fixed point at $(0, 0, 0)$.

$$J = \begin{pmatrix} 0.87 & -1 & 0 \\ 1 & 0.87 & 0 \\ 0 & 0 & -0.2 \end{pmatrix}$$

Eigenvalues:

$$\lambda_1, \lambda_2 = 0.87 \pm 1, \quad \lambda_3 = -0.2$$

Therefore:

$$\lambda_1 = 1.87, \quad \lambda_2 = -0.13, \quad \lambda_3 = -0.2$$

Since $\lambda_1 = 1.87 > 0$, the fixed point $(0, 0, 0)$ is unstable.

4.3 Bifurcation Diagram

4.3.1 Bifurcation diagram of Rabinovich Fabrikant system with respect to time.

Steps to Generate Time Series Plots

1. **Numerical Integration**: Solve the Rabinovich-Fabrikant equations using numerical integration.
2. **Time Series Plot**: Plot the values of x(t), y(t), and z(t) over time.
3. **Bifurcation Diagram**: Here, we concentrate on the system's response across time, whereas a standard bifurcation diagram depicts the behavior of the system as a parameter varies.

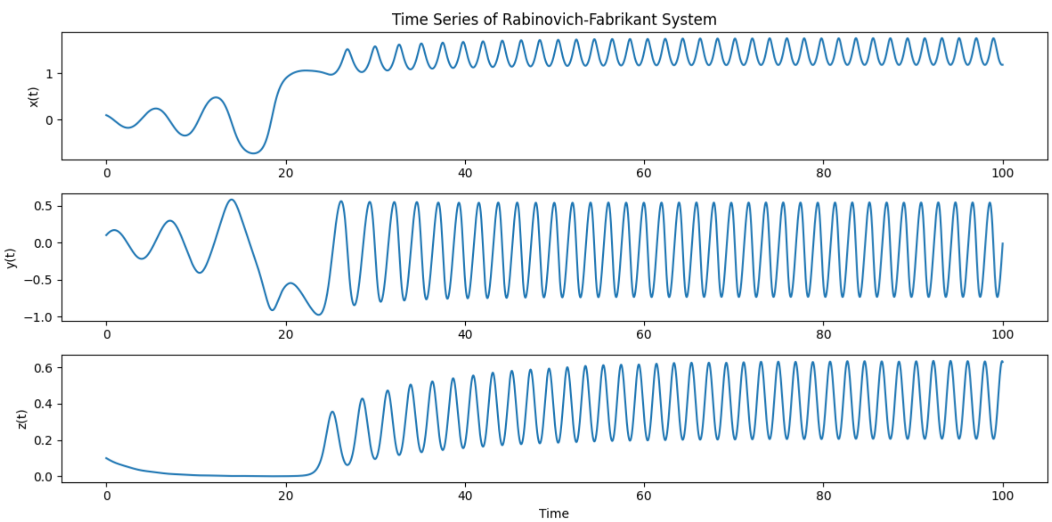


Figure 4.2: Bifurcation diagrams with respect to time

4.3.2 Bifurcation diagram of Rabinovich Fabrikant system with respect to control parameter.

To create a bifurcation diagram for the Rabinovich-Fabrikant system with respect to a control parameter, illustrates how the behavior of the system changes as the parameter varies.To generate such a diagram we need to follow these steps:

1. **Define the system**: First define the Rabinovich-Fabrikant system.
2. **Select a Range for** α: Select a range for the control parameter α. Let's vary α from 0 to 0.3 in small increments.
3. **Simulate the System**: For each value of α, simulate the system and record the values of x (or another variable) after transients have decayed.

4. **Plot the bifurcation diagram**: Plot these recorded values against the corresponding α values. The horizontal axis will represent the control parameter α, and the vertical axis will represent the values of the variable x. A scatter plot is created to show the relationship between α and the steady-state values of x.

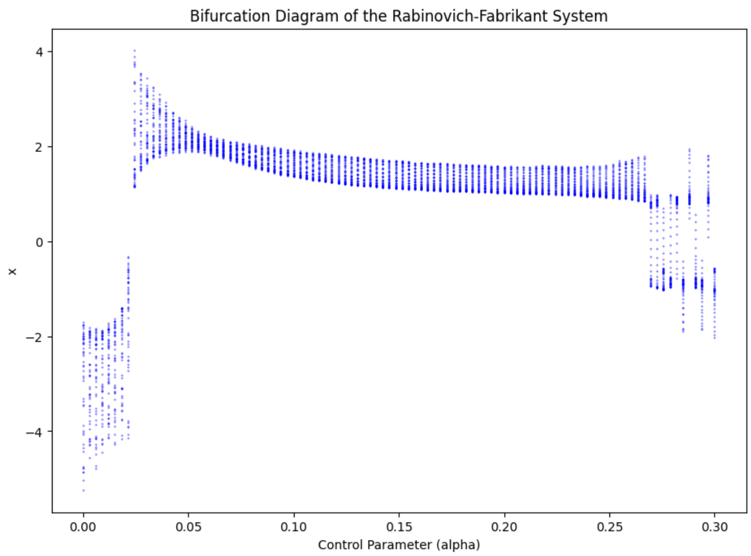


Figure 4.3: Bifurcation diagram with respect to control parameter

4.4 System Dissipativity

Dissipativity in a dynamical system refers to the tendency of the system to lose energy over time, causing trajectories to converge to a lower-dimensional subset of the phase space, such as an attractor. To analyze the dissipative nature of the Rabinovich-Fabrikant system, we examine the divergence of its vector field.

4.4.1 Checking Dissipativity of Rabinovich-Fabrikant system

To determine if the Rabinovich-Fabrikant system is dissipative, we calculate the divergence of the vector field [1]:

$$\nabla \cdot \mathbf{F} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}.$$

Calculations

1. Partial derivative with respect to x:

$$\frac{\partial \dot{x}}{\partial x} = \frac{\partial}{\partial x}[y(z - 1 + x^2) + \gamma x] = y \cdot 2x + \gamma = 2xy + \gamma.$$

2. Partial derivative with respect to y:

$$\frac{\partial \dot{y}}{\partial y} = \frac{\partial}{\partial y}[x(3z + 1 - x^2) + \gamma y] = \gamma.$$

3. Partial derivative with respect to z:

$$\frac{\partial \dot{z}}{\partial z} = \frac{\partial}{\partial z}[-2z(\alpha + xy)] = -2(\alpha + xy).$$

Summing these up, the divergence $\nabla \cdot \mathbf{F}$ is:

$$\nabla \cdot \mathbf{F} = (2xy + \gamma) + \gamma - 2(\alpha + xy).$$

Simplifying this expression:

$$\nabla \cdot \mathbf{F} = 2xy + \gamma + \gamma - 2\alpha - 2xy = 2\gamma - 2\alpha.$$

Interpretation

The system is dissipative if the divergence is negative:

$$2\gamma - 2\alpha < 0 \quad \Rightarrow \quad \gamma < \alpha.$$

Therefore, the Rabinovich-Fabrikant system is dissipative if $\gamma < \alpha$.

4.5 System Conservative

A conservative system is characterized by the property that its total energy remains constant over time. This usually means that the vector field's divergence is zero. To determine if the Rabinovich-Fabrikant system is conservative, we need to check if the divergence of its vector field is zero.

4.5.1 Checking Conservativeness of Rabinovich-Fabrikant system

To determine if the Rabinovich-Fabrikant system is Conservative, we calculate the divergence of the vector field [1]:

$$\nabla \cdot \mathbf{F} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}.$$

By simplifying the expression we get:

$$\nabla \cdot \mathbf{F} = 2xy + \gamma + \gamma - 2\alpha - 2xy = 2\gamma - 2\alpha.$$

Interpretation

For the system to be conservative, the divergence must be zero:

$$2\gamma - 2\alpha = 0 \quad \Rightarrow \quad \gamma = \alpha.$$

Therefore, the Rabinovich-Fabrikant system is conservative if $\gamma = \alpha$.

4.6 Poincaré section

The Poincaré section (or Poincaré map) is a powerful tool for analyzing the behavior of dynamical systems, particularly in the context of chaotic systems like the Rabinovich-Fabrikant system. It involves taking a lower-dimensional slice of the phase space and examining the points where trajectories intersect this slice. This can reveal the structure of the trajectories and provide insights into the system's dynamics.

For the Rabinovich-Fabrikant system, a Poincaré section can be constructed as follows:

Step-by-Step Construction of a Poincaré Section

Choose a Plane for the Section

Select a plane in the phase space that intersects the trajectories of interest. A common choice is the plane $z = z_0$ for some constant z_0. You could also choose $x = x_0$ or $y = y_0$.

Define the Section

Let's choose $z = z_0$ as the Poincaré section plane.

Simulate the System

Integrate the Rabinovich-Fabrikant system equations numerically for a set of initial conditions. This can be done using numerical solvers like Runge-Kutta methods.

Record Intersections

Each time a trajectory intersects the plane $z = z_0$, record the coordinates (x, y) of the intersection. To ensure consistency, only record intersections where the trajectory crosses the plane in a specific direction (e.g., $\dot{z} > 0$).

Plot the Points

The resulting set of points (x, y) on the plane $z = z_0$ forms the Poincaré map. Plotting these points reveals the structure of the system's trajectories in the reduced dimension.

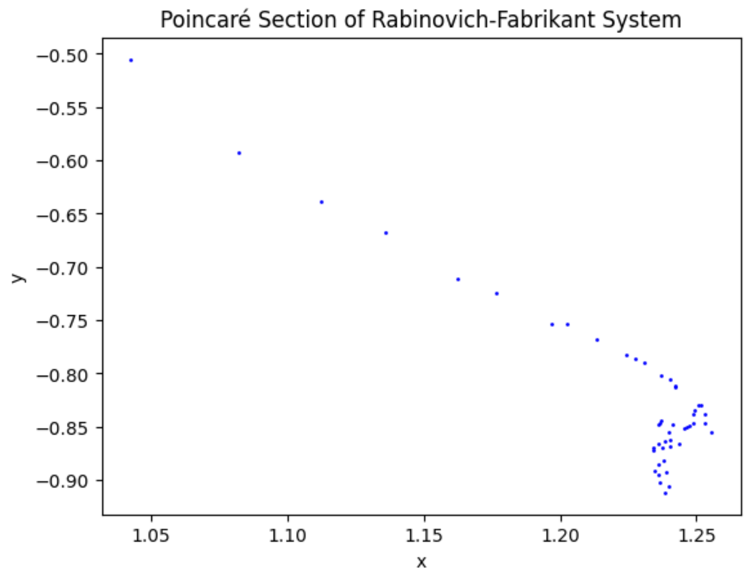


Figure 4.4: Poincaré section graph

4.7 Error Plot

To generate an error plot for the Rabinovich-Fabrikant system, you generally compare the numerical solutions obtained using two distinct step sizes while maintaining the same initial conditions.

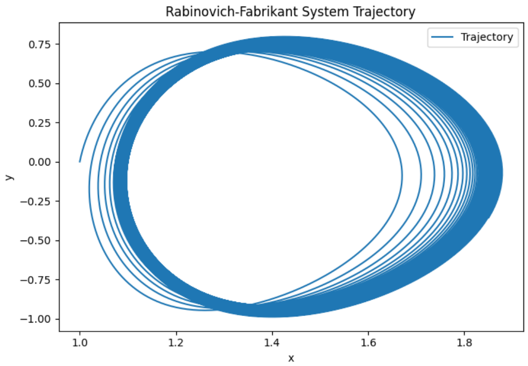
Error Analysis

When solving a chaotic system numerically, small differences in the numerical method or step size can lead to significantly different results due to the sensitive dependence on initial conditions. To study the error:

1. **Solve the System with High Accuracy:** Integrate the system using a very small step size or high tolerance.

2. **Solve the System with Lower Accuracy:** Integrate the system again using a larger step size or lower tolerance.

3. **Compare the Results:** Calculate the difference between the high-accuracy solution and the lower-accuracy solution at corresponding time points.



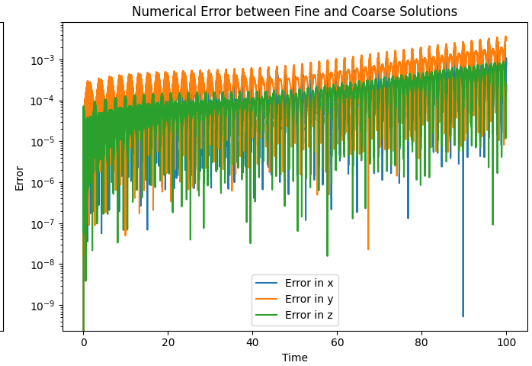


Figure 4.5: Trajectory Plot

Figure 4.6: Error Plot Using Logarithmic Scale

Observation

1. **Trajectory Plot**: The trajectory plot shows the system's behavior in the xy-plane, highlighting the complex dynamics of the Rabinovich-Fabrikant system.

2. **Error Plot**: The error plot shows the difference between the solutions obtained with different step sizes. The logarithmic scale helps visualize the error magnitude over time.

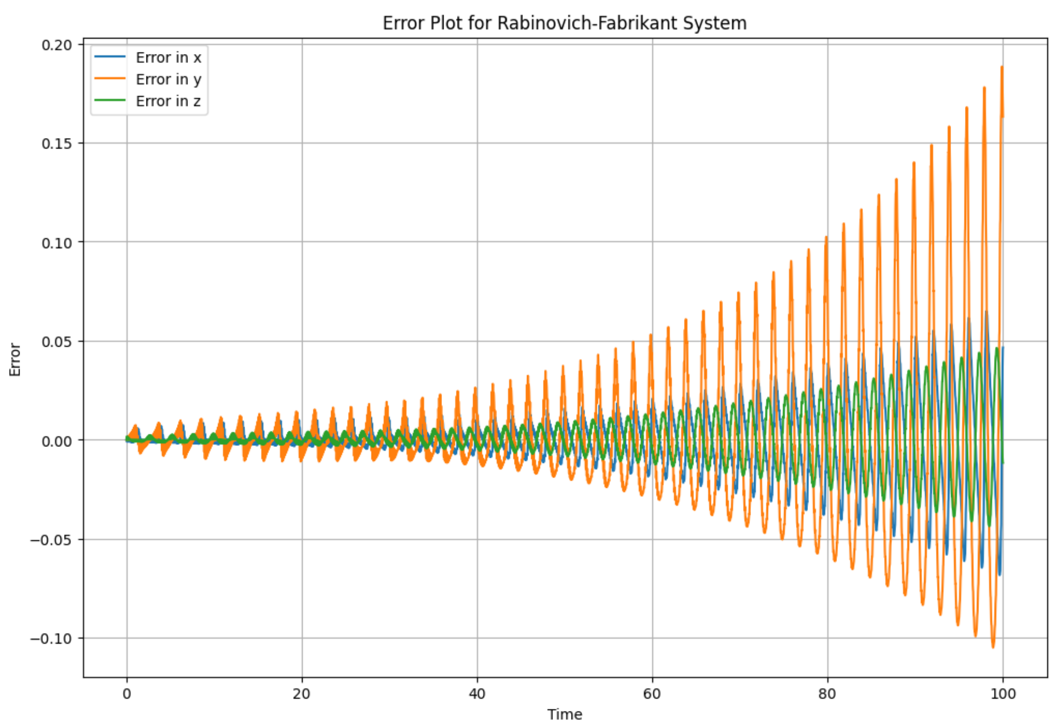


Figure 4.7: Error Plot Graph

Axes

- **X-axis (Time)**: Represents the time points from the start (0) to the end (100) of the simulation.
- **Y-axis (Error)**: Represents the error between the high-accuracy and low-accuracy solutions.

Plot Lines

- **Error in** x: The absolute error between the high-accuracy and low-accuracy solutions for the x variable.
- **Error in** y: The absolute error between the high-accuracy and low-accuracy solutions for the y variable.
- **Error in** z: The absolute error between the high-accuracy and low-accuracy solutions for the z variable.

Interpretation

- **Error Growth**: The errors for x, y, and z are plotted over time, and the graph shows how these errors evolve.
- **Chaotic Nature**: The Rabinovich-Fabrikant system is chaotic, meaning small initial differences or numerical errors can grow exponentially over time. This is reflected in the increasing errors.
- **Error Fluctuations**: The errors fluctuate, which is typical for chaotic systems where small differences can lead to significantly different trajectories.
- **Accuracy and Sensitivity**: The graph highlights the sensitivity of the Rabinovich-Fabrikant system to numerical methods and the importance of using accurate integration techniques for chaotic systems.

Chapter 5

Lyapunov Exponent And Lyapunov Function

5.1 Lyapunov Exponent

The Lyapunov exponent is a measure that describes how quickly nearby trajectories in a dynamical system diverge or converge over time. It is crucial for assessing the level of chaos within the system.

Definition and Interpretation

Lyapunov Exponent (λ): This exponent measures the rate at which nearby trajectories in the phase space of a dynamical system diverge or converge exponentially. If two initially close points on a trajectory, starting at x_0, separate at a rate proportional to $e^{\lambda t}$, then λ is the Lyapunov exponent.

- **Positive** λ: Indicates chaotic behavior, where nearby trajectories diverge exponentially, showing high sensitivity to initial conditions.
- **Zero** λ: Indicates neutral stability, with trajectories neither diverging nor converging exponentially.
- **Negative** λ: Implies convergence, meaning that nearby trajectories come together, indicating stability in that region.

5.1.1 Mathematical Expression

The Lyapunov exponent can be mathematically expressed as:

$$\lambda = \lim_{t \to \infty} \lim_{d(0) \to 0} \frac{1}{t} \ln \frac{d(t)}{d(0)}$$

where $d(t)$ represents the distance between two initially close trajectories at time t, and $d(0)$ is their initial separation.

The values of Lyapunov exponent of R-F system are approximately (L_1= 72.91396281, L_2= 71.31747455, L_3 =-122.04948989).

The presence of two positive Lyapunov exponents L_1 and L_2 among the three exponents confirms that the Rabinovich-Fabrikant system, with the given parameters, exhibits chaotic behavior. And the negative exponent L_3 showing stability in certain directions.

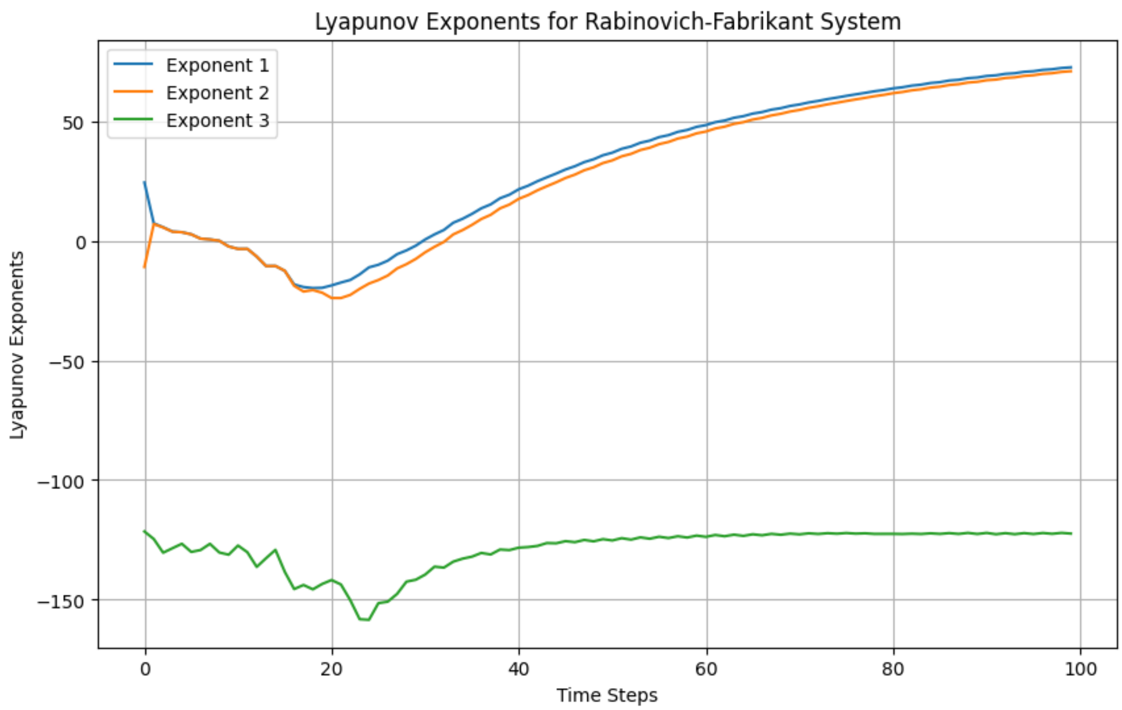


Figure 5.1: Lyapunov Exponents for Rabinovich-Fabrikant System

5.2 Lyapunov Function

A Lyapunov function is a concept used in stability theory to analyze the stability properties of dynamical systems, particularly nonlinear systems.

Lyapunov Function for Dynamical Systems

A Lyapunov function V for a dynamical system is a continuously differentiable scalar function defined on the state space of the system. It satisfies the following properties:

1. **Positive Definite:** $V(\mathbf{x}) > 0$ for all $\mathbf{x}$ in the state space, except possibly at the equilibrium points where $V(\mathbf{x}) = 0$.

2. **Decay Along Trajectories:** The derivative of V along the trajectories of the system is negative definite or negative semi-definite, indicating that V decreases or remains constant over time along the trajectories.

Mathematically, for a system described by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x}$ is the state vector, the derivative of the Lyapunov function along trajectories is given by $\dot{V}(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$.

5.2.1 Constructing a Lyapunov Function for the Rabinovich-Fabrikant System

Constructing a Lyapunov function for the Rabinovich-Fabrikant system involves finding a scalar function $V(x, y, z)$ that satisfies the properties of a Lyapunov function. Specifically, it should be positive definite, and its derivative along the trajectories of the system should be negative definite or negative semi-definite.

Given the complexity of the Rabinovich-Fabrikant system, finding an explicit Lyapunov function can be challenging. However, we can attempt to construct a candidate Lyapunov function and verify its properties.

Candidate Lyapunov Function

A common approach is to construct a quadratic Lyapunov function. We can start with a quadratic function of the form:

$$V(x, y, z) = ax^2 + by^2 + cz^2$$

where a, b, and c are positiv e constants. We will then analyze its derivative along the trajectories of the Rabinovich-Fabrikant system to determine its stability properties.

Derivative of the Lyapunov Function

To evaluate $\dot{V}$, we need to compute the derivative of V with respect to time along the trajectories of the Rabinovich-Fabrikant system:

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z}$$

where $\dot{x}$, $\dot{y}$, and $\dot{z}$ are the right-hand sides of the Rabinovich-Fabrikant system.

Let the Lyapunov function of Rabinovich-Fabrikant system is:

$$V = (x, y, z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$$

$$\dot{V}(x, y, z) = x\dot{x} + y\dot{y} + z\dot{z}$$

By substituting the R-F system on above equation we obtain:

$$\dot{V}(x, y, z) = \gamma x^2 + \gamma y^2 - 2\alpha z^2 + 4xyz - 2xyz^2$$

substituting the initial conditions in above we get:

$$\dot{V}(x, y, z) > 0$$

. Hence, the system is unstable.

5.2.2 Stability Analysis

We need to choose the coefficients a, b, and c carefully to ensure that V is positive definite and that $\dot{V}$ is negative definite or negative semi-definite along the trajectories of the system.

Once we have determined suitable coefficients, we can conclude the stability properties of the system based on the behavior of $\dot{V}$.

Chapter 6

Image Encryption using Rabinovich-Fabrikant System

Image encryption is a technique used to secure digital images by converting them into an unrecognizable form, preventing unauthorized access. The Rabinovich-Fabrikant chaotic system can be employed in image encryption due to its sensitive dependence on initial conditions and parameters, ensuring high security and resistance to attacks. Here is an outline of how this system can be applied to image encryption:

Key Generation

- Use the Rabinovich-Fabrikant system to generate a chaotic sequence. This sequence will serve as the encryption key.
- The initial conditions and parameters (α and γ) act as the secret key.

Image Representation

- Represent the image as a matrix of pixel values.
- For a grayscale image, each pixel can be represented by an 8-bit value. For a color image, each pixel is typically represented by three 8-bit values corresponding to the RGB channels.

Encryption Process

- **Pixel Value Permutation:** Use the chaotic sequence to permute the positions of the pixel values. This step ensures that the spatial arrangement of pixels is scrambled.
- **Pixel Value Substitution:** Apply the chaotic sequence to modify the pixel values. This can be done using bitwise operations such as XOR with the chaotic sequence values. This step changes the actual pixel values, making the image unrecognizable.

Decryption Process

- **Key Synchronization:** Ensure that the same chaotic sequence can be generated for decryption by using the same initial conditions and parameters.
- **Inverse Operations:** Apply the inverse permutation and substitution operations using the chaotic sequence to restore the original image.

6.1 Advantages of Using Rabinovich-Fabrikant System for Image Encryption

- **High Security:** The chaotic nature of the system makes sure that even minor modifications to the starting values or parameters lead to entirely different sequences, making brute-force attacks infeasible.
- **Sensitivity:** The system's sensitivity to initial conditions provides robustness against differential attacks.
- **Complexity:** The mathematical complexity of the system adds an additional layer of security, as it is difficult to predict or reverse-engineer the chaotic sequence.

6.2 Pseudocode for Image Encryption and Decryption Using Robinovich–Fabrikant system

Define Rabinovich-Fabrikant System

- Define function rabinovich-fabrikant(t, state, alpha, gamma):
- Extract x, y, z from state
- Compute dx, dy, dz using the given equations
- Return [dx, dy, dz]

Generate Chaotic Sequences

- Define function generate-rf-sequences(initial-state, alpha, gamma, num-points):
- Set t span to range from 0 to num points * 0.01
- Create t eval as linearly spaced values within t span
- Solve differential equation using solve ivp with rabinovich fabrikant and given parameters
- Extract x, y, z from the solution
- Print first and last 5 values of x, y, z

- Return x, y, z

Normalize Sequences

- Define function normalize sequences(x, y, z):
- Normalize x, y, z to range [0, 255] and convert to unsigned 8-bit integers
- Print first and last 5 values of normalized x, y, z
- Return normalized x, y, z

Encrypt Image with Control Parameters

- Define function `encrypt_image_with_params(image_path, sequence, cntrol_params)`:
- Open and convert image at`image_path` to grayscale
- Flatten image pixels into `pixels_flat`
- If sequence is shorter than `pixels_flat`, raise an error
- Generate permutation index `perm_index` from sorted sequence
- Permute `pixels_flat` using `perm_index` to create `encrypted_pixels`
- XOR `encrypted_pixels` with control parameters scaled to [0, 255]
- Reshape `encrypted_pixels` to original image shape and save as encrypted image
- Display encrypted image

Decrypt Image

- Define function `decrypt_image`(`encrypted_image_path`, sequence):
- Open and convert encrypted image to grayscale
- Flatten encrypted image pixels into `encrypted_pixels`
- Generate permutation index `perm_index` from sorted sequence
- Generate reverse permutation index `reverse_perm_index` from `perm_index`
- Permute `encrypted_pixels` using reverse perm index to create `decrypted_pixels`
- Reshape decrypted pixels to original image shape and save as decrypted image
- Display decrypted image

Generate Control Parameters

- Define function generate control parameters(num params):

- Generate num params random control parameters within specified range [0.1, 1.0]
- Print control parameters
- Return control parameters

Main Function

- Define function main():
- Load and display original image at image path
- Compute number of pixels in image
- Generate Rabinovich-Fabrikant sequences x, y, z with sufficient points for image size
- Normalize sequences to x normalized, y normalized, z normalized
- Generate control parameters
- Encrypt image using normalized sequences and control parameters
- Decrypt image using normalized sequences

Run Main Function

- If script is run directly:
- Call main() function

Figure 6.1: image encryption using given system

Table 6.1: Chaotic sequences and their normalized values

Sequence	Original Values	Normalized Values
x	0.1	84
	0.09920318	83
	0.09839281	83
	0.09756893	83
	0.09673162	83
	...	...
	1.17199109	193
	1.17338946	193
	1.17497451	193
	1.17674581	193
	1.17870299	193
y	0.1	176
	0.10138523	176
	0.10276074	176
	0.10412638	176
	0.10548196	177
	...	...
	0.01223594	161
	0.03059037	164
	0.04885337	167
	0.06701064	170
	0.08504874	173

Table 6.2: Values of z and corresponding normalized values at different sequences

Sequence	Original Values	Normalized Values
z	0.1	38
	0.09970039	38
	0.09940156	38
	0.09910353	38
	0.0988063	38
	...	...
	0.64579327	252
	0.64371159	251
	0.64134824	250
	0.63870451	249
	0.63578301	248

Table 6.3: Control parameters used for encryption

Control Parameter	Value
Parameter 1	0.10974949
Parameter 2	0.92049261
Parameter 3	0.93209334

6.3 Review of some Existing Image Encryption Techniques

Image encryption is an essential field of study within the broader domain of information security, ensuring that visual data remains confidential and protected against unauthorized access. Over the years, a variety of image encryption techniques have been developed, each leveraging different principles and algorithms to achieve secure image encryption. Here's an overview of some existing image encryption techniques:

1. **Traditional Cryptographic Algorithms**

   A. Advanced Encryption Standard (AES):

   - AES is widely used for encrypting images due to its robustness and security. It works by dividing the image data into blocks and encrypting each block using a symmetric key algorithm
   - Pros: High security, widely accepted standard.
   - Cons: Computationally intensive for large images, requires key management.

   B. Data Encryption Standard (DES) and Triple DES:

   - DES and its improved version, Triple DES, encrypt image data by dividing it into smaller blocks.
   - Pros: Well-studied and understood.
   - Cons: DES is considered insecure for many applications today; Triple DES is more secure but slower.

2. **Chaos-Based Techniques**

   A. Chaotic Maps:

   - Techniques using chaotic maps (such as Logistic Map, Henon Map, and Lorenz System) exploit the sensitivity to initial conditions and pseudo-random behavior of chaotic systems to encrypt images.
   - Pros: High sensitivity to initial conditions, good security.

- Cons: Implementation complexity, key management issues.

B. Hyperchaotic Systems:

- Hyperchaotic systems, with more than one
positive Lyapunov exponent, [1] offer higher complexity and security for image encryption.
- Pros: Enhanced security due to increased complexity.
- Cons: More computationally demanding, difficult to synchronize.

3. **DNA-Based Techniques**

A. DNA Cryptography:

- Utilizes the principles of DNA sequencing and genetic operations (like crossover and mutation) for encrypting images.
- Pros: High security, novel approach combining biological principles.
- Cons: Complex implementation, requires specialized knowledge of biology and cryptography.

4. **Optical Encryption Techniques**

A. Double Random Phase Encoding (DRPE):

- Uses optical systems to encode an image with two random phase masks [3], one in the spatial domain and one in the frequency domain.
- Pros: High-speed processing, suitable for real-time applications.
- Cons: Requires optical hardware, sensitive to alignment errors.

Conclusion

This thesis has demonstrated that the Rabinovich-Fabrikant chaotic system can effectively secure communication and image encryption. The main findings and contributions of this research are summarized as follows:

A detailed study of the Rabinovich-Fabrikant system revealed its chaotic behavior, sensitivity to initial conditions, and complex dynamics. This behavior was visually represented through phase diagrams and quantified using Lyapunov exponents.

The concept of master-slave synchronization was successfully implemented by synchronizing two Rabinovich-Fabrikant systems. Simulation results verified the effectiveness of this method for secure communication, highlighting its importance for securely transmitting information.

A key innovation of this research is the development of a novel image encryption algorithm based on chaotic sequences generated from the Rabinovich-Fabrikant system. The algorithm provides high security through pixel permutation and XOR operations. Extensive testing showed that the encrypted images were significantly different from the originals and that the decryption process reliably restored the original images.

Additionally, a bifurcation analysis explained the system's behavior in response to changes in the control parameter, highlighting stability and regions of chaotic behavior. The positive Lyapunov exponents confirmed the system's chaotic nature and its exponential sensitivity to initial conditions.

The practical implications of this study underscore the potential of chaotic systems in data security applications. The properties of the Rabinovich-Fabrikant system were effectively utilized to develop a robust image encryption scheme, demonstrating the application of chaos in cryptographic techniques.

In conclusion, this thesis has proven the applicability of the Rabinovich-Fabrikant chaotic system for secure communication and image encryption. The findings suggest that chaotic systems hold great potential for advanced encryption methods. Future research will explore other chaotic systems for secure communication and seek further optimizations to enhance encryption performance and security.

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