COMPOSITION OPERATORS ON FUNCTION SPACES

A Project Dissertation submitted in partial fulfilment of the requirementsfor the degree of

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CANDIDATE'S DECLARATION

We, Pooja and Poonam Saini, who are currently pursuing a Master of Sciencein Applied Mathematics with Roll Number 2K22/MSCMAT/29 and 2K22/MSCMAT/30 respectively, hereby declare that the project dissertation submitted by us to the Departmentof Applied Mathematics at Delhi Technological University to fulfil the requirement for theaward of the degree of Master of Science in Applied Mathematics, is original and has not been copied from any source. Furthermore, this work has not been previously used as the basis for conferring a degree, diploma, associate's degree, fellowship, or any other similartitle or honour.

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CERTIFICATE

We hereby bear witness that the Project Dissertation submitted by Pooja, Roll Number 2K22/MSCMAT/29 and Poonam Saini, Roll number 2K22/MSCMAT/30 of the Department of Applied Mathematics, Delhi Technological University, Delhi in partial fulfilment of the requirement for the award of the degree of Master of Applied Mathematics, is a record of the project work completed by the student under my supervision. To the best of my knowledge, neither a portion nor the entirety of this work has ever been submitted to this university or any other institution for a degree or diploma.

Place: Delhi

Date: 06 June 2024

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Abstract

This thesis investigates the complex relationship between composition operators and function spaces, attempting to understand their behaviour, properties, and practical applications. Composition operators are useful tools in a variety of mathematical disciplines, including functional analysis and operator theory. The paper looks at the theoretical foundations of composition operators, examining their impact on various function space structures such as Banach and Hilbert spaces. The emphasis is on understanding how composition operators alter the properties and characteristics of these function spaces. Furthermore, this study investigates the practical implications of composition operators in signal processing, control theory, and approximation theory. This thesis gives useful insights into the uses of composition operators in several scientific and technical disciplines by investigating their effectiveness in tackling real-world situations. This thesis advances our understanding of composition operators in function spaces through rigorous analysis and investigation, setting the path for future research and applications in a variety of mathematical and scientific disciplines.

Acknowledgement

We are incredibly lucky to have had this support and guidance from many people through-out the course of my project because it was essential to its accomplishment and successfulcompletion.

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We would like to end by thanking God Almighty for seeing to it that no unforeseen problems occurred while we were working.

Contents

CHAPTER 1

INTRODUCTION

1.1Definitions and Historical Background

Let F_X be a vector space over the [1]field K (where K = R or C) for any x ∈ X, and let X be a non-empty set[1]. Under linear operations defined pointwise, the Cartesian product of $\prod_{x \in X} F_X$ the family (F_X: x ∈ X) is a vector space. Thefamily (F_X: x ∈ X) is known as a vector-fibration over X, and each element of $\prod_{x \in X} F_X$ is known as a cross-section over X. A topological vector space of the cross-sections[1] over X is indicated by L(X). Let T : X → X be a mapping such that, whenever f ∈ L(X), f o T ∈ $\prod_{x \in X} F_X$. The composition transformation caused by T refers[1] to the linear transformation f → f oT from L(X) to T_X. C_T [1]denotes this metamorphosis. Suppose π is a mapping defined on X such that[1] If f ∈ L(X), then f → π. f o T is a linear transformation from L(X) to $\pi \prod_{x \in X} F_X$. The transformation caused by π and T is known as the weighted composition v transformation[1] (W_π,T).

Three key circumstances arise while studying these operators:

(i) The underlying space X is a measure space; the inducing mappings are measurable transformations.
(ii) The underlying [1]space X is a region in C or C^n and the inducing maps are holomorphic functions.
(iii) The underlying space X is a topological space with v continuous functions.

L(X) is assumed to be a topological vector space of [1]measurable functions in the first scenario, such as L^P –spaces[1]; in the second scenario, L(X) is assumed to be a topological vector space of analytic functions, such as a Dirichlet[1], Hardy, or Bergman space.
space: in the third instance, a topological vector v space of continuous functions is assumed to be L(X)[1].

These areas can be defined into three v broad categories:

(i) L^P – spaces.
(ii) Functional Banach spaces of functions.
(iii) Locally convex function spaces.

1.2 L^P -Spaces

Assume (X, Y, m) is a measure space and p is a real number such that $1 \leq p < \infty$. Let ℓ^P(m) be the set of all complex-valued measurable functions on X such that If $|F|^P$ is m-integrable. ℓ^P(m) is a complex linear space that supports pointwise addition and scalar multiplication[1]. If N^P(m) represents the set of all null functions on X, it is a subspace[1] of ℓ^P(m). Let L^P(m) be the quotient space ℓ^P (m)/ N^P(m).The element in is L^P a coset of the type f + N^P(m), which belongs to l^P(m). The coset f + N^P(m) is represented as [f]. [1]Thus, two ℓ^P(m) functions, g and h, belong to the same coset if and only if g and h are virtually always the same. [1]On L^P(m) we define a norm as:

$$\| f \|_P = (\int | f |^P \ dm)^{1/p}$$

The Minkowski inequality indicates that L^P(m) is a normed linear space[1] with the specified norm[1]. Under this norm, L^P(m) is complete[1]. Thus, L^P(m) is a Banach space . L^P(m) and L^P(m) is conjugate space[1] , with conjugate indices p and q. For p = 2, L^P(m) is a [1]Hilbert space with the inner product defined as:

$$\langle [f], [g] \rangle = \int f \bar{g} \ dm$$

If X contains a non-empty subset of measure zero, [1]the members of L^P(m) are not functions on X, but rather equivalence classes of functions. Two components of ℓ^P (m) are considered equal if they agree practically everywhere. Under this [1]We view L^P(m) as a Banach space of functions. A complex valued measurable function f on X is considere essentially bounded if the set (x: x ∈ X and { | f(x) | > M} has a measure[1] greater than M. The value is zero[1]. The essential supremum of f is the lowest such M, shown as $\| f \|_\infty$. Let ℓ^∞((m) be the set of basically bounded functions on X. ℓ^∞(m) is a linear space. L^∞(m) represents the quotient space ℓ^∞(m)/ N^∞, where N^∞ is the subspace[1] of null functions. Using the basic supremum norm, L^∞(m) becomes a Banach space. The sign ℓ^∞ represents the Banach space of all bounded sequences of complex numbers[2].

1.3 Functional Banach Space for Functions

Assume X is a non-empty set, and H(X) is a Banach space of complex-valued functions with pointwise addition and scalar multiplication. Let x ∈ X. Let δ_X be the mapping from H(X) to f(x). [11]Then it's clear that δ_X is a linear functional on

H(X); it is known as the evaluation functional induced by x. H(X) is a functional Banach space if each evaluation function δ_x is continuous, i.e., [3]if $\delta_X \in H^*$ (X) for every x ∈ X, where H^*(X) is the dual space of H(X). If H(X) is a functional Hilbert space, the Riesz-representation theorem allows us to discover[3] a unique F_X ∈ H(X), such that

$$^{[11]}g(x) = \delta_X(g) = (g, f_X).$$

For each g ∈ H(X). The function f_x is known as the kernel function[3] of X induced by x. Consider K(X) ={ F_X: x ∈ X}. [1]Then K(X) is a subset of H(X). The complex function K defined on X × X is as follows:

$$K(x, y) = \langle f_x, f_y \rangle$$

is represent as the replicating kernel of H(X).

Examples :

The following are some common instances of functional Banach spaces[11].

(1.3.1) ℓ^P – Spaces.

Let X be any countable set, and m be the counting measure specified on its power set. L^P(m)[11], often known as ℓ^P(X), is a functional Banach space for $1 \leq p \geq \infty$. The continuity of the evaluation functionals arises from the fact that

$$^{[3]}|\ (f)\ | = |\ f(x)\ | \leq \|\ f\ \|$$

Both the unitary space C^n and the classical sequence space[3] ℓ^P are functional Banach spaces. For p = 2, ℓ^P(X) is a functional Hilbert space. The reproducing kernel of ℓ^2(X) is given by

$$K(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

L^2 replicating kernel corresponds to the diagonal[3] of ℕ×ℕ.

(1.3.2) Space of bounded functions

Let H_b (X) showing the vector space of all complex-valued bounded functions on X. [1]For f ∈ H_b(X), define ‖ f ‖ as

$$\|\ f\ \| = \sup\{\ |\ f(x)\ |\ : x \in X$$

Using this norm, H_b(x) is a functional Banach space[11] since

$$| f(x) | \leq \| f \|$$

1.4 Locally Convex Function Spaces

Assume X is a topological space, E is a topological vector space, and A(X, E) is the vector space containing all linear functions from X to E defined pointwise. Then by a locally convex space of functions on X, we mean a [3]seminormed linear space is formed by combining the subspace F(X, E) of A(X, E) [1]with a family of seminorms. If E = K, we write F(X) as F(X, K). Not all [3]locally convex spaces of functions are Banach spaces[11] or normed linear spaces. For instance, [1]the space J(X, E) of continuous E-valued[1] functions with compect-open topology, where X is non-compact and E is a locally convex space, is locally convex but not normable.

We define the weighted spaces of continuous E-valued functions as follows:

$V_0 J(X,E)$ ={f ∈ $J(X,E)$: v f vanishes[3] at infinity on X for each v ∈ V}.
$JV_P(X,E)$ ={f ∈ $J(X,E)$: vf(X) is precompact[1] in E for all v ∈V}, and
$V_b J(X,E)$ ={f ∈ $J(X,E)$: vf(X) is bounded in E for all v ∈V}. Obviously, $JV_0(X,E)$ and $JV_P(X,E)$ and $JV_b(X,E)$are vector spaces and
$JV_p(X,E)$ $JV_b(X,E)$ while the upper semicontinuity of the weights yields that
$JV_0(X,E)$4 $JV_P(X,E)$. [1]Let v ∈ V, q ∈ cs(E) and f ∈ J(X,E). If we define

$$\| f \|_{v,q} = \sup \{ q(v(x)\, q(f(x)) : x \in X\}$$

The seminorm $\|.\|_{v,q}$ can be applied to JV_b(X, E), JV_p(X, E), or JVo(X,E). The seminorm family { $\|.\|_{v,q}$: v ∈ V, q ∈ cs(E) }[3] defines a Hausdorff locally convex topology on these spaces. This topology will be designated.

The vector spaces $JV_0(X,E)$ and $JV_b(X,E)$ with wv are referred[3] to as weighted locally convex spaces for vector-valued continuous functions.

Example 1.4.1. Let X be a locally compact [3]Hausdorff space, and let E be a locally convex space.
Then,
(i) JV_0^2 (X, E) = JV_P'(X, E) = JV_P'(X, E) = (J(X, E), k), where k denotes the compact -openvtopology :
(ii) JV_0^2 (X, E) = JV_P^2(X, E) = JV_P^2(X, E) = (J(X, E), k);

(iii) JV_0^3 (X, E) $= J_0(X, E), u)$,
JV_0^3 (X, E) $= J_P(X, E), u)$,and[3]
JV_0^3 (X, E) $= J_b(X, E), u)$,where u denotes the topology of uniform convergence on X, and.
(iv) JV_0^4 (X, E) $= JV_P^4(X, E) = JV_b^4(X, E) = J_b(X, E), \beta)$, [3]where p denotes the strict topology.

To introduce weighted spaces of cross-sections, we need the following definitions. Let {F, : x ∈ X} denote a vector fibration over X. Then add a weight to X.

We define a function w on X that is a seminorm on F forveach x in X. [3]We'll use the notation w to represent the seminorm w(x) for each x in X. We define w ≤ w' as $w_x \le w_x'$, for any x € X. Let W represent[1] a set of weights on X. W is considered directed upward if for any pair w, w' € W and ⅄> 0, there exists w" ≤ W [2]such that h w ≤ w". If f is a [3]cross-section across X and w is a weight on X, we shall refer to w. If, the positive-valued function on X that transforms x into $w_x[f(x)]$[2]. The weighted spaces of cross-sections over X with respect to the system of weights W are defined as: LWo(X) = {f ∈ L(X): w[f]} is upper[1] semicontinuous and disappears at infinity on X for each w ∈ Wand $LW_b(X)$={f ∈ L(X): w[f] is[3] a bounded function on X for each w € W}[3]. It is obvious that LWo(X) and $LW_b(X)$ are vector spaces, and LWo(X) subset $LW_b(x)$. Now, consider w ∈ W[3] and f ∈ L(X). If we define

$$|| f || = \sup\{w_x[f(x)] : x \in X\}$$ [3]

The seminorm $|| . ||_w$ can be applied to either $LW_b(X)$ or LWo(X)[3], and the family { || . || : w ∈ W} defines a Hausdorff locally convex topology on both spaces. We will refer to this topology as τW, and the vector spaces[11] as LWo(X). and $LW_b(X)$ with ἰw is referred to as the weighted locally convex spaces of cross sections. The form has closed, completely convex neighborhwds at its origin.

$$B_w = \{ f \in LW_b(X) : || f ||_w \le 1 \}$$ [3]

CHAPTER 2

COMPOSITION OPERATORS ON L^P -SPACES

2.1 Definitions, Characterizations And Example.

Let (X, Y, m) represent a measure space. A mapping T from X to X is considered measurable if $T^{-1}(s) \in$ Y for every S in Y. A measurable transformation T is non-singular if m($T^{-1}(s$))= 0 for any m(S) = 0. If T is non-singular, the measure mT^{-1}, defined as m$T^{-1}(s)$ = m$T^{-1}(s)$ for all S ∈ Y, is absolutely continuous on Y in relation to m. If m is an σ -finite measure, the Radon-Nikodym theorem states that there is a non-negative function f_T in L' (m) that

$$mT^{-1}(S) = \int_s f_T \, dm$$

Every S ∈ Y. The function f_T is known as the Radon-Nikodym derivative of m with respect to mT^{-1}. A non-singular transformation T from X to itself results in a linear transformation C_T on L^P(m) into the linear space of all measurable functions on X, defined as

$$C_T f = f \circ T$$

For every f ∈ L^P(m). If C_T is continuous from L^P(m) to itself, it is considered a composition operator on L^P(m) induced by T.

Theorem 2.1.1. Assume (X, Y, m) is an σ-finite measure space and T : X → X is a measurable transformation. If b > 0, T generates a composition operator C_T on L^P(m).

$$mT^{-1}(S) = b\, m(S) \qquad \text{for all } S \in Y.$$

Proof. Assume C_T is the composition operator generated by T. If S ∈ Y and m (S) < ∞, then $\chi_s \in L^P$(m).

$$mT^{-1}(S) = \| C_T \chi_s \|^P \leq \| C_T \|^P \| \chi_s \|^P = \| C_T \|^P m(S)$$

Let b = $\| C_T \|^P$. Then

$$mT^{-1}(S) \leq b\, m(S)$$

If $m(S) = \infty$, then inequality will be trivial.
Assume the condition is true. If $mT^{-1} << m$, the Radon-Nikodym derivative (f_T) of mT^{-1} with regard to m exists.

$$f_T \leq b \quad \text{a.e}$$

Let $f \in L^P(m)$**.** Then

$$\| C_T f \|^P = \int | foT |^P dm = \int | f |^P dm\, T^{-1} = \int | f |^P f_T\, dm \leq b \| f \|^P$$

This demonstrates that C_T is a bounded operator in $L^P(m)$. This concludes the proof of the theorem.

Example 2.1.1: let X be a locally compact abelian group and m represent the Haar measure on the σ-algebra of Bore1 sets. Let you $y \in X$. Then specify $T_y: X \rightarrow X$ as

$$T_y(x) = yx$$

For each $x \in X$. C_{Ty} is a composition operator on $L^P(m)$ for $1 \leq p \leq \infty$. Assume X is the real line with standard topology and addition as the group operation. Then $T_y(x) = x + y$. Koopman's work on classical mechanics introduced the composition operators C, sometimes known as translation operators.

Theorem 2.1.2. Let (X, Y, m) be a standard Bore1 space, and A be an operator on $L^P(m)$.
Then A is a (generalized) composition operator if and only if $ƙ^P$ is A-invariant, that is, $Aƙ^P \subset ƙ^P$.

Proof. Assume A is a (generalized) composition operator for $L^P(m)$. A measurable set $Y \in$ Y and a measurable transformation T from Y to X result in $A= C_T$. If $\chi_s \in ƙ^P$, then $A \chi_s \in L^P(m)$. But

$$A\chi_s = C_T\, \chi_s = \chi_{T^{-1}(s)}$$

Thus $A \chi_s \in ƙ^P$.
Assuming $Aƙ^P \subset ƙ^P$. let $S \in Y$ be of finite measure. Then $\chi_s \in ƙ^P$. Hence $A \chi_s \in ƙ^P$. There exists $W \in Y$ such that $A \chi_s = \chi_w$. Let us define $\phi_o(S)= W$. Thus, ϕ_o is defined on the collection of sets of finite measures. If S_1 and S_2 are disjoint measurable sets of finite measures, then

$$A (\chi\, s_1 \cup s_2) = A (\chi_{s_1} + \chi_{s_2})\ \chi_{s_2}\ s_2$$
$$= A\ \ \chi_{s_1} + \chi_{s_2}\ A$$
$$= \chi_{w_1} + \chi_{w_2}$$

This represent that m ($W_1 \cap W_2$) = 0 and

$$\phi_o(S_1 \cup S_2) = \phi_o(S_1) \cup \phi_o(S_2)$$

It can be demonstrated that ϕ_o maintains intersection and difference. Given that m is an σ -finite measure, there exists a sequence $\{S_i\}$ of pairwise disjoint measurable sets of finite measures.

$$x = \bigcup_{i=1}^{\infty} Si.$$

Let $X_i = \phi_o(S_i)$, i.e., A $\chi_{si} = \chi_{xi}$ for i ∈ ℕ, and let $X' = \bigcup_{i=1}^{\infty} Xi$. If S is an arbitrary member of Y, $\phi_o(S)$ can be written as:

$$\phi_o(S) = \bigcup_{i=1}^{\infty} \phi o \ (S \cap S_i)$$

$\phi_o : Y \rightarrow Y'$ is an σ-homomorphism, whereas Y' is an σ -algebra of measurable subsets of X'. This σ -homomorphism generates Cp: $Y/ £ \rightarrow Y'/ £'$, which is defined as

$$\phi (S \Delta £) = \phi_o(S) \Delta £'.$$

According to Theorem, there exists a measurable transformation T: X' → X such that $\phi = h'_T$. If m (S) equals ∞, then

$$A\ \chi_s = \chi_{T^{-1}(s)} = C_T\ \chi_s$$

Thus, A and C_T agree on $ƙ^P$ and, by extension, on $L^P(m)$. This indicates that A equals C_T. This concludes the proof of the theorem.

2.2 Invertible Composition Operators

If π is a bounded complex-valued measurable function on X, the mapping M π on L^2 (m) defined by $M\pi\ f= \pi . f$ is a continuous operator with a range in L^2 (m). This operator M π, is called the multiplication operator induced by π.
If C_T is a composition operator on L^2 (m), then C_T. C_T^* is a multiplication operator, and $C_T\ C_T^*$ is similar to a multiplication operator. The following theorem explains these findings.

Theorem 2.2.1 Let C_T be the composition operator on L^2 (m). Then
(i) $C_T^*\ C_T = M_{fT}$.

(ii) $C_T C_T^*$ = $M_{fT\ o\ T}$p, where P is L^2 (m) projection onto the C_T range closure.
(iii) C_T has dense range if and only if $C_T C_T^*$; = $M_{fT\ o\ T}$.

Proof. (i) Let f,g ∈ L^2 (m). Then

$$\langle C_T^* C_T f, g \rangle = \langle C_T f, C_T g \rangle = \int f \bar{g}\, dm\, T^{-1}$$

$$= f_T\ f \bar{g}\, dm$$

$$= \langle M_{fT} f, g \rangle$$

Thus C_T^* C_T = M_{fT}.

(ii) Assume f ∈ L^2 (m). Then Pf belongs to the closure of C_T range. Hence, there exists a sequence {C_T f_n} in the range of C, which converges to P f in norm. Thus

$$C_T C_T^* Pf = \lim_n C_T\, C_T^*\, C_T\, f_n$$

$$= \lim_n C_T(f_T\, f_n)$$

$$= M_{fToT}\, Pf$$

We can deduce that f - P f is in the orthogonal complement of C_T range, which is equivalent to the kernel of $C_T C_T^*$f = $C_T C_T^*$Pf. Thus $C_T C_T^*$f = $C_T C_T^*$Pf for all f ∈ L^2 (m). Hence $C_T C_T^*$ = M_{fToT} P

(iii) If C_T has a dense range, then (ii) P equals I, the identity operator.
Hence $C_T C_T^*$ = M_{fToT} P. Since f_T o T not equal 0 at., $C_T C_T^*$ is an injection. Since C_T^* and $C_T C_T^*$ share the same ,We now got the desired outcome. This concludes the proof of the theorem.

Theorem 2.2.2: Let C_T be a composition operator on L^2(m)[7]. Then the following are equivalent.
(i) C_T is an injection.
(ii) f and f o T have the same essential range for any f ∈ L^2 (m)
(iii) m << m o T^{-1}
(iv) f_T varies from zero practicalle everywhere.

Proof: (i) ⟹ (ii) Assume C_T represents an injection. The essential range of f o T is always the same as the essential range of f in L^2 (m). To demonstrate reverse inclusion, let a be in the essential range of f. Let G be the neighborhood of a. Then, using the notion of essential range, m (T^{-1} (f^{-1}(G))) ≠ 0. As C_T is an injection, we can conclude that m (T^{-1} (f^{-1}(G))) = 0. Thus, a falls within the fundamental range of f oT.

(ii) ⟹ (iii): Let S ∈Y be such that m T^{-1}(S) = 0. The essential range of $C_T \chi_s$ is equivalent to the singleton set (0). According to (ii), the basic range of χ_s is equal to zero. This means that m (S) equals 0. Thus, m << m o T^{-1}.

(iii) ⟹ (iv): This implication stems from the following equation:

m T^{-1}(S) = $\int_s f_T\, dm$

(iv) ⟹ (i): Assume f_T differs from zero practically everywhere. It is well-known that the multiplication operator M_{fT} is an injection. As per portion (i) of Theorem 2.2.1, C_T^* C_T is an injection. Therefore, C_T is an injection. This concludes the proof of the theorem.

Corollary 2.2.1: Let C_T be a composition operator on L^2(m). Let T be right invertible and its right inverse be non-singular. The C_T scan is then administered as an injection.

Let Y_1 and Y_2 be two σ-subalgebras in Y. Y_1 and Y_2 are considered equal (written as $Y_1 = Y_2$) if for any $S_1 \in Y_1$, there exists $S_2 \in Y_2$, such that $S_1 = S_2$, and vice versa. If T is a measurable transformation, then $T^{-1}(Y)$ is an σ-subalgebra of Y, where

$$T^{-1}(Y) = \{T^{-1}(S) : S \in Y\}$$

The L^2-space with respect to the σ-subalgebra $T^{-1}(Y)$, denoted as L^2 (X, $T^{-1}(S)$, m), is a subspace of L^2 (m). The range of any composition operator is a subspace (not necessarily closed) of this space.

Each composition operator is dense in L^2 (X, $T^{-1}(S)$, m). We will demonstrate this in the following theorem.

2.3 Compact Composition Operators

In a separable Hilbert space, an operator's compactness means it converts weakly convergent sequences into norm convergent sequences. For example, if $x_n \longrightarrow x$ are weakly convergent, then $Ax_n \longrightarrow Ax$ in the Hilbert space's norm. This section covers compact composition operators on L^2 (m). There aren't many compact composition operators. No composition operator exists on the second level of a non-atomic measure space is compact. No composition operator is compatible with ℓ^2, the L^2-space of an atomic measurement space. However, some weighted sequence spaces have compact composition operators.

Let (X, Y, m) be a measure space, with $\mathcal{E} > 0$ and π a complex-valued measurable function on X. The set (x: x ∈ X and | π (x) | > $\mathcal{E}$) is denoted by X_ε^π. Let Z_ε^πbe defined by.

$$Z_\varepsilon^\pi = \langle \, f\chi_{X_\varepsilon^\pi} : \; f \in L^2 (m) \rangle$$

Then Z_ε^π is a subspace of L^2(m). If an element from L^2 (m) vanishes outside of X, it belongs to Z. The theorem describes compact multiplication operators on L^2 (m) based on Z dimension.

Theorem 2.3.1: Let M_π be a multiplication operator on L^2(m). Then M_π is compact if and only if Z_ε^π has finite dimension for all $\mathcal{E} > 0$.

Proof. Let $M_\pi$be compact. Since Z_ε^π is invariant under M_π, it follows that the restriction of M_π to Z_ε^π is also compact. As x is constrained away from zero on X_ε^π, we can conclude that M_π , Z_ε^π . Thus Z_ε^π is finite-dimensional.

In contrast, if X_ε^π is finite-dimensional for every $\mathcal{E}$, it is also finite-dimensional for every n ∈ ℕ . Let x be defined as.

$$\pi_n = \pi \; \chi_{X_{1/n}^\pi}$$

Then M_{π_n}, is a finite rank operator, and M_{π_n}+ M_π is a norm. Hence, M_π is a compact operator. This concludes the proof.

Theorem 2.3.2 : (i) states that a multiplication operator M_π on L^2(m) is compact if π = o a.e. on

(ii) If M_π is an injective multiplication operator on L^2 (m), then it is compact. indicates that (X, Y, and m) is an atomic measure space.

(iii) If C_T is a composition operator on L^2 (m), T^{-1} (X_2) = X equals X when C_T is compact.

(iv) If m(X) = ∞, the compactness of a composition operator C implies that m(X_2) = ∞.

Proof. (i) Define m_2 as the limitation of the measure m on X_2, and $m_1 = m - m_2$. L^2(m_1) is an invariant subspace of M_π , resulting in π = 0 a.e. on X_1.

(ii) If M_π is compact, the kernel of M_π contains L^2(m_1). Because M_π is one-to-one, L^2 (M_1) equals zero. Hence, $m_1 = 0$. Thus, m equals m_2.

(iii) If C_T is compact, then M_{fT} is also compact. Hence, by section (i), f_T is zero practically everywhere on X_1.

$$X = T^{-1}(X_2)$$

(iv) If C_T is compact and $m(X_2) < \infty$, part (iii) and Theorem 2.1.1 provide a contradiction.

From the previous theorems, it is clear that the hope for the existence of the compact .Composition operators occur when the underlying measure space is atomic. Sequence spaces are examples of L^2-spaces within atomic measure spaces. This section covers compact composition operations on Hilbert spaces of sequences. Let T: $\mathbb{N} \longrightarrow \mathbb{N}$ be a mapping, with $\varepsilon > 0$. Then definethe set N_ε as:

$$N_\varepsilon = \{ n : n \epsilon \mathbb{N} \text{ and } m\, T^{-1}(\{n\}) > \varepsilon\, m(\{n\}) \} \quad N\varepsilon$$

Theorem 2.3.3. Let C_T be a composition operator for ℓ^2(w). C_T is compact if and only if N contains a finite subset for all $\varepsilon > 0$.

Proof. Assume (f_i) is a weakly convergent sequence in ℓ^2 (w), with $\varepsilon > 0$. Assume N_ε is finite with k components. Theorem 2.1.1 states that if b > 0, then $m\, T^{-1}(\{n\}) > \varepsilon\, m(\{n\})$ for all n ε $\mathbb{N}$. Thus

$$\| C_T f_i \|^2 = .\int_N | f_i |^2 \, dm\, T^{-1} = \int_{N\varepsilon} | f_i |^2 \, dm\, T^{-1} + \int_{N/N\varepsilon} | f_i |^2 \, dm\, T^{-1}$$

$$\leq b\, k\, | f_i (n_i) |^2 \, m(\{n_s\}) + \| f_i \|^2$$

where $| f_i (n_r) | = \max \{ | f_i (n_i) | : n_t \in N_\varepsilon$ and $m(\{n_s\}) = \max\{m(\{n_t\}) : n_t \in N_\varepsilon\}$ goes to zero pointwise, we can identify j ϵ N such that for i > j. we have N

$$\| C_T f_i \|^2 \leq \varepsilon_1\, bk\, .\, m(\{n_s\}) + \varepsilon \| f_i \|^2$$

Since weakly convergent sequences are norm constrained, we may conclude that the sequence { $\| C_T f_i \|$} converges to zero. This demonstrates that C_T is compact.

In contrast, suppose Nε has an unlimited number of elements for any ε > 0. C_T is bounded away from zero at the end of the span {e_j : j ϵ Nε }. According to Problem, C_T is not compact since the range of its limitation is a closed, infinite-dimensional subspace within C_T. This concludes the proof of the theorem.

2.4 Normality Of Composition Operators

In Hilbert space, an operator A is considered normal if it commutes with its adjoint A^*, and quasinormal if it commutes with A^*A. If A^*A $-AA^*$ is positive, then A is considered a hyponormal operator. If either If A or A^* is a hyponormal, then A is considered seminormal. Unitary means A^*A = AA^* = I.

Theorem 2.4.1: Let (X, Y, m)[7] be a typical Borel space, and let C_T be a composition operator onL^2(m). Then the following are equivalent.
(i) C_T is unitary.
(ii) T is an injection with $f_T = 1$ a.e., (iii) C_T is invertible with $f_T = 1$ a.e., and (iv) C is a composition operator.

Proof: (i) ⟹ (ii) If C_T is unitary, then

$$C_T^* C_T = C_T C_T^* = I$$

Hence M_{fT} = I. Thus, f_T equals 1 a.c. Theorem 2.2.14 states that C_T is invertible, which implies that T is an injection.
(ii) ⟹ (i) $f_T = 1$ a.e., C_T is an isometry with closed range. If T is an injection, then C_T has a discrete range, according to Corollary 2.2.1. Hence, C_T is invertible.

(iii) ⟹ (iv) Given that C_T is invertible and $f_T = 1$ a.e., we obtain

$$C_T^* = M_{fT}\, C_T^{-1} = C_T^{-1} = C_{T^{-1}}$$

(iv) ⟹ (i) Assume C_T^* = C_U for a measurable transformation U. Then

$$M_{fT} = C_T^* C_T = C_U C_T = C_{ToU}$$

Thus, using the argument presented before the theorem, we obtain

$$f_T = 1 \text{ a.e}$$

This demonstrates how C_T is an isometry. If C_T has a dense range, it will be unitary. To prove that C_T has a dense range, simply show that $T^{-1}(Y)$ = Y. Let S ϵ Y be finite[7] in measure. If $\chi_s \,\epsilon$ ran C_T, then there is a h ϵ L^2(m). Such that

$$C_T h = \chi_s$$

Because C_T is an injection, Corollary[7] states that h = χ_{s_2} for some S_2 ϵ Y. Hence, S =$T^{-1}(s_2)$ ϵ $T^{-1}(Y)$. If χ_s is not in the range of C_T, we can write

$$\chi_S = C_T\, g + f$$

where f ϵ (ran C_T) $^{\perp}$. Now we have

$$C_U\ \chi_S = C_T^* C_T\ g + C_T^*\, f$$
$$= g$$

Thus, g equals $\chi_{T^{-1}(S)}$. It is proven that $\chi_S = \chi_{T^{-1}\left(U^{-1}(S)\right)}$. Hence S $\epsilon T^{-1}(Y)$. It is proven that S ϵ $T^{-1}(Y)$ for any S ϵ Y. Hence Y $\subset$ $T^{-1}(Y)$ $\subset$ Y . Thus

$$Y = T^{-1}(Y)$$

This proves the statement.

Theorem 2.4.2. Let (X, Y, m) be an a-finite measure space with C_T as a composition operator on L^2 (m).

(i) C_T is hyponormal if $\| \sqrt{f_T}\ f \| \geq \| \sqrt{f_T}$ o T pf $\|$ for everyone f ϵ L^2(m).

(ii) C_T^* is hyponormal if and only if f_T o T $\geq f_T$ a.e. and the completion of the σ-algebra given by the set of type S interception) X_o^{fT} for S ϵ Y is contained in T^{-1} (Y), where

$$X_o^{fT} = \{ x : f_T(x) > 0 \}$$

Proof: (i) C_T is hyponormal if and only if $C_T^* C_T - C_T C_T^* \geq 0$. Hyponormal if and only if.

$$\langle (C_T^*\, C_T - C_T\, C_T^*)\, f\, , f \rangle \geq 0$$

According to Theorem 2.2.1, C_T is considered hyponormal if and only if

$$\langle M_{fT}\, f\, , f \rangle \geq \langle M_{f_{T_o T}}\ pf\, , f \rangle \qquad \text{for all } f\ \epsilon\ L^2(m)$$

Based on this, we can conclude that C_T is hyponormal if and only

$$\| \sqrt{f_T}\ f \| \geq \| \sqrt{f_T}\, o\, T\ pf \| \qquad \text{for all } f\ \epsilon\ L^2(m)$$

This demonstrates (i) (because P is the projection of L^2(m) on the C_T, $P^2 = P$, and P(f_T o T f) = f_T O T pf

(ii) Assume C_T^* is hyponormal. The kernel of C_T^* is found in the kernel of C_T . Assume S is a finite measure set that does not belong to $T^{-1}(Y)$ and has a correlatio with X. Then χ_S is not within the C_T range's closure. There is a function f in the orthogonal complement of the C_T range closure that has avalue of zero (f, χ_S) $\neq 0$. Since f ϵ ker C_T^* subset ker C_T equal ker M_{fT}, we have

$$f_T \, f = 0$$

Thus, we arrive at a contradiction. Hence, S ϵ $T^{-1}(Y)$. Let

$$S_1 = \{(x : (f_T \circ T)(x) < f_T(x)\}$$

Then S_1 ϵ $T^{-1}(Y)$. Using the hyponormality of C_T^* , it can be proved that $m(S_1) = 0$. Hence f_T o T $\geq f_T$ a. e. For the converse, suppose conditions are true.
Let f ϵ $L^2(m)$[7]. Then f can be written as

$$f = f_1 + f_2$$

where f_1 is the closure of ranC_T and f_2 its orthogonal complement. It is possible to prove that

$$\| C_T^* f \|^2 - \| C_T f \|^2 = \int (f_T \circ T - f_T) \, | f_1 |^2 \, dm$$

Since f_T o T $\geq$ f_T, we have the hyponormality of C_T^*.

2.5 Weighted Composition Operators

The weighted composition operator $W_{\pi,T}$ on a function space H(X) over a set X is a continuous linear transformation from H(X) to itself, defined as $W_{\pi,T}$(f) = π. f o T, where π is a function in X and T is a [7]self map of X. If π induces the multiplication operator M_π on H(x), and T induces the composition operator C_T on H(X), then $W_{\pi,T} = M_\pi C_T$. However, the weighted composition operator $W_{\pi,T}$ may be induced by the pair (π,T), but not by T. For example, if π (x) = 0 for every x ϵ X and T: X $\rightarrow$X is a map, then $W_{\pi,T}$ is a weighted composition operator whether T causes an operator or not. The composite function f o T is multiplied by R to obtain the function $W_{\pi,T}$ (f). Multiplying by π and composing the function π.f with T yields the operator f $\rightarrow$ (π. f)o T, denoted as $W_{T,\pi}$.

We suppose : X $\rightarrow$ C is a bounded measurable function and T: X $\rightarrow$ X is a non-singular measurable transformation. For more generalized weighted composition operators, we can use the support of π as the domain of T.
Now, define the measure m_T^π on Y as Since m(S) = 0, mT^{-1} (S) = 0, and m_T^π (S) = 0,

we can conclude that m_T^{π} << m. Define f as the Radon-Nikodym derivative of m with respect to m, with 0 equaling $\phi = (f_T^{\pi})^{1/p}$. If f_T is essentially bounded, $W_{\pi,T}$ is also bounded on $L^P(m)$. However, the opposite is not true. The following theorem defines $W_{\pi,T}$ boundedness as the boundedness of ϕ.

Theorem 2.5.1 states that a weighted composition operator $W_{\pi,T}$ on L^2 (m), is compact if and only if $mT^{-1}(X_1) = 0$.

$$\frac{m_T^{\pi}(\{x_i\})}{\alpha_i} \to 0 \ \ as \ \ \to \ \infty$$

This limit is assumedvto be 0 if X_2 is finite.

Proof. Let $X_o = \{x \in X_1 : \ \phi(x) > 0\}$.
Now,

$$\int \ \ |\pi|^P \, dm = \ m_T^{\pi}(X) = \ \phi^P \, dm$$

As a result, mT^{-1} $(X_1) = 0$ if and only if $m(X_o) = 0$. That is, $\phi(x) = 0$ a.e. on X_1 if and only if $\pi(x) = 0$ a.e. on T^{-1} (X_1) . Now,

$$\alpha_i \, \phi^P (X_i) = \ \phi^P \, dm = m_T^{\pi} (\{X_i\})$$

and hence

$$\phi(x_i) = \ \left\{ \left\langle \frac{m_T^{\pi} (\{Xi\})}{\alpha_i} \right\rangle \right\}^{1/p}$$

$W_{\pi,T}$ is compact only when $m(X_o) = 0$ and $\lim_{i \to \infty} \phi(x_i) = 0$. If $m(X_o) = 0$ and $\phi(x_i) \to 0$, then for each $\varepsilon > 0$, the set X_{ε}^{ϕ} is a union of finitely many atoms with measure zero. As a result of theorem, Z_{ε}^{ϕ} has finite dimensions. $W_{\pi,T}$ is compact.

CHAPTER 3

COMPOSITION OPERATORS ON FUNCTIONAL BANACH SPACES

3.1 General Characterizations

If X is a non-empty set and H(X) is a functional Banach space of complex-valued functions, then the evaluation functionals δ_x, defined as $\delta_x(f) = f(x)$, are continuous and belong to $H^*($ (X), the conjugate space of H(X). If T : X $\rightarrow$X is a mapping with f oT belonging to H(X) whenever f belongs to H(X), then the [11]mapping f into f oT is a linear transformation from H(X) into itself. According to the closed graph theorem, this linear transformation is continuous and bounded. C_T represents the composition operator on H (X) caused by T.

If A is an operator on H(X), then its adjoint, A^*, is an operator on H^*(X).
defined as:

$$(A^*F)(f)=F(Af)$$

for every[11] F ϵ $H^*($ (X) and f ϵ H(X). It is clear that A^* is the composition operator[7].
on $H^*($ (X) caused by the operator A. If C_T is a composition operator on H(X),
Then clearly, Δ is invariant under C_T^* really

$$C_T^* \, \delta_x = \delta_{T\,(x)}$$

Theorem 3.1.1: Let A be an operator on H(X) and let H(X) be a functionalvBanach space over a non-empty set X. If A is invariant under A^*, that is, $A^*(\Delta)$ subset Δ, then A is a composition operator if and only if.

Proof: Assume that for some A = C_T. Let $\delta_x \in \Delta$ and f $\in$ H(X). (C_T^* δ_x) (f) = δ_x (C_T f) = δ_x (f o T) = $\delta_{T(x)}$ (f) is the result then. Consequently, C_T^* δ_x = $\delta_{T(x)} \in \Delta$
Alternatively, consider that Δ is invariant under A^*. Assume x$\in$ X. Next, $\delta_x \in$ A, and hence , A^* δ_x,$\in$ Δ. As a result, T(x) $\in$ X exists such that $A^*\delta_x = \delta_{T(x)}$. It is clear what mapping T takes x to T(x) is.

$$(foT)(x) = f(T(x)) = \delta_{T(x)}(f) = (A^*\delta_x\ )(f) = (Af)(x)$$

This now demonstrates that C_T = A. This concludes the theorem's proof.

3.2 Operators Of Composition On Spaces $H^P(D)$, $H^P(D^N)$ AND $H^P(D_N)$

Theorem 3.2.1.

(i) A composition operator C_T on Hp(D) for $1 \leq p \leq \infty$ is induced by each holomorphic map T from D into itself. And

$$\| C_T \|^P \leq \frac{1+|T(0)|}{1-|T(0)|} \qquad p \neq \infty$$

(ii) The following are identical if A is a non-zero bounded operator on $H^P(D)$ and $f_n(z) = z^n$ for $n \in Z$:

(a) A is a composition operator;

(b) for every $n \in Z$, $A f_n = (Af_1)^n$

(c) for every bounded analytic function f and g in $H^P(D)$, A(f.g) = Af.

(iii) T is a conformal automorphism of D if and only if the composition operator C_T on $H^P(D)$ is invertible.

Proof. (i) Rudin's results (state that an analytic function f on D is only included in $H^P(D)$ if and only if If $|f|^P$ is the least harmonic majorant of f and possesses a harmonic majorant, then $\|f\|^P = \psi_f(0)$. Hence, fo T is analytic on D and

$$| (foT)(z)|^P = | f(T(z)) |^P \leq \psi_f(T(z)) \qquad \text{for all } z \in D$$

$\psi_f o\, T$ is a harmonic majorant for f o T and consequently f oT $\in H^P$ (D) if T: D → D is analytic and f $\in H^P$ (D). This demonstrates how the composition operator C_T is induced by T. In this instance,

$$\| C_T f \|^P = \| foT \|^P = \psi_f o\, T(0) \leq \psi_f o\, (T(0))$$

we therefore get

$$\| C_T \|^P \leq \frac{1+|T(0)|}{1-|T(0)|} \| f \|^P$$

(ii) Suppose $Af_n = (Af_1)^n$ for $n \in \mathbb{Z}_+$ according to Harnack's inequality. Assuming T = Af_1, T is a member of H^P (D). We currently have every $n \in \mathbb{Z}_+$
.

$$\| T^n \|^{1/n} = \| Af_n \|^{1/n} \leq \| A \|^{1/n}$$

It is possible to demonstrate this by taking the limits of both sides as $n \rightarrow \infty$

$$\| \tilde{T} \|_\infty \leq 1$$

where $\tilde{T}$ represents the radial limit of T, which is almost always present on the unit circle.
T cannot be a constant function of unit modulus since it translates D into D according to the maximum modulus principle. Consequently, T generates the composition operator C_T, which concurs with A on f_n's meaning that $A=C_T$. On the other hand, $Af_n = (Af_1)^n$ follows naturally if A is a composition operator. This demonstrates how (a) and (b) are equivalent.
This makes the equivalency of (a) and (c) obvious.

(iii) Assume f and g are bounded analytic functions in H^P (D) and that A is the inverse of C_T.

$$C_T A(fg) = A(fg)oT = f.g = (C_T Af)(C_T Ag)$$
$$= (Af.Ag)o\, T$$

hence $(A(fg)-Af.Ag)oT = 0$. T range is considered to be an open set since T is non-constant as C_T. is therefore invertible, therefore we get

$$A(fg) = Af \,.\, Ag$$

As a result, by (ii), an analytic function U from D onto itself exists, making $A = C_U$. Given that,

$$(C_U C_T f_1)(z) = (ToU)(z) = z = (UoT)(z) \quad \text{for all } z \in D$$

we can argue that T has an analytic inverse. T is a conformal automorphism as a result. It is simple to prove the opposite.

Theorem 3.2.2: Let T: $D^n \to D^n$ be a holomorphic function with $1 < p < \infty$.
(i) C_T is a composition operator on $H^P(D^n)$ if and only if $u_T(S(G)) < c\, m_n(G)$ for every open set G in ∂D^n and a [11]constant c > 0. $|z_i|$ $|z_i|^2$

(ii) If $\sup_{z \in D^n} \prod_{i=1}^{n} \frac{(1-|z_i|^2)}{1-|T_i(z)|^2}$ is infinite, T does not provide the composition operator on $H^P(D^n)$, where

$$T(z) = (T_1(z), T_2(z), \ldots\ldots T_n(z)$$

Proof. (i) We infer that u_T is a well-defined measure since $\int f\, du_T = \int (f o\, \tilde{T})\, dm_n$ generates a continuous linear Dn functional on C ($\overline{D^n}$). . If $f \in H^P(D^n)$ intersection C($\overline{D^n}$). then

$$||foT||_P^P = \int_{\partial D^n} |f|^P o\, \tilde{T}\, dm_n$$

$$= |f|^P du_T$$

If u_T is a Carleson measure, the foregoing assertion implies that C_T is continuous on H and so on $H^P(D^n)$ intersection C($\overline{D^n}$)as it is dense in $H^P(D^n)$. If C_T is bounded, the following argument implies that u is a Carleson measure. This describes the proof of component (i).

(ii) It is sufficient to assume p = 2. Let $z \in D^n$. Define $g_z : D^n \to \mathbb{C}$ as

$$g_z(w) = \prod_{i=1}^{n} \frac{1}{1-\overline{w_i}\, z_i}$$

Then

$g_z \in H^2(D^n)$, and thus

$$||g_z||^2 = g_z(z) = \prod_{i=1}^{n} \frac{1}{1-|z_i|^2}$$

we also have

$$\| g_{T(z)} \|^2 = g_{T(z)}(T(z)) = \prod_{i=1}^{n} \frac{1}{1-|T_i(z)|^2}$$

Also

$$\| g_{T(z)}\|^2 = (g_{T(z)} o T)(z) = (C_T\, g_{T(z)}) (z)$$

Therefore

$$\| g_{T(z)}\|^2 \le \| C_T\, g_{T(z)}\|_2 \langle \prod_{i=1}^{n} \frac{1}{1-|z_i|^2} \rangle^{1/2}$$

Therefore, it follows that

$$\| C_T\|^2 \ge \| g_{T(z)}\|^2 \prod_{i=1}^{n}(1-|z_i|^2) = \prod_{i=1}^{n} \frac{(1-|z_i|^2)}{1-|T_i(z)|^2}$$

This is true for every $z \in D^n$ thus,

$$\| C_T\|^2 = \sup_{z \in D^n} \prod_{i=1}^{n} \frac{(1-|z_i|^2)}{1-|T_i(z)|^2}$$

This concludes the proof of Part II.

Theorem 3.2.3: Let C_T be a composition operator on H^P (D).

(i) C_T is compact if and only if the norm-bounded sequence $\{f_n\}$ in H^P (D) that converges uniformly on compact subsets of the unit disc also converges to zero in the norm.

(ii) C_T is compact, implying that $|\tilde{T}(e^{\iota\theta})| < 1$ a.e.

(iii) C_T is not compact if T has an angular derivative at some point .

(iv) C_T is Hilbert-Schmidt if and only if $1 / (1-|\tilde{T}|^2)$ is integrable in the Lebesgue measure on ∂D at p = 2.

Proof. (i) This conclusion is valid over multiple Banach spaces of analytic functions, and the proof is straightforward. It can be located at .

(ii) Because T maps D to D, it is clear that $|\tilde{T}(e^{\iota\theta})| < 1$ a.e. Define f_n on D as $f_n(t)=z^n$ where n is in $\in$ N. $\{f_n\}$ is a norm-bounded sequence that converges to zero uniformly on compact subsets of D. If $|\tilde{T}(e^{\iota\theta})| = 1=1$ for a set of non-zero measures,

$$\| C_T f_n \|^2 = \| f_n \circ T \|^2 = \int |T(e^{\iota\theta})|^{2n} d\theta \nrightarrow 0$$

as $n \to \infty$. According to (i), C_T is not compact.

(iii) Assume T's angular derivative at eie equals 1 without loss of generality. The angular derivative requires a constant $k > 0$ to exist, as defined.

$$\frac{|1-T(t)|}{|1-t|} \leq k \qquad \text{for } -1 < t < 1$$

Let

$$f_n(z) = \frac{1}{\sqrt{n(1-z)^{(n-1)/n}}} \qquad \text{for } z \in D$$

In H^2 (D), $\{f_n\}$ is a weak null sequence, while $\{f_n \circ T\}$ is bounded from zero. Hence, C_T is not compact .

(iv) C_T is Hilbert-Schmidt if and only if $\sum_{n=1}^{\infty} \| C_T\ f_n \|^2 < \infty$, where fn is the same as specified in (2). Since

$$\sum_{n=1}^{\infty} \| C_T\ f_n \|^2 = \sum_{n=1}^{\infty} \int \left(|\tilde{T}(e^{\iota\theta})|^{2n}\right) d\theta = \int \frac{1}{1-|\tilde{T}(e^{\iota\theta})|^2} d\theta$$

We finish the results.

3.3 Composition Operators On $H^P(P^+)$

P^+ denotes the upper half plane, defined as {w: w $\in$ C and Im w > 0}, where Im w represents the imaginary part of w. Then the hardy space The definition of $H^P(P^+)$ for $1 \leq p < \infty$ is:

$$H^P(P^+) = \{ f : f \text{ is analytic on } P^+ \text{ and } \sup_{y>0} \int_{-\infty}^{\infty} |f(x+\iota y|^p dx < \infty$$

The pointwise vector space operations and the norm are defined as

$$\| f \|^P = \sup_{y>0} \int_{-\infty}^{\infty} |f(x + \iota y|^p \, dx$$

$H^P (P^+)$ becomes a functional Banach space.

Let $L(z) = \frac{\iota(1+z)}{1-z}$ maps D to P^+ and ∂D to the real line, with L^{-1} given as

$$L^{-1} = (w-\iota)/(w+\iota)$$

Let Q be defined as

$$(Qf)(x) = (1/\sqrt{\pi})(f \text{ o } L^{-1})(x)/(x + \iota)$$

Then Q is a well-known isometric isomorphism from L^P (m) to L^P ($-\infty$, ∞). Let t: D → D be an analytic map, and T=Loto L^{-1} . Let $\beta(z) = \frac{1-t(z)}{1-z}$ for Z ∈ D. In , it was demonstrated using Poisson integrals in the disk and upper half plane that C_T is a composition operator on $H^P (P^+)$ if and only if M_β $A(P^+)$ refers to any analytic mappings T that take P^+ into itself and only have a pole at - as their singularity. The following theorem presents findings on composition operators on $H^2(P^+)$.

Theorem 3.3.1.

(i) Assume T ∈ $A(P^+)$. T thus induces a composition operator on If (P) if and only if it has a pole at zero.
(ii) C_T is a composition operator on $H^2 (P^+)$, then

$$\sup_{w \in p^+} \{(\text{Im } w)/(\text{Im } T(w))\} \leq \| C_T \|^2$$

(iii) C_T is invertible if induced by T ∈ $A(P^+)$ and T is a conformal automorphism of P.

Proof. (i) Assume that T has a pole at ∞ . If T is analytic in a neighborhood of ∞, the function t = L^{-1}oTo L is analytic in a neighborhood of 1 with $\tilde{\iota}$ (l)=l. T induces a composition operator C_T on $H^2 (P^+)$ via an earlier statement. If f oT ∈ $H^2 (P^+)$ for every f $\in H^2(P^+)$, (foT)(w) tends to zero as w $\to \infty$ within each half-plane Im w $\geq \delta > 0$. Since the function from w to $1/(\iota + w)$ belongs to $H^2(P^+)$, we can conclude that it tends to zero at w $\to \infty$. Thus, T has a pole at ∞.

(ii) The replicating kernel K for $H^2(P^+)$ is given as

$$K(w,u) = \frac{\iota}{2\pi(w-\overline{u})}$$

Also, $\| f_u \|^2 = (f_u , f_u) = K(u,u) = \frac{\iota}{4\pi Im\, u}$

We know that $C_T^* f_w = f_{T(w)}$ for all w $\in P^+$.

Therefore , $\frac{Im\, w}{Im\, T(w)} = \|f_{T(w)}\|^2 / \| f_w \|^2 = \| C_T^* f_w\|^2 / \| f_w \|^2$

Hence, $\sup\left(\frac{Im\, w}{Im\, T(W)} : w \epsilon\, p^+\right) \leq \| C_T \|^2$

(iii) If T is a conformal automorphism, T^{-1} is analytic and has a pole at ∞, just as T does. C_T^{-1} is a composition operator on $H^2(P^+)$ that is the inverse of C_T. Suppose C_T is invertible. We know that

$$M_\beta C_t = P\, Q^{-1} P^{-1}\, C_T\, \hat{P} Q P^{-1}$$

where t = L^{-1} o T o L, β (z) = 1- t(z)/1-z, P is the Poisson integral in the disc, and $\hat{P}$ is the Poisson integral in the upper half plane . Thus, $M_\beta C_t$ is invertible.
Because M_β is subnormal and surjective, we can deduce that it is invertible. C_t is invertible, hence t is a conformal automorphism. Thus, T is a conformal automorphism. This concludes the proof of the theorem.

Example 3.3.1 (i): Let a > 0 and $w_0 \in P^+$. Then, define

$$T(w) = aw + w_0 \qquad \text{for } w \in P^+$$

T induces a composition operator on $H^2(P^+)$, as per section (i) of the preceding theorem. found that the composition operator's norm is $\sqrt{1/a}$.

(ii) Let n be a positive integer, and

$$T(w) = \frac{\iota((w+\iota)^{n+1} + w(w-1)^{n})}{(w+\iota)^{n+1} - w(w-\iota)^{n}}$$

for w in belong P^+. T maps P^+ to P+, resulting in a composition operator on H^2 (P^+).

(iii) T is a linear fractional transformation defined as

$$T(w) = \frac{aw+b}{cw+d}$$

where a, b, c, and d are real values with ad - bc > 0 and c not equal 0. T maps P^+ to P+, but does not provide a composition operator on H^2 (P^+) as the point at infinite is not a pole of T.

Theorem 3.3.2: Let T: $P^+ \to P^+$ be a holomorphic mapping that induces the Composition operator C_T on H^2 (P^+). Then
(i) C_T is not compact if lim $y \to 0$ T(x+iy) exists a priori and is a real number for any real number x.

(ii) C_T is not compact if there is a k > 0 such that | $(\iota$+nT(w))/(ι +nw) | $\leq$ k T(x+iy) exists and belongs to P^+. Denote this limit for every w belong P^+ and n belong N.

(iii) Assume lim $y \to 0$ T(x+iy). C_T is a [3]Hilbert-Schmidt composition operator if and only if.

$$\int_{-\infty}^{\infty} (Im\, T_*(x))^{-1}\, dx < \infty$$

Proof. (i) For n belong $\mathbb{Z}_+$, define the function fn on P^+ as

$$f_n(w) = (1/\sqrt{\pi})[(w-\iota)^n / (w+\iota)^{n+1}]$$

Then f_n is a weak null sequence in $H^2(P^+)$. The algorithm demonstrates that

$$\| C_T f_n \|^4 = 1/\pi \int_{-\infty}^{\infty} \frac{1}{1+(T_*(x))^2}\, dx$$

$\{C_T f_n\}$ does not converge to zero in the norm. Hence, C_T is not compact.

(ii) For $n \in N$, define f on P^+ as

$$f_n(w) = \frac{1}{\sqrt{n(\frac{\iota}{n}+w)}}$$

Then f_n is a pointwise null sequence that is norm bounded, making it a weak null. Now, for $w \in P^+$ we have

$$| (C_T f_n)(w) |^2 = | n^{-1/2} (\tfrac{\iota}{n} + T(w))^{-1} |^2$$

$$= n^{-1} | (\tfrac{\iota}{n} + T(w))^{-1} (\tfrac{\iota}{n} + w) (\tfrac{\iota}{n} + w)^{-1} |^2$$

$$\geq k^{-2} | f_n(w) |^2$$

Hence $\| C_T f \|^2 \geq k^{-2}\, \pi$, because $\| f_n \| = \sqrt{\pi}$, for $n \in N$. Thus, C_T is not compact.
(iii) Assume n is a non-negative integer. Define the function f_n on P^+ as

$$f_n(w) = (w + \iota)^n / \sqrt{\pi}\, (w + \iota)^{n+1}$$

The family $\{ f_0, f_1, f_2, ...)$ provides an orthonormal foundation for $H^2(P^+)$. The composition operator is Hilbert-Schmidt if and only if $\sum_{n=0}^{\infty} ||C_T f_n||^2 < \infty$. However,

$$\sum_{n=0}^{\infty} ||C_T f_n||^2 = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} |(f_n o T)_*(x)|^2\, dx = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} | f_n(T_*(x)) |^2\, dx$$

A simple computation shows that C_T is Hilbert-Schmidt if and only if $\int_{-\infty}^{\infty} (\mathrm{Im}\, T_*(x))^{-1}\, dx < \infty$. This concludes the proof of the theorem.

3.4. Composition Operators On ℓ^P-Spaces

Let w = (w_n) denote a sequence of non-negative real numbers. In Chapter I, the weighted sequence space L^P(w) for $1 \leq p < \infty$ was defined as the Banach space of all complex number sequences (α_n) with $\sum_{n=1}^{\infty} w_n|\alpha_n|^P < \infty$. If the weight sequence (w) has non-zero terms, the space ℓ^P (w) is a functional Banach space. Thus, ℓ^P (w) is classified as both L^P-spaces and functional Banach spaces. If p equals 2, then ℓ^P (w) is a Hilbert space. L^P(w), often known as ℓ^P, is a typical example of a sequence space when w = 1 for all n in N. Hilbert himself explored a Hilbert space known as ℓ^2.

Theorem 3.4.1. Assume T: N $\rightarrow$ N is a function.

(i)T induces a composition operator on ℓ^P (w) if and only if there is k > 0 such that

$$\sum_{i \in T^{-1}(n)} w_i \leq k\, w_n$$

for every n ∈ N. In this scenario, $\| C_T \|^P$ equals the inf of such k's[7].
(ii) T is an injection if it creates an isometric composition operator on ℓ^P.
(iii) T produces an invertible composition operator on ℓ^P if and only if it is invertible.

Proof. (i) Let S be a subset of N. Then define.

$$m(S) = \sum_{i \in S} w_i$$

Then m becomes a measure on the σ-algebra of all subsets of N, and ℓ^P (w) is just L^P(m). Result of theorem states that each singleton set (n) has a non-zero measure, which leads to the proof.

(ii) T induces a composition operator on ℓ^P if and only if there is k > 0 [7]such that n ∈ N. If T is an injection, # T^{-1}((n)) is either 0 or 1; thus, if k = 1, the above inequality is satisfied. It is clear that 1 is the infimum of such k's satisfying the above inequality.
Hence, $\| C_T \| = 1$. The converse is clear.

(iii) Assume T is invertible. For every n ∈ N, there is a U such that (ToU) (n) = (UoT) (n). As U is an injection, C_U is an isometric composition operator on ℓ^P , and $C_T C_U = C_U C_T = I$. Hence, C_T is invertible. Suppose C_T is invertible. If T is not an injection, T(n) = T(m) for unique n and m. Hence, every sequence (α_i) in the range of C_T has a, = G. Then C_T is not upon, resulting in a contradiction. Hence, T is an injection. If T is not a surjection, the kernel of C_T must be non-trivial, resulting in a contradiction. Therefore, T is onto. This shows that T is invertible. This concludes the proof.

Theorem 3.4.2 states that T: N → N is a map with C_T as a composition operator on ℓ^P (w).
(i)The adjoint C_T^* is as follows:

$$(C_T^*x)(n) = \begin{cases} 1/w_n \left(\sum_{i\,\epsilon T^{-1}(n)} w_i\, x_i \right., & if\ \ T^{-1}(n)\ is\ non\ empty \\ 0, & if\ \ T^{-1}(n) is\ empty \end{cases}$$

(ii) C_T^* is a composition operator on ℓ^P if and only if C_T is invertible or unitary.

Proof. (i) Let x and y be in on ℓ^P (w). Then

$$(C_T\, x , y) = \sum_{n=1}^{\infty} w_n\, (x o\ T)(n).\overline{y(n)}$$

$$= \sum_{n=1}^{\infty} \sum_{i\,\epsilon T^{-1}(n)} w_i\ x(n)\ \overline{y(\iota)}$$

$$= \sum_{n=1}^{\infty} x_n \quad \sum_{i\,\epsilon T^{-1}(n)} w_i\ \ \overline{y(\iota)}$$

$$=\sum_{n=1}^{\infty} w_n\, x(n)\ \overline{C_T^* y)(n)}$$

$$= (x, C_T^* y)$$

Therefore adjoint of C_T is C_T^*.

(ii) Let $C_T = C_U$ for some U, then for every n belong N

$$\chi_{(T(n))} = C_T^*\ \ \chi_{(n)} = C_U\, \chi_{(n)} = \chi_{U^{-1}(n)}$$

where $\chi_{(s)}$ represents the sequence with a value of 1 on S and 0 elsewhere. Thus, $U^{-1}(n)$ equals (T(n)). As a result, U is invertible, and C_U follows suit. This demonstrates that C_T is invertible. If C_T is invertible, then he ranges of x ∈ ℓ^P and x oT are the same. Hence

$$\| x \|^2 = \sum_{n=1}^{\infty} |x_n|^2$$

$=\sum_{n=1}^{\infty} |x_{T(n)}|^2$

$= \| C_T x \|^2$

Thus, C_T is unitary, and $C_T^* = C_{T^{-1}}$. This concludes the proof of the theorem.

Theorem 3.4.3: Let T: N → N be a function[7] with C_T as a composition Operator on ℓ^2 (w).

(i) C_T is compact if and only if

$$\frac{1}{w_n} \sum_{i \epsilon T^{-1}(n)} w_i \to 0 \quad \text{as } n \to \infty$$

(ii) C_T is Hilbert-Schmidt if and only if the sequence $\frac{1}{\sqrt{w_{T(n)}}}$ belongs in ℓ^2 (w).

Proof. Corollary provides an obvious proof for (i). (i) re-states the corollary based on weights.
(ii) The Hilbert-Schmidt operator can be used to the orthonormal basis for ℓ^2 (w) represented by the functions $f_n = e_n / \sqrt{w_n}$ for n belong $\mathbb{Z}_+$.

Theorem 3.4.4: Let C_T be a composition operator on ℓ^2. Then
(i) C_T has an invariant subspace.
(ii) C_T has a decreasing subspace if N has two separate elements that are not in the same orbit as T. Two elements of N are considered in the same orbit of T when they can be reached by composing T and T^{-1} several times.

Proof. Theorem states that if C_T is invertible, it is also unitary and so normal. Thus, it possesses an invariant subspace. If C_T is not invertible, it cannot be both surjective and injective. If C_T is not surjective, its range, which is a closed subspace of ℓ^2 , is an invariant subspace of C_T. If C_T is not injective, the kernel represents an invariant subspace of C_T. This completes the proof for part (i).

(ii) Assume mo and no are different components of N that are not in the same orbit as T. Let F = (n: n belong N, where n and no do not share the same circle of T). Let M be the span[3] of e_n' s, where n belong F: e_n represents the sequence with n^{th} entry 1 and rest 0. M is a proper, closed subspace of ℓ^2 that is invariant under C_T and C. Thus, M is a reducing subspace of C_T. This concludes the proof of the theorem.

CHAPTER 4

SOME APPLICATIONS OF COMPOSITION OPERATORS

In chapter II, we examined composition operators on several function spaces. Natural operators, whether explicit or implicit, can be found in various fields of mathematics, including classical. Topics covered include mechanics , ergodic theory, dynamical systems, Markov processes, semigroup theory, isometries, and homomorphisms. This chapter will cover some of these interactions and their applications. Isometries are crucial for understanding certain mathematical structures. Isometrics onvsuitable function spaces are typically associated with compositionvoperators.

4.1 Isometries And Composition Operators

We start with the standard Banach-Stone theorem. If X and Y are compact Hausdorff spaces with a continuous map T: X→Y, the composition operator C_T: C(Y) →C(X) [9]is an isometry if and only if T is surjective, and a surjective isometry if T is a homeomorphism. If[1] T: X→Y is a homeomorphism and π: X→C is a continuous function with |x| = 1 for every x∈X, the weighted composition operator $W_{\pi,T}$ = C(Y) → C(X) becomes a surjective isometry. The Banach-Stone theorem states that any surjective isometry [9]from C(Y) to C(X) is a weighted composition operator, which contradicts the previous tatement. The Banach-Stone theorem requires some definitions.

Definition 4.1.1: In a vector space E, the set (t x_1 + (1 - t)x_2: 0 < t < 1) is the open line[11] segment between x_1 and x_2. The line segment $x_1 \neq x_2$ is considered appropriate. If K is a convex subset of E, then Point Y∈K is considered an extreme point if no open line segment contains y and is wholly within K. The symbol ext K represents the set of all extreme points of K. If E is a normed linear space, $(E)_1$ represents its closed unit ball.

If X is a topological space, M(X) refers to the normed linear space of all complex-valued regular Borel measures with the total variation norm. If X is a compact Hausdorff space, we may show that the set[11] of all extreme points of $(M(X))_1$ is ($\alpha\delta_x$: |α| = 1 and x ∈ X), while the set of all extreme points of P(X), the set of probability measures on X, is (δ_x: x ∈ X).

Theorem 4.1.1. [The Banach-Stone Theorem]. Let X and Y be compact Hausdorff spaces with a surjective isometry. There is a homeomorphism T: X → Y and a function x in C(X) where | X | = 1 for all X∈X or

$$(A_f)(x) = z(x)_f(T(x)), \text{ for all } f \in C(Y)$$

and every x ∈ X ($A = W_{\pi,T}$).

Proof. Assume A: C(Y) → C(X)[1] is a surjective isometry. The adjoint operator A^*: M(X) →M(Y) is a surjective isometry. A^* is a weak*-homeomorphism from $(M(X))_1$ to $(M(Y))_1$ that distributes Over convex combinations. Additionally, it suggests that

$$^{[9]}A^* (\text{ext}(M(X))_1) = \text{ext}(M(Y))_1.$$

This theorem implies that for each x ∈ X, there is a unique T(x) in Y and a unique scalar π(x) with |π(x)| = 1 and $A^*(\delta_x) = \pi(x)\delta_{T(x)}$. Thus, x: X → C and T: X → Y are unique well-defined functions. First of all, we demonstrate that the function IC is continuous. Let (x_α) be a net in X that equals x_α →x. Then clearly, $\delta_{x_\alpha} \to \delta_x$ (weak*) in M(X). Furthermore, we have $A^*(\delta_{x_\alpha}) \to A^*(\delta_x)$ (weak*) in M (Y). Specifically, $\pi(x_\alpha)\delta_{T(x_\alpha)} \to \pi(x)\delta_{T(x)}$. For example, $\pi(x_\alpha) = A^*(\delta_{x_\alpha})(1) \to A^*(\delta_\alpha)(1) = \pi(x)$.

This shows that π is a continuous map. We will demonstrate that the map T: X → Y is a homeomorphism. Since the map $x \to \delta_x$ is a homeomorphism from X into (∇(X), wk*), we can argue that $T(x_\alpha) \to T(x)$ This shows that T is continuous. To demonstrate that T is an injection, let x_1 and x_2 ∈ X be So that $x_1 \neq x_2$. Then $\overline{\pi(x_1)}\,\delta_{x_1} \neq \overline{\pi(x_2)}\,\delta_{x_2}$, implying that $T(x_1) \neq T(x_2)$. We can now solve y∈Y. The surjectivity of A* allows for the existence of $\mu \in M(X)$ with $A^*(\mu) = \delta_y$. In view of (1). This implies that $\mu \in \text{ext}(M(X))_1$. For some x ∈ X and $\beta \in C$, $\beta = \beta\delta_x$, where $|\beta| = l$. This means that $\delta_y = A^*(\beta\delta_x) = \beta\pi(x)\delta_{T(x)}$. Furthermore, it follows that $\beta = \overline{\pi(x)}$ and T(x) = y. Thus, T: X → Y is a continuous bijection and must be a homeomorphism. Let $f \in C(Y)$ and x ∈ X. Then

$$\delta_x(A f) = A^*(\delta_x)(f) = \pi(x)\delta_{T(x)}(f) = \pi(x)f(T(x)).$$

Thus, $(A f)(x) = \pi(x)f(T(x))$. Hance we prove the theorem.

Assume S is a subspace of the Banach space C(X) that separates points of X. Define Ls as a linear map from S to E. Assume the norm on S is defined by one of the following formulas:

(F_1) $\| f \| = \max \{\| f \|_\infty, \| L_s f \|\}$. for $f \in S$, where $\| . \|_\infty$, is the usual supremum normon C(X),

(F_2) $\| f \| = \| f \|_\infty + \| L_s f \|$, for $f \in S$,

(F_3) $\| f \| = \sup\{|f| + |L_s f(x)|: x \in \boldsymbol{X}\}$ $f \in S$, assume that E = C(X)

For example[9]. Ls: C^1[O,l] → C[O,l] may be defined as Ls(f) = f'. Similarly, Ls: AC[O, I] →L^1[O, 1] can be defined as Ls(f) = f'. AC[O, 1] refers to the space of all absolutely continuous functions on [0, 1].

Theorem 4.1.2: Assume X is a compact Hausdorff space, and S is a subspace of C(X).

(i) S is dense in C(X)

(ii) the norm on S is supplied by a map L: S → E via the formula (F_3).

(iii) S has the constant function 1 and $L_S(1) = 0$.

(iv) $\dim(L_S(S)) \geq 2$

(v) E is strictly convex

(vi) For any unimodular function $\theta \in S$ such that Ls(θ) = 0, the map $f \to f / \theta$ is a well-defined isometry of S onto itself.

Assume M is a subspace of C(Y), where Y is a compact Hausdorff space that meets conditions (i)–(vi). Any isometry A from S to M has the following form:

$$A(f) = \theta .\, f \circ T,\ f \in S$$

where T is a homeomorphism from Y onto X and $0 \in M$ is a unimodular function with $L_M(\theta) = 0$.

Proof. assumption (vi), it is sufficient to demonstrate that A(l) is a unimodular function on Y with $L_M(A(l))=0$. The P-property of an element g ∈ S is defined as ‖g‖ = 1 and $\beta \in \partial D$ such that

$$\|g+\beta f\| = \| g \| + \| f \| .$$

This property is retained by the isometry A. Assume f ∈ S and $\beta \in \partial D$, with $\| f \|_\infty = sup_{x\in X} \operatorname{Re}(\beta f(x))$. Using the norm definition for S, we get:

$$\| 1+\beta f \| = \| 1+\beta f \|_\infty + \| L_s f \| = 1 + \| f \|_\infty + \| L_s f \| = 1 + \| f \| .$$

To finish the proof, we need to show that if g ∈ S satisfies the P-property, it is a unimodular function on X with $L_S(g) = 0$. First, we prove that | g | = c on X given a constant c. Then, using the norm definition on S, we get $c = \| g \|_\infty = \| g \| = 1$

Assume that there exists a $x_0 \in X$ in which $| g(x_0) | < \| g \|_\infty$. Then assume $f \in S$, $\| f \|_\infty = \| g \|_\infty - | g(x_0) |$ and $| f(x) | \leq \| g \|_\infty - | g(x) | + 1/2(\| g_1 \|_\infty - |g(x_0)|)$, $x \in \boldsymbol{X}$ for $\beta \in \partial D$ it shows that $\| 1+\beta f \|_\infty \leq \| g \|_\infty + 1/2\| f \|_\infty$. Then

$$\|g+\beta f \| = \| g+\beta f \|_\infty + \| L_s(g+\beta f) \|$$

$$\leq \| g \|_\infty + 1/2\| f \|_\infty + \| L_s(g) \| + \| L_s(f) \|$$

$$= \| g \| + \| f \| - 1/2\| f \|_\infty < 1 + \| f \|.$$

Hence | g | equals constant. Assumption (iv) states that if L(g) is greater than zero, L_s(f) and L_S(g) are not proportionate. If h and h' are strictly convex, the equation $\| h + h' \| = \| h \| + \| h' \|$ holds proportionate, therefore for any $\beta \in \partial D$ all, we have

$$\|g+\beta f \| = \| g+\beta f \|_\infty + \| L_s(g+\beta f) \|$$

$$\leq \| g \|_\infty + \| f \|_\infty + \| L_s(g) \| + \| L_s(f) \|$$

$$= 1 + \| f \|.$$

This demonstrates that Ls(g) equals 0. Hence, the proof of the theorem is accomplished.

Proposition 4.1.1 A bounded linear operator on $L^P(X,E)$ with $1 \leq p < \infty$ can translate functions of (almost) disjoint supports to functions of (almost) disjoint supports. Then there is an n-set homomorphism ϕ of Y and a strongly [1]measurable map Ψ from X into B(E) such that

$$(Af)(.) = \Psi(.)\big(\phi(f)\big)(.) \quad \text{for every } f \in L^P(X, E).$$

Proof. Assume (e_n) is a countable, linearly independent subset of E with a dense linear span (K) within E. Let K_0 be the set of all linear combinations of (e_n) with complex rational coefficients. Assume A fulfill the Hypothesis for the proposition. Now we will fix $S \in Y$. Next, define the set function $\phi : Y \to Y$ as

$$\phi(S) = \cup_n supt(A(\chi_S e_n)).$$

$A(\mathcal{X}_{S_1} e_n)$ and $A(\mathcal{X}_{S_2} e_m)$ are virtually disjoint for all n and m when S_1 and S_2 are disjoint sets. So $\phi(S_1)$ and $\phi(S_2)$ are disjoint the equation $A(\mathcal{X}_{S_1} e_n) + A(\mathcal{X}_{S_2} e_n) = A(\mathcal{X}_{S_1 \cup S_2} e_n)$ suggests that $\phi(S_1 \cup S_2) = \phi(S_1) \cup$

$\phi(S_2)$ within a null set. This can be extended to countable unions of disjoint sets as A is continuous. The extension to countable unions of any set is straightforward. This demonstrates that ϕ is a set homomorphism. Let X_0 represent the kernel of ϕ is a set-isomorphism if and only if A is one-to-one, as its null space is the space of functions that vanishes (a.e.) on X. Assume u and 8 correspond to (6) and (7). We can select $\theta(x) > 0$ for every $x \in \phi(X)$ or $\theta(x) = 0$ for every $x \notin \epsilon\phi(X)$. Set $f_n = A(1_{e_n})$, where 1_{e_n} is the constant function on X, take values e_n. We can suppose that f_n (x) = 0 for all n and $x \notin \phi(X)$.. Define

$$\Psi(x)e_n = f_n(x) \quad \forall\ x \in X, \quad n \in N.$$

After extending $\Psi(x)e_n$ linearly to K, we obtain

$$\Psi(x)\left(\sum_{i=1}^{k} \lambda_i e_i\right) = \sum_{i=1}^{k} \lambda_i f_i(x).$$

Thus, for any $y \in K, \Psi(.)(y) = A(1_y)$ a.e. We will show that Ψ(x) is a bounded operator on E. Assume $S \in Y, S_1 = S \backslash X_0$, and $y \in K_0$. Then

$$\begin{aligned}\int_{\phi(S)} \| \Psi(x)y \|^P du(x) &= \int_{\phi(X)} \| A(1_y)(x) \|^P du(x)^{[11]} \\ &= \int \| A(\chi_{\phi(X)}y)(x) \|^P du(x) \\ &= \| \mathcal{X}_{S_1} y \|^P \\ &\leq \| A \|^P u(S_1) \| y \|^P \\ &= \| A \|^P \| y \|^P \int_{\phi(S)} (\theta(x))^P du(x)\end{aligned}$$

Thus $\| \Psi(x)y \| \leq \| A \| \| y \| |\theta(x)|, \quad x \in \phi(X)$. Now for $x \notin \phi(X)$ the inequality is trivial. So the null set S_0,

$$\| \Psi(x)y \| \leq \| A \| \| y \| |\theta(x)|, \quad y \in K_0, \quad x \notin S_0$$

If $y = \sum_{i=1}^{n} \lambda_i e_i$ and E_n is the linear span of e_1,..., e_n, then the restriction of $\varphi(x)$ to E_n is a linear map between two finite dimensional spaces and hence bounded. Additionally, because $K_0 \cap E_n$ is dense in E_m, the norm $\| \Psi(x) \| \leq \| A \| \theta(x)$ applies. This demonstrates that (8) holds for every $y \in K$, extending $\Psi(x)$ to a bounded linear function. Operator on E: $\| \Psi(x) \| \leq \| A \| \theta(x)$ computed Ψ(x) for x c. Finally, we demonstrate that w: X → B(E) is strongly measurable. To achieve this, let y ∈ E and $y_n \subset K$ be such that $y_n \to y$. The continuity of practically all Ψ (x) and A leads to the conclusion that $\Psi(.)y = A(1_y)$, indicating that w is measurable. Setting $(A_1 f)(.) = \Psi(.)\big(\phi(f)\big)(.)$ yields

$$\int \parallel (A_1 f)(x) \parallel^P du(x) \leq \int \parallel \Psi(x) \parallel^P \parallel (\phi(f))(x) \parallel^P du(x)$$

$$\leq \parallel A \parallel^P \int \big(\theta(x)\big)^P \parallel \phi(f)(x) \parallel^P du(x)$$

$$\leq \parallel A \parallel^P \parallel f \parallel^P$$

This demonstrates that A_1 is a bounded linear operator on L^P (X, E). It is established that A_1 agrees with A on constant functions. Additionally, since

$$A(\chi_S y) = \chi_{\phi(S)} A\big(1_y\big) = \Psi(.)\chi_{\phi(S)} y = A_1\big(\chi_{\phi(S)} y\big),$$

We infer that A agrees with A_1 on simple functions, which implies that $A = A_1$. This concludes the demonstration of the proposition.

Theorem 4.1.3. Let $p \neq 0$ and let A be an isometry of H^P into H^P. [9]Then there is a non-constant inner function T and a function Ψ in H^P such that

$$Af = \Psi C_T, \quad f \in H^P.$$

(1)

T and Ψ are related by

$$\int_S |\Psi|^P du = \int_S 1\backslash P(T) du \qquad S \in Y_T$$

(2)

where P is the Poisson kernel induced by T. Conversely, when a non-constant inner function T and a function Ψ in H^P are related by (2). (1) defines an isometry of H^P into H^P

Proof. Assume that A is an isometry from H^P into H^P, andset Ψ = A(1).

Then $\Psi \in H^P$ such that $\Psi \neq 0$ and w cannot disappear on any positive p-measure set. Let u be the measure for which $d\nu = |\Psi|^P du$. So ν and u are Mutually completely continuous. If a linear transformation L: $H^P \to L^P(u\nu)$ is defined by L f = A f 1 w, it is an isometry of H^P into L^P (u), and L(1) = 1.

Let fo be the inner function defined as $f_0(z) = z$. Then

$$\int |L(f_0^n)|^P d\nu = 1.$$

We get

$$\int |L(f_0^n)|^2 d\nu = 1.$$

This implies that $|L(f_0^n)|$ = 1 as p $\neq 0$. Thus L takes the algebra generated by fo into $L^\infty(\nu)$. From Proposition 4.1.1. (ii) we know that L is multiplicative on this algebra, therefore for a polynomial g we have $L\big(\mathrm{g}(f_0)\big) = \mathrm{g}(\mathrm{L}f_0)$

$$A\big(\mathrm{g}(f_0)\big) = \Psi C_T \mathrm{g}, \tag{3}$$

Where T = Lf_0. Since $\Psi \in H^P$, we have Ψ = F.G, where F is an inner function and G is an outer function in H^P. Furthermore, (3) suggests that F. T^n is an inner function for $n \geq 0$. Now we'll show that T is an inner function. Let A be the closed subspace of L^2 defined by $f_0^j T^k (j, k \geq 0)$. Jcc remains invariant when multiplied by f_0, but not by jo, as F(μ) is contained within H^2. Thus, $\mu = \theta(H^2)$, where | θ | = 1.Polynomials in f_0 and T may approximate f/TkB, which is in Jcc for j, k = 2, and 8. Since f_0 generates a dense algebra in H^P and A is bounded, This implies that

$$Af = \Psi f o T, \quad \forall \;\; f \in H^P,$$

where $\Psi \in H^P$ and T are non-constant inner functions. Additionally, for $f \in H^P$, we have

$$\int |\Psi|^P |foT|^P du = \int |\, f \,|^P \, du.$$

(4)

Let $S = T^{-1}(S_1)$, where $S_1 \in Y$. follows that

$$\int_S |\Psi|^P du = \int_{S_1} du$$

(5)

Because the characteristic function of S_1 can be approximated by the moduli of functions in H^P, and we get from (9),

$$\int_{S_1} du = \int_{S_1} 1\backslash P \; d\nu = \int_S 1\backslash P(T) du.$$

(6)

Thus, (5) and (6) denote the intended form (2). In contrast, if T is a non-constant inner function and $\Psi \in H^P$ is connected by (2), then (5) is a result of (1) and (6). Furthermore, if A is restricted to basic functions, (5) demonstrates that A, defined by (1). is anisometry between L^P and itself. Thus, A is an isometry of L^P.

A takesthe algebra formed by f_0 into H^P, hence it is obvious that A takes H^P into H^P. This concludes the proof.

Remark. The previous result does not apply to vector-valued functions, although the isometries of H^1(D) and H^∞(D) do. If E is both consistently convex and uniformly smooth
L in demonstrated that every is ometry [9]A of H^∞ (D, E) onto H^∞ (D, E) takes the form

$$Af = \Psi(C_{Tf}), \qquad f \in H^\infty \text{ (D, E)}$$

where Ψ is an isometry from E onto E and T is a conformal map of D onto itself.

4.2 Ergodic Theory And Composition Operators

Let G be a group with the identity e, and let X be any non-empty set. Assume u: G × X → X is a mapping such that u(e, x) = x and u(st, x) = u(s, u(t, x)) for any x ∈ X and s, t ∈ G. Then u is known as an action of G on X or a motion on X caused by G. If x ∈ X, the function u^x: G → X defined as $u^x(t)$ = u(t, x) is called a motion through the point x. The range of this function is called the orbit of x, denoted by the symbol orb(x). If t ∈ G, then the function u,: X → X defined as u_t = u(t, x) is a bijection, with $(u_t)^{-1} = u_{t^{-1}}$. If G is a topological group, X is a topological space, and the mapping u: G × X → X is continuous, [9]the triple (G, X, u) is considered a transformation group. The transformation group (ℤ,X,u) is known as discrete. A dynamical system is defined as (ℝ, X, u) where ℤ represents the discrete topology of integer addition and ℝ represents the typical topology of real number addition. Substituting $\mathbb{Z}^+$ and for ℤ and ℝ results is semidynamical systems.

Definition 4.2.1: Let (X, Y, u) be a probability measure space. The operator A on L^1(u) is considered doubly stochastic if:

(i) $Af \geq 0,\ \ when\ \ f \geq 0,$

(ii) $\int_X Af\, du = \int_X f\, du,$

(iii) $Af = f$, when f is constant.

If T: X → X is a measure-preserving transformation, then $\int_X C_T f\ du = \int_X f du T^{-1} = \int_X f du$, indicating that C_T is doubly stochastic. In standard Borel spaces, only the composition operators are doubly Some cases use v stochastic isometric operators. We will give the following theorem.

Theorem 4.2.1: Let (X, Y, u) be a standard Borel probability measure space. Let A be a doubly stochastic operator on $L^1(u)$ that is an isometry on $L^2(u)$. Then there exists a measure-preserving transformation T: X → X with A = C_T.
Proof. Let S ∈ Y. Then $A\chi_S \geq 0$ and $\int_X A\chi_S du = \int_X \chi_S du = u(S) \leq 1\ Then\ 0 \leq A\chi_S \leq 1$. As A is an isometry on $L^2(u)$, we have

$$\int_X (A\chi_S)^2 du\ = \langle A\chi_S, A\chi_S \rangle = \langle \chi_S, \chi_S \rangle$$

$$= \int_X \chi_S du = \int_X A\chi_S du. \qquad \{(A\chi_S)^2 = A\chi_S\}$$

finally,

$$u(S) = \int_X \chi_S du = \int_X C_T \chi_S du = uT^{-1}(S)$$

For each S ∈ Y, we conclude that T is measure-preserving. This concludes the proof of the theorem.

Note. If C_T is unitary on $L^P(u)$, the doubly stochastic operator A generates discrete measurable dynamical systems on X and on $L^P(u)$, p ≥ 1.

If T: X → X is a measure-preserving transformation, the family (T^n: n ∈ $\mathbb{Z}^+$) creates a discrete, measurable semidynamical system. It turns out. The orbit of practically every point in a measurable subset S of X has a non-empty intersection with S. Poincare's classical theorem demonstrates this.

Theorem 4.2.2. [The Poincare Recurrence Theorem]. Let T be a measure-preserving transformation on a finite measure space (X, Y, u) and S = Y. For practically any s ∈ S, there is n ∈ $\mathbb{Z}^+$ such that T^n(s) ∈ S.

Proof. Suppose the theorem's conclusion is incorrect.

$$F = \{s \in S\ :\ \ T^n \notin S, \qquad \forall\, n \in \mathbb{Z}^+\}$$

has non-zero measure

$$F = S \cap T^{-1}(X \backslash S) \cap T^{-2}(X \backslash S) \dots \dots \dots$$

If x ∈ F, then T^n (x) ∉ F for every n ∈ ℕ. Hence, F ∩ T^{-1}(F) = 0 for all n ∈ ℕ. Because T is measure preserving and u(X) <∞ , we have a contradiction With this contradiction, the proof of the theorem is complete.

Examples of Measure Preserving Transformations :

(i) Assume X = ℝ and u is the Lebesgue measure. Define T_t : ℝ → ℝ as T_t (x) = x + t, where x ∈ ℝ. The family (T_t : t ∈ ℝ is a collection of measure preserving transformations that leads to quantifiable dynamical systems ℝ and L^P(u) for p ≥ 1

(ii) Let X = [0, 1] and Y be the a-algebra of all Borel sets. Assume 0 < a < 1 and T_a (x) represents the fractional part of x + a. T_a : X → X is a measure-preserving transformation.

Corollary 4.2.1: Let T be a measure-preserving transformation on a probability measure space (X, Y, u), and let $f \in L^P(u)$. If the sequence (g_n) converges to g in the L^P-norm, then g is a fixed point in C_T.Where

$$g_n = 1\backslash n \sum_{k=0}^{n-1} C_T^k f \ , \qquad \forall\ n \in \mathbb{N}$$

Outline for the evidence. Let $\varepsilon > 0$. Then there exists a $f' \in L^\infty(u)$ such that $\| f - f' \| < \varepsilon\backslash 4$. Let $g_n' = 1\backslash n \sum_{k=0}^{n-1} C_T^k f'$. According to the convergence theorem the sequence { g_n' } converges to $\bar{f}'$ in the L^P-norm, with $\bar{f}'(x) = \lim_n \sup g_n'(x)$. Now

$$\| g_n - g_{n+m} \|_P \leq \| g_n - g_n' \|_P + \| g_n' - g_{n+m}' \|_P + \| g_{n+m}' - g_{n+m} \|_P$$

$$\leq \varepsilon\backslash 4 + \varepsilon\backslash 2 + \varepsilon\backslash 4 = \varepsilon,$$

for an appropriate option of n. Since $\{g_n\}$ is a Cauchy sequence, there exists a g e L^P(μ) such that $g_n \to$ g in the L^P norm. C_Tg can be proven to equal g.

Definition 4.2.1: Let T be a measure-preserving transformation on a measure space (X, Y, μ). T is considered ergodic if T^{-1}(S) = S, which means that either μ(S) = 0 or μ(X \ S) = 0. A doubly stochastic operator A on L^1(μ) is considered ergodic if its sole fixed points are constant functions (i.e. $Af = f, f \in L^1$(μ), implying that f is a constant function a.e.).

Theorem 4.2.3: Let T be a measure-preserving transformation on a probability measure space (X,Y.μ).If the composition operator C_T is ergodic, then T is also ergodic.

Proof. Assume the composition operator C_T is ergodic and that T^{-1}(S) = S, where S e. The equation $\chi_{T^{-1}(S)} = \chi_S$• leads to $C_T\chi_S = \chi_S$·, indicating that $\chi_S = c$ a.e. given a constant c. This implies that μ(S) = 0 or μ(X \ S) = 0. This demonstrates that T is ergodic. SupposeT is ergodic. Assume $C_T f = f$ for $f \in L^1$(μ). Let $k \in \mathbb{Z}$ and n $\in \mathbb{Z}^+$. Let

$$X_n^k = \{x \in X \ : k\backslash 2^n \leq f(x) < k + 1/2^n\}.$$

Then $T^{-1}(X_n^k) = X_n^k$. Hence $u(X_n^k) = 0$ or $u(X\backslash X_n^k) = 0$. Since $X = \bigcup_{k\in\mathbb{Z}} X_n^k \quad \forall\, n.$

Definition 4.2.1: An ergodic transformation that preserves the measure T has a discrete spectrum if the orthonormal basis for L^2(μ) is made up of C_T eigenfunctions. T_1 and T_2 are said to be conjugate if there exists an a-algebra automorphism Φ on Y/y such that $\Phi h_{T_1} = \Phi h_{T_2}$, where Φh_T is an a-homomorphism induced by T.

4.3. Homomorphisms And Composition Operators

For compact Hausdorff spaces X and Y, C(X) and C(Y) are Banach algebras of continuous complex-valued functions with supremum norm topology. C(X) and C(Y) are c•-algebras with maximal ideal spaces homeomorphic to X and Y, respectively. If T: Y→ X is a continuous map, then we know that it induces the composition operator Cr: C(X) → C(Y), which is an •-homomorphism. It is discovered that every non-zero •-homomorphism from C(X) to C(Y) is a composition operator.

Theorem 4.4.1. Let X be a compact Hausdorff space and T:X → X be a continuous map. If μ is a probability measure on Borel subsets of X, then T is a measure-preserving transformation with respect to u.

Proof (outline). Set P = {μ ∈ M(X): μ ≥ 0 and ‖ u ‖= 1}. The Banach-Alaoglu theorem states that P is w*-compact. It is possible to show that P is $C_T{}^*$ -invariant, non-empty convex subset of M(X). According to the Kakutani Markov fixed point theory, there exists a μ ∈ P such that

$$\int_X C_T f\, du = F_u(C_T f) = (C_T^* u)(f)$$

$$= F_u(f)$$

$$= \int_X f\ du.$$

It shows that

$$u(T^{-1}(S)) = u(S)$$

for each Borel set S. This demonstrates that T is a transformation that preserves the measure µ. With this, the proof outline is complete.

CHAPTER 5

Conclusion

Finally, this thesis looked into the composition operator on functional spaces. These conclusions rely on sophisticated mathematical notions from functional analysis, specifically measure-preserving transformations, composition operators, and Hardy spaces. The document finishes with a theorem demonstration and proofs concerning measure-preserving transformations and other mathematical features. The document's conclusions are summarised below. This study has provided us with a thorough comprehension of the mathematical ideas underlying thecomposition operator on function spaces. The composition operator has numerous and diverse uses. The capacity to compose operators in thefrequency domain has resulted in function spaces in communication systems, allowing for measure-preserving transmission and better function spaces.

Bibliography

1. Abramovich, Yuri A., Multiplicative Representaion of Disjointness Preserving Operators, Mathematics Proceeding A. 86 (1983), 265-279.
2. Adams, G.T., The Bilateral Bergman Shift, Memoirs Amer. Math. SOC., 355 (1986). 01-75.
3. Baillon, J.B., Un Theoreme de Type Ergodique Pour less Contractions Non Lineares dans un Espace de Hilbert, C. R. Acad. Sci. Paris, 280 (1975), 1511-1514.
4. Banach, S., Theorie des Operations Lineaires, Monografie Matematyczne, Warsaw, 1932.
5. Cambern, M., and K. Jarosz, Isometries of Spaces of Weak*-Continuous Functions, Proc. Amer. Math. SOC., 106 (1989), 707-712.
6. Campbell, J.T., and J.E. Jamison, On Some Classes of Weighted Composition Operators, Glasgow Math. J., 32 (1990), 87-94.
7. Kamowitz, H., R.K. Singh and D. Wortman, Compact Composition Operators on Weighted Spaces of Continuous Functions @reprint).
8. Kamowitz, H., and D. Wortman, Compact Weighted Composition Operators on Sobolev Related Spaces, Rocky Mountain J. Math., 17 (1987), 767-782.
9. Kolaski, C.J., Surjective Isometries of Weighted Bergman Spaces, P roc. Amer. Math. SOC., 105 (1989). 652-657.
10. MacCluer, B.D., Compact Composition Operators on $H^P(B_N)$ Michigan Math. J., 32 (1985), 237-248.
11. Pathania, R.S., A Study of Composition Operators and Multiplication Operators on Functional Banach Spaces, Thesis, University of Jammu, 1991.

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