CONTRIBUTIONS TO SOME TOPICS IN THE THEORY OF SIGNED GRAPHS

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by

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under the supervision of

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DECLARATION

This is to certify that the thesis entitled **"Contributions to Some Topics in the Theory of Signed Graphs"** which is being submitted for the award of the Degree of **Doctor of Philosophy** to the Department of Applied Mathematics, Delhi Technological University, Delhi, is based on theoretical work carried out by me under the supervision of Dr. Mukti Acharya and Dr. Sangita Kansal. The results embodies in the thesis have not been submitted to any other University or Institution for the award of any degree or diploma. This thesis does not contain other person's data, graphs or other information, unless specifically acknowledged.

Date:

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CERTIFICATE

This is to certify that the thesis entitled **"Contributions to Some Topics in the Theory of Signed Graphs"**, submitted by **Ms. Rashmi Jain** to the Department of Applied Mathematics, Delhi Technological University, Delhi, for the award of the Degree of ***Doctor of Philosophy*** is a bonafide record of research work carried out by her under our supervision. The contents of the thesis, in full or parts, have not been submitted to any other Institution or University for the award of any degree or diploma.

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GENERAL INTRODUCTION

There are several reasons for the acceleration of interest in graph theory. It is worth to mention that there are applications of graph theory in various areas of physics, chemistry, communication science, computer technology, electrical and civil engineering, architecture, operational research, genetics, psychology, sociology and economics. Graph theory is also intimately related to many branches of mathematics, including group theory, matrix theory, numerical analysis, probability, topology, combinatorics etc. The fact is that graph theory serve as a mathematical model for any system involving a binary relation. Due to diagrammatic representation, graphs have an intuitive and aesthetic appeal. Although there are many results in this field of an elementary nature, there is also an abundance of problems with enough combinatorial subtlety to challenge the most sophisticated mathematician.

Signed graph theory forms one of the most vibrant areas of research due to its link with behavioral and social sciences as evident from the published literature; research-level journals like the Journal of Mathematical Sociology, Journal of Mathematical Psychology, Social Networks, Journal of Mathematical Chemistry etc., are only a few notable ones that can be mentioned in this context. In our excursion of research, we were mainly driven to carry out work in the area of signed graphs derived under some operations, which mainly deals with the structural reconfigurations of the structure of dynamical systems under prescribed rules and rules are designed to deal with a variety of interconnections among the elements of the system. We were able to obtain some theoretical results which led us to the area of research in the theory of signed graphs with a hope to build necessary conceptual resources for applications.

1.1 Brief History of Graph Theory

Graph theory may be said to have its beginning in 1736 when Leonhard Euler solved, or rather proved unsolvable, the Königsberg bridge problem. Euler (1701-1782) became the father of graph theory as well as topology when in 1736 he settled a famous unsolved problem, called the Königsberg Bridge Problem. There were two islands linked to each other and to the banks of the Pregel river by seven bridges. The problem was to begin at any of the four land areas, walk across each bridge exactly once and return to the starting point.

One can easily try to solve this problem empirically but all attempts must be unsuccessful, for the tremendous contribution of Euler in this case was negative. In proving that the problem is unsolvable, Euler replaced each land area by a point and each bridge by a line joining the corresponding points, thereby producing a "graph" (as shown in **Figure 1.1**).

The paper by Leonhard Euler [28] on seven bridges of Königsberg (published in 1736) is regarded as the first paper in the history of

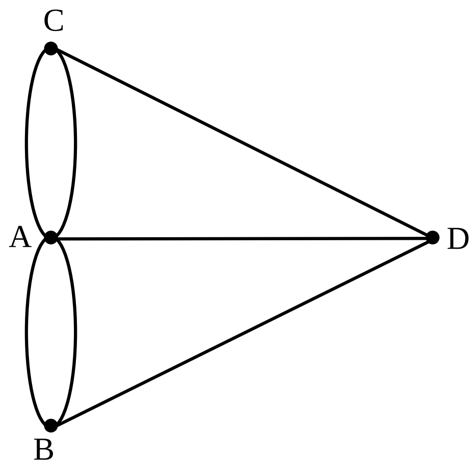

Figure 1.1: The graph of the Königsberg Bridge Problem

graph theory. It took 200 years before the first book on graph theory was written. This was Theorie der endlichen und unendlichen Graphen (Teubner, Leipzig, 1936) by König in 1936. For the history of early graph theory, see N.L. Biggs, R.J. Lloyd and R.J. Wilson, Graph Theory 1736-1936, Clarendon Press, 1986. Euler's formula relating the number of edges, vertices and faces of a convex polyhedron was studied and generalized by Cauchy [22] and L'Huillier [47]. It was the origin of combinatorial topology.

Another important factor of common development of graph theory and topology came from the use of the techniques of modern algebra. The first example of such a use comes from the work of the physicist Gustav Kirchhoff, who published in 1845 his Kirchhoff's circuit laws for calculating the voltage and current in electric circuits. More than one century after Euler's paper on the bridges of Königsberg, Kirchhoff [43] developed the theory of trees in order to solve the system of simultaneous equations which give the current in each branch and around each circuit of electric network.

Although a physicist, he thought like a mathematician when he abstracted an electrical network with its resistances, condensers, inductances etc. and replaced it by its corresponding combinatorial structure consisting only of points and lines without any indication of the type of electrical element represented by individual lines. Thus, in effect, Kirchhoff replaced each electrical network by its underlying graph and showed that it is not necessary to consider every cycle in the graph of an electric network separately in order to solve the system of equations. Instead, he pointed out by a simple but powerful construction that the independent cycles of a graph determined by any of its "spanning trees" will suffice.

In 1857, Cayley [23] discovered the important class of graphs called trees by considering the changes of variables in the differential calculus. Later, he was engaged in enumerating the isomers of the saturated hydrocarbons C_nH_{2n+2}, with a given number n of carbon atoms. Of course, Cayley restated the problem as to find the number of trees with p points in which every point has degree 1 or 4. Enumerative graph theory then rose from the results of Cayley and the fundamental results were published by Pólya between 1935 and 1937 and the generalization of these were given by De Bruijn [26]. In 1857, Sir William Hamilton invented a game consisting of a solid regular dodecahedron, in which 20 vertices are labeled by the names of some cities and the objective of game was to find a path visiting all cities exactly once. This concept is used in solving very famous travelling salesman problem.

The most famous unsolved problem in graph theory and perhaps in all of mathematics was the celebrated Four Color Conjecture (FC-C). This problem remained unsolved for more than a century. In 1969 Heinrich Heesch [37] published a method for solving the problem using computers. A computer-aided proof produced in 1976 by Kenneth Appel and Wolfgang Haken (see [13–15]) makes fundamental use of the notion of "discharging" developed by Heesch. The proof involved checking the properties of 1,936 configurations by computer and was not fully accepted at the time due to its complexity. A simpler proof considering only 633 configurations was given twenty years later by Robertson, Seymour, Sanders and Thomas. The FCC had an interesting history, but its origin remains somewhat vague. There have been reports that Möbius was familiar with this problem in 1840, but it is only definite that the problem was communicated to De Morgan by Guthrie, an apprentice to a geographer in Germany, in about 1850. The first of many erroneous proofs of the conjecture was given by Kempe in 1879. In particular, the term graph was introduced by Sylvester [68] in a paper published in Nature.

The psychologist Lewin [46] proposed that the "life space" of an individual can be represented by a planar map. It was pointed out that Lewin was actually dealing with graphs. This viewpoint led the psychologist at the Research Center for Group Dynamics, University of Michigan, interpretation of a graph as a 'sociogram', in which people are represented by vertices and their interpersonal relations by edges.

People related to theoretical physics discovered graph theory for its own purpose. In the study of statistical mechanics by Uhlenbeck [70], vertices represent the molecules and two adjacent vertices indicate nearest neighbor interaction of some physical kind, for example, magnetic attraction or repulsion. In a similar interpretation by Lee and Yang [45], the vertices stand for small cubes in the Euclidean space, where each cube may or may not be occupied by a molecule. Then, two vertices are adjacent whenever both spaces are occupied.

In computer science, graphs are used to represent networks of communication, data organization, computational devices, the flow of computation etc. One practical example: The link structure of a website could be represented by a directed graph. The vertices are the web pages available at the website and a directed edge from page A to page B exists if and only if A contains a link to B. A similar approach can be taken to problems in travel, biology, computer chip design and many other fields.

Graph theory is also widely used in sociology as a way, for example, to measure actor's prestige or to explore diffusion mechanisms, notably through the use of social network analysis softwares. Under the umbrella of Social Network graphs there are many different types of graphs: Starting with the Acquaintanceship and Friendship Graphs, these graphs are useful for representing whether n people know each other. Next, there is the influence graph. This graph is used to model whether certain people can influence the behavior of others. Finally there's a collaboration graph which models whether two people work together in a particular way.

In mathematics, graphs are useful in geometry and certain parts of topology, e.g., Knot Theory. Algebraic graph theory has close links with group theory. A graph structure can be extended by assigning a weight to each edge of the graph. Graphs with weights or weighted graphs, are used to represent structures in which pairwise connections have some numerical values. For example, if a graph represents a road network, the weights could represent the length of each road. A network can be defined as a graph in which nodes and/or edges have attributes (e.g. names). Network theory is a part of graph theory. Network theory has applications in many disciplines including statistical physics, particle physics, computer science, electrical engineering, biology, economics, operations research, climatology and sociology. Applications of network theory include logistical networks, the World Wide Web, Internet, gene regulatory networks, metabolic networks, social networks, epistemological networks etc.

In nut shell, in real life, to study the dynamics of any system it is necessary to know the interaction pattern among the submodules of the system with the description of how precisely one is able to represent the positive and negative aspects of various links interconnecting the submodules. In most of the engineering and technological systems, a proper understanding of such networks called generally the structural (modular) configuration of the systems, is essential in their proper operation by means of taking care of various risk factors, optimal operating conditions, maintenance etc.

1.2 Preliminaries

Formally, a combinatorial graph or simply a graph is a discrete structure (some times called network, depending on the context) formed by a set V whose elements are called vertices and a collection E, which is essentially a set unless mentioned otherwise, of elements called edges, each of which is a 2-subset of elements of V. Generally, a graph is an ordered pair $G = (V, E)$, where $V = V(G)$ and $E = E(G)$ are the vertex set and the edge set of the graph G respectively. For standard terminology and notations in graph theory, we follow Harary [31] and [73]. Further, G is called finite if V is a *finite* set and *infinite* otherwise. A finite and an infinite graphs are shown in **Figure 1.2**. Unless mentioned otherwise, all graphs to be treated in this thesis are finite.

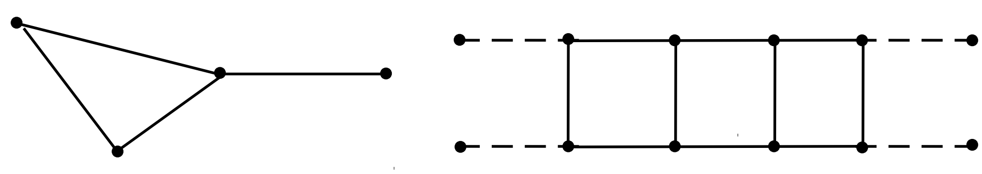

Figure 1.2: Finite and infinite graphs

A *walk* in a graph G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, ..., e_n, v_n$, beginning and ending with vertices, in which every edge is preceded and followed by two adjacent vertices. It is closed if $v_0 = v_n$. A closed walk in which all the vertices are distinct is called *cycle*. It is written as $C_n = (v_1, v_2, ..., v_n, v_1)$. A closed walk in which all the edges are distinct is called a *circuit*.

All graphs to be treated in this thesis are simple, connected and finite. A *cut-vertex* of a graph is one whose removal leaves the graph disconnected. For a graph G, $C(G)$ denotes its cut-vertex set. A *non-separable graph* is connected, nontrivial and has no cut vertices. A *block* of a graph is a maximal nonseparable subgraph. A *clique* of a graph is its maximal complete subgraph.

Signed Graphs

In most of the applications, one needs to assign weights to the edges of a graph, whence one calls such a 'weighted' configuration an undirected network; weights as such may be taken from any arbitrary set of labels which might, in particular, be real numbers or subsets of a set or elements of an algebraic structure such as a group or a semigroup, etc., depending on real-life applications for which network models have to be constructed. We are fascinated by the variety of applications of networks in which each edge carries an element of the involutory group underlying the Galois field $GF(2) = \{-1, 1\}$ as label; in literature, such a network has been referred to as a *signed graph* or simply a *sigraph* (e.g., see [4, 5, 50, 77]).

Formally, a signed graph is an ordered pair $S = (S^u, \sigma)$, where $S^u := G = (V, E)$ is a graph called the underlying graph of S and $\sigma : E(S^u) \longrightarrow \{+, -\}$ is a function, called the *signature* (or *sign* in short) of S. Alternatively, the signed graph can be written as $S = (V, E, \sigma)$, where V, E and σ are in the above sense (see [24, 33]). Signed graphs first were introduced by Harary [33]. Further, $E^{+}(S) =$

$\{e \in E(S^u) : \sigma(e) = +\}$ and $E^{-}(S) = \{e \in E(S^u) : \sigma(e) = -\}$. The elements of $E^{+}(S)$ ($E^{-}(S)$) are called *positive* (*negative*) *edges* of S and the set $E(S) = E^{+}(S) \cup E^{-}(S)$ is called the edge set of S.

In a pictorial representation of a signed graph S, its positive edges are shown as solid line segments ('Jorden curves' drawn on the plane) and negative edges as dashed line segments (see [50]). An example of a signed graph is displayed in **Figure 1.3**, where solid line segments represent edges that are assigned the weight '+' and broken line segments represent those that are assigned the weight '−'.

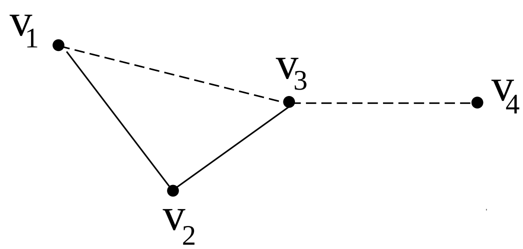

Figure 1.3: A signed graph S

A signed graph in which all the edges are positive, is called *all-positive signed graph* (*all-negative signed graph* is defined similarly). A signed graph is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise. Thus, one can treat a graph as a signed graph in which each edge is positive. A *positive (negative) section* in a signed graph S means a maximal connected subgraph of S consisting only positive (negative) edges of S; in particular, a positive (negative) section in a heterogeneous cycle of S is essentially a maximal all-positive (all-negative) path in the cycle [30].

This natural generalization of graphs arose while building a prototype model of cognitive interpersonal relationship in a social group, where the individuals in the group are taken as vertices and interpersonal relationship between any two individuals A and B is treated as an edge, a negative edge AB is interpreted as a negative relationship (such as enmity, dislike etc.) and a positive edge AB is interpreted as a positive relationship (such as friendly, like etc.) between A and B. Such a model was first presented by Harary [33], while voluminous literature on this model is still accumulating.

The positive (negative) degree of a vertex is the number of positive (negative) edges incident that vertex. By $d(v)$, we denote the degree of $v \in V(S)$, $d(v) = d^+(v)+d^-(v)$, where $d^+(v)$ ($d^-(v)$) denote the positive (negative) degree of v. A vertex v of even (odd) degree is called *even* (*odd*) vertex. The *edge degree* of an edge uv, denoted by $d_e(uv)$, is the total number of edges adjacent to uv. Clearly, $d_e(uv) = d(u)+d(v)-2$.

The *negation* of a signed graph S, denoted $\eta(S)$, is obtained by negating the sign of every edge of S, i.e., by changing the sign of every edge to its opposite [32]. An example of a signed graph and its negation are shown in **Figure 1.4**.

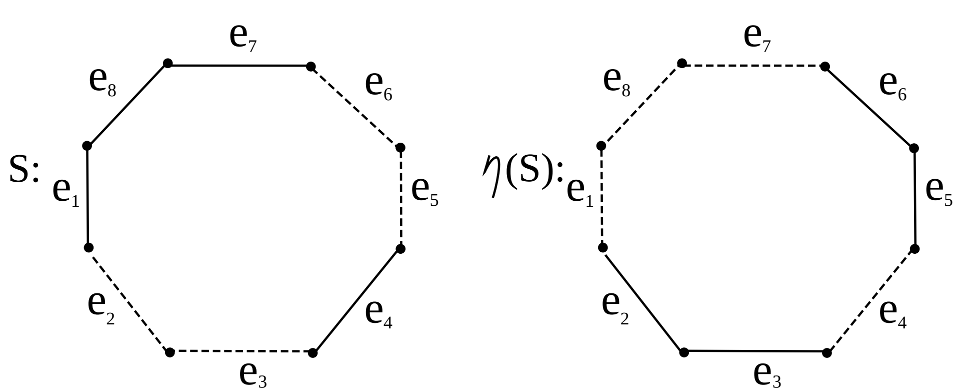

Figure 1.4: A signed graph S and its negation $\eta(S)$

A cycle in a signed graph is said to be positive (negative) if the product of the signs of its edges is positive (negative); that is, it contains an even (odd) number of negative edges. A signed graph is said to be *balanced*, if every cycle in it is positive [33]. An example of balanced signed graph S is exhibited in **Figure 1.5**. A signed graph S is said to be *totally unbalanced* if every chordless cycle in S is negative.

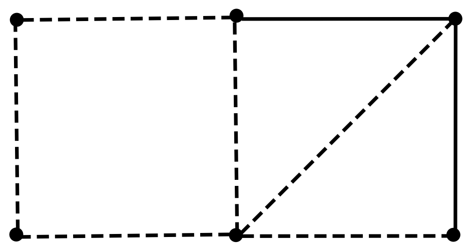

Figure 1.5: A balanced signed graph S

Signed graphs provide models for investigating balance in connection with various kinds of social relations. Since empirical social networks always involve uncertainty because of errors due to measurement, imperfect observation or sampling, it is desirable to incorporate uncertainty into signed graph models. Frank and Harary [29] introduced a stochastic signed graph and investigated the properties of some indices of balance involving triads. In particular, they considered the balance properties of a graph which is randomly signed and of one which has been randomly sampled from a large population graph.

Harary [33] observed that signed graphs serve as apt prototype models for the study of the notion of structural balance in a social group endowed with dyadic interactions, generalizing such considerations earlier by Heider [38] in triads; Harary [33] generalized the Heider's notion of cognitive balance in social systems as basically a structural feature of the underlying network configured in such a way that every cycle of the network contains an even number of negative edges (e.g., see [4, 33, 36, 50]). A *chordless cycle* of a graph G is a cycle in G that has no cycle chord, i.e., the cycle is an induced subgraph.

Harary derived the following structural criterion called partition criterion for balance in signed graphs:

Theorem 1.2.1. [33] *A signed graph S is balanced if and only if its vertex set $V(S)$ can be partitioned into two subsets V_1 and V_2 (one of them possibly empty) such that every positive edge joins two vertices in the same subset and every negative edge joins two vertices from different subsets.*

The social groups corresponding to V_1 and V_2 in Theorem 1.2.1 may be regarded as subgroups of similar ideologies in the social system. Thus, a social system is called balanced if all relations between people are positive or if we can divide the group into two subgroups so that every positive relation occurs between individuals in the same subgroup and every negative relation occurs between individuals in different subgroups. No additional restriction applies if neither a positive nor a negative relation (i.e., indifference) exists between individuals in the same or different subgroups. Thus, tendency towards balance in any social system, implying that an unbalanced system contains excessive stress or tension. The system tends to adjust so as to relieve the tension. For example, certain individuals within the group changing their points of view. Thus, there is a tendency for the group to split into two factions such that within each faction the only relations are positive and between factions the only relations are negative.

Theorem 1.2.2. [33] *A signed graph S is balanced if and only if for each pair of distinct vertices u and v in S, all paths joining u and v have the same sign.*

Cartwright and Harary [21] studied such considerations in the wider aspect and it led to the formulation of independent theory of social networks. In the recent literature, the study can be seen by Doreian and Mrvar [27] and Zheng et. al. [79]. Harary and Kabell developed a simple algorithm to get balanced signed graphs and also enumerated them (see [34, 35]).

The following important lemma on balanced signed graph is given by Zaslavsky:

Lemma 1.2.1. [75] *A signed graph in which every chordless cycle is positive, is balanced.*

A *marking* of S is a function $\mu : V(S) \longrightarrow \{+, -\}$. The following characterization of balanced signed graphs given by Sampathkumar [54], is well known:

Theorem 1.2.3. *A signed graph S is balanced if and only if there exits a marking μ of S such that every edge uv of S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

Originally, the notion of 'marking' vertices of a graph $G = (V, E)$ was envisaged by Sampathkumar [54] as a vertex analogue of the notion of 'signing' the edges of G. In fact, Sampathkumar tried to produce an analogue of the notion of balanceness in a signed graph by defining a marked graph (G, μ) to be *p-balanced* if every connected component of G contains an even number of negative vertices. The notion of balanceness was independently discovered by Katai and Iwai [41]. They gave an elegant algorithm to derive minimum and minimal 'balancing sets' of an arbitrary signed graph S whose underlying graph is planar, where by a balancing set of S they meant a set of edges of S which when negated results into a balanced signed graph S'.

The cardinality of a minimum balancing set of a signed graph S in general is called the *frustration index* (see [77]), for its connection with another famous long-standing unsolved problem called the 'Ising Problem' in the study of stability of energy levels in ferromagnetic materials (see [42, 69, 71, 72]). Subsequently, Acharya and Acharya [5] proposed an extension of the Katai-Iwai procedure to produce minimum and minimal balancing sets of an arbitrary sigraph S using certain linear algebraic method, the success of which lies in the truth of a conjecture posed in that paper and which still stands unresolved.

In the intervening period of all the above development, Beineke and Harary [17, 18] came up with the right analogue of the notion of balanceness in a signed graph S, called *consistency* in marked graphs. Motivated from Harary's characterizations of balance in a signed graph (signed digraph; [36]), they called a marked digraph (D, μ) consistent if for any two vertices u and v in D, all $u-v$ walks have the same sign, where the sign of a walk is defined as the product of the marks of its vertices. For a symmetric digraph or equivalently for a graph G this turns out to be equivalent to saying that it is consistent if every cycle in G contains an even number of negative vertices. Interestingly, while they obtained a neat characterization of consistent marked digraphs, they threw the open problem of characterizing consistent marked graphs. Acharya [2] came up with a first characterization of consistent marked graphs by transforming the problem to a problem on signed graphs and then solving the same. Almost simultaneously, Rao [49] came up with a constructive characterization of consistent marked graphs, besides some tedious algorithms to deciding whether a graph is consistent. Subsequently, a much simpler and elegant characterization of bipartite consistent marked graphs was obtained by Acharya [3] and more handy characterization of a consistent marked graph in general was obtained by Hoede [39].

Sampathkumar introduced the idea of marking the vertices with signs derived from the signs of the edges in [54], given as

$$\mu_\sigma(v) := \prod_{vw \in E(S)} \sigma(vw).$$

This marking is called canonical marking. In this thesis, a vertex $v \in V(S)$ of $d^-(v)$ even or $\mu_\sigma(v) = +$ is called *positive vertex* (*negative vertex* is defined similarly).

Lemma 1.2.2. [54] *In any canonically marked signed graph there are an even number of vertices marked negative.*

A cycle in a marked signed graph S_μ is said to be *consistent* if it contains an even number of negative vertices with respect to canonical marking μ and a marked signed graph S_μ is said to be consistent if every cycle in it is consistent [59]. An example of consistent signed graph is shown in **Figure 1.6**.

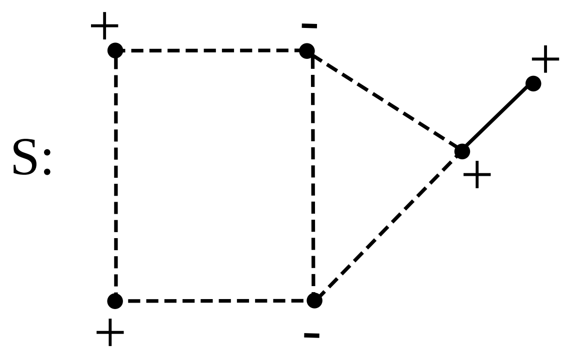

Figure 1.6: A consistent signed graph S

Similarly, a cycle of a signed graph is called *canonically consistent (or $\mathcal{C}$-consistent)* if it contains an even number of negative vertices with respect to canonical marking and a signed graph is said to be $\mathcal{C}$-consistent if every cycle in it is $\mathcal{C}$-consistent. A canonically consistent signed graph S is shown in **Figure 1.7**.

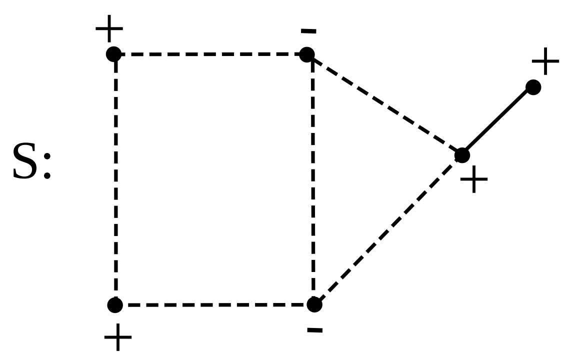

Figure 1.7: A canonically consistent signed graph S

The concept of consistency was motivated by communication networks [53]. If binary messages are sent through a network with vertices having negative marking, reversing messages and vertices having positive marking, leaving them unchanged, then a consistent marked graph has the following consistency property: If a message is sent from the vertex u to the vertex v through two different vertex disjoint paths and u, v have same signs, then v will receive the same message no matter which path is followed. In similar manner, consistent marked graphs have utility in social networks, networks whose vertices are people. If some people always lie and some always tell the truth, a consistent social network has the property that if a message is sent from u to v and they have the same sign, then v will receive the same message independent of the path followed.

The problem of characterizing consistent marked graphs was subsequently settled by Acharya [2, 3] and Rao [49]. Roberts [51] discussed the problem of characterizing graphs that can be consistently marked using at least one negative sign, reduced the problem to blocks and solved it for blocks whose longest cycle has length at most five. Further, Roberts [52] established a relation between balanced signed graphs and consistent marked graphs. Roberts and Xu [53] investigated several characterizations of a consistent marked graph. Very recently, Joglekar et al. [40] gave an algorithmic characterization of consistency in marked graphs.

Acharya and Sinha obtained consistency of signed graphs that satisfy certain signed graph equations in [7, 59, 66]. Also, Sinha and Garg have discussed consistency of several signed graphs in [60, 61, 63–65]. Recently there has been new interest in the canonical vertex signature in connection with derived signed graphs from a signed graph, in particular a line signed graph (see [6,7,57,78]). Sinha and Garg have established structural characterizations of signed graph S so that its line signed graphs $L(S)$ and $\times$-line signed graphs $L_\times(S)$ are $\mathcal{C}$-consistent in [60].

A marked signed graph S_μ is called *cycle-compatible* if for every cycle Z in S, the product of signs of its vertices equals the product of signs of its edges, i.e.,

$$\prod_{e \in E(Z)} \sigma(e) = \prod_{v \in V(Z)} \mu(v). \tag{1.1}$$

Signed graphs S_1 and S_2 are said to be *isomorphic*, written as $S_1 \cong S_2$, if there is a graph isomorphism $f : S_1^u \to S_2^u$ that preserves edge signs.

The idea of *switching* of a signed graph was introduced by Abelson and Rosenberg [1] in connection with structural analysis of social networks and may be formally stated as follows: Switching of S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $\mathbb{S}_\mu(S)$ and is called *μ-switched* signed graph or just switched signed graph.

Further, a signed graph $S_1 = (S_1^u, \sigma)$ *switches* to a signed graph $S_2 = (S_2^u, \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written as $S_1 \sim S_2$ whenever there exists a marking μ of S_1 such that $\mathbb{S}_\mu(S_1) \cong S_2$ (see **Figure 1.8**).

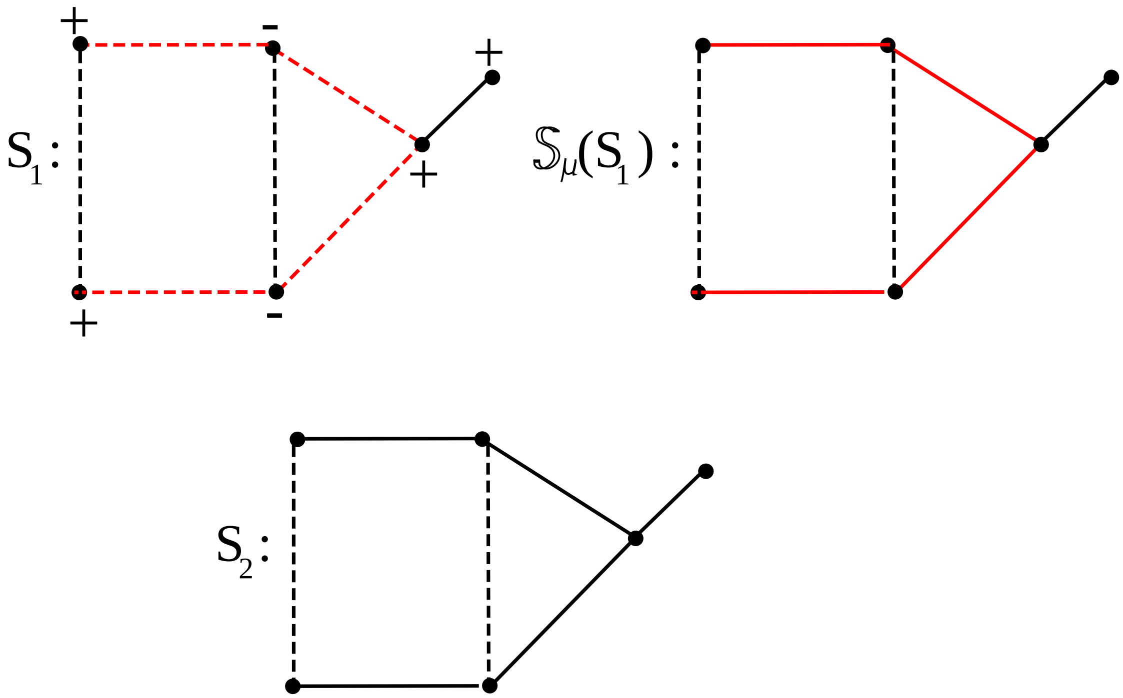

Figure 1.8: Two signed graphs S_1 and S_2 such that $S_1 \sim S_2$

Since the definition of switching does not change the underlying graphs of the respective signed graphs, $S_1 \sim S_2$ implies that $S_1^u \cong S_2^u$. Deeper mathematical aspects, significance and connections of this notion to other fields may be found in the mathematical bibliography by Zaslavsky [76].

Two signed graphs S_1 and S_2 are said to be *weakly isomorphic* (see [67]) or *cycle isomorphic* (see [74]) if there is a graph isomorphism $f : S_1^u \to S_2^u$ such that the sign of every cycle Z of S_1 equals the sign of $f(Z)$ in S_2 (i.e., f preserves both vertex adjacencies and the signs of the cycles of S_1 and S_2).

The concept of cycle (weak) isomorphism is illustrated in the Figure **Figure 1.9**. For example, in this Figure, cycles (u_1, u_4, u_5) and

(v_1, v_4, v_5) are negative and cycles (u_3, u_5, u_6) and (v_3, v_5, v_6) are positive.

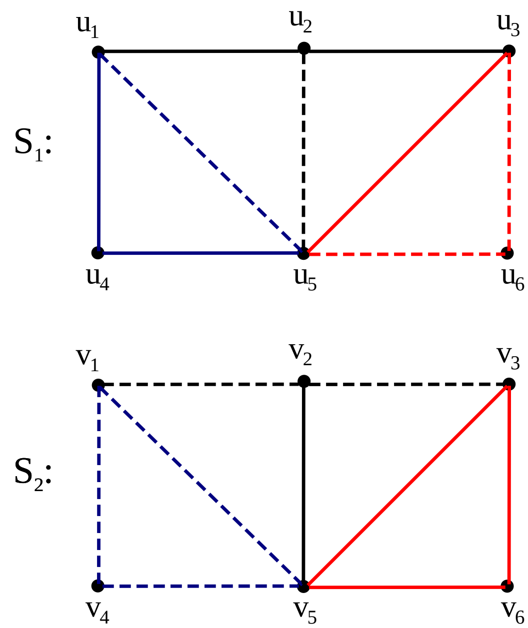

Figure 1.9: Two cycle isomorphic signed graphs

The following result is well known:

Theorem 1.2.4. [67, 74]] *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

The concept of switching of signed graphs is closely related to the concept of structural balance of signed graphs as evident from the following theorem:

Theorem 1.2.5. *A signed graph $S = (S^u, \sigma)$ is balanced if it is switching equivalent to its underlying graph S^u.*

Suppose B is a set and $F = \{B_1, B_2, ..., B_k\}$ is a nonempty family of distinct nonempty subsets of B whose union is B; the ordered pair

(B, F) is called a *hypergraph* [19, 20]. The *intersection graph* of F is denoted by $\Omega(F)$ and is defined by $V(\Omega(F)) = F$, with B_i and B_j adjacent whenever $i \neq j$ and $B_i \cap B_j \neq \phi$.

The *line graph* of a graph $G = (V, E)$, denoted $L(G)$, is the intersection graph $\Omega(E(G))$ [31]. *Line-cut (or, in short, lict) graph* of a graph G, denoted by $L_c(G)$, is a graph whose vertex set is $E(G) \cup C(G)$, where $C(G)$ is the set of cut vertices of G, in which its two vertices are adjacent if they correspond to adjacent edges of G or one corresponds to an edge e_i of G and the other corresponds to a cut-vertex c_j of G such that e_i is incident with c_j and *litact graph* of a graph $G = (V, E)$, denoted here $L_{ct}(G)$, is the graph having vertex set $E(G) \cup C(G)$ in which its two vertices are adjacent if the corresponding members of G are adjacent or incident [44].

There are three notions of a *line signed graph* of a signed graph $S = (S^u, \sigma)$ in the literature, viz., $L(S)$, $L_\times(S)$ and $L_\bullet(S)$, all of which have $L(S^u)$ as their underlying graph; only the rule to assign signs to the edges of $L(S^u)$ differ. Every edge ee' in $L(S)$ is negative whenever both the adjacent edges e and e' in S are negative [16], an edge ee' in $L_\times(S)$ has the product $\sigma(e)\sigma(e')$ as its sign [9] and an edge ee' in $L_\bullet(S)$ has $\mu_\sigma(v)$ as its sign, where $v \in V(S)$ is a common vertex of edges e and e' [8] and $\mu_\sigma(v)$ is the product of signs of the edges incident to v.

Mathad and Narayankar [48] extended the definition of lict graph to lict signed graph as follows:
The *lict signed graph* (for convenient, we call this signed graph here product-lict signed graph or $\times$-lict signed graph) of a signed graph $S = (S^u, \sigma)$, is the signed graph $L_{\times_c}(S) = (L_c(S^u), \sigma')$, where for every

edge uv of $L_c(S^u)$

$$\sigma'(uv) = \begin{cases} \sigma(u)\sigma(v) & \text{if } u, v \in E(S); \\ \sigma(u) & \text{if } u \in E(S) \text{ and } v \in C(S) \end{cases}$$

Acharya et al. in [11,12] introduced the lict signed graph $L_c(S)$ and the $\bullet$-lict signed graph $L_{\bullet_c}(S)$ as follows:
The *lict signed graph* of a signed graph $S = (S^u, \sigma)$, is a signed graph $L_c(S) = (L_c(S^u), \sigma')$, where for every edge uv of $L_c(S^u)$

$$\sigma'(uv) = \begin{cases} -, & \text{if } u \in E^{-}(S) \text{ and } v \in E^{-}(S) \text{ or negative cut-vertex}; \\ +, & \text{otherwise.} \end{cases}$$

The dot-lict signed graph (or $\bullet$-*lict signed graph*) of a signed graph $S = (S^u, \sigma)$, is a signed graph $L_{\bullet c}(S) = (L_c(S^u), \sigma')$, where for every edge uv of $L_c(S^u)$

$$\sigma'(uv) = \begin{cases} \mu_\sigma(p), & \text{if } u, v \in E(S) \text{ and } p \text{ is their common vertex}; \\ \mu_\sigma(v), & \text{if } u \in E(S) \text{ and } v \in C(S) \end{cases}$$

The *jump graph* of a graph G is a graph $J(G) = (E(G), E')$, where $ee' \in E'$ if and only if e and e' are non-adjacent edges of G [25].

The *jump signed graph* (c.f.: [10]) of a signed graph $S = (G, \sigma)$ is a signed graph $J(S) = (J(G), \sigma')$ where for any edge ee' in $J(G)$, $\sigma'(ee') = \sigma(e)\sigma(e')$

The *semitotal line graph* $T_1(G)$ of a graph $G = (V, E)$ is the graph whose vertex set is $V \cup E$ and two vertices are adjacent in G if and only if they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it [55].

The *semitotal signed graph* [62] of a signed graph $S = (G, \sigma)$ is a signed graph $T_1(S) = (T_1(G), \sigma')$, where for any edge uv of $T_1(G)$,

$$\sigma'(uv) = \begin{cases} \sigma(u)\sigma(v), & \text{if } u, v \in E(S); \\ \sigma(v), & \text{if } u \in V(S) \text{ and } v \in E(S)\ . \end{cases}$$

1.3 Overview of the Thesis

The thesis comprises eight chapters including the last chapter on future scope. The thesis has been organized as follows:

In **Chapter 1**, we not only introduce the basic terminology and notations required to go through the thesis and that are not commonly found in standard textbooks of Graph Theory but also mention fundamental or well known concepts and results which are required to understand the work reported in succeeding Chapters.

In **Chapter 2**, we characterize lict graphs. We hope, this result will have far-reaching applications in understanding the dynamics of ad-hoc networks.

The main concern of **Chapter 3** is to define two new types of lict signed graphs of a signed graph S that are lict signed graph $L_c(S)$ and dot-lict signed graph $L_{\bullet_c}(S)$ and to establish structural characterizations of line-cut signed graphs $L_c(S)$, $L_{\times_c}(S)$, $L_{\bullet_c}(S)$ and also line signed graphs $L_\times(S)$ and $L_\bullet(S)$.

In **Chapter 4**, we characterize signed graphs on K_p, $p \geq 2$, on cycle C_n and on $K_{m,n}$ which are $\bullet$-lict signed graphs or $\bullet$-line signed graphs and we also characterize signed graphs S so that $L_{\bullet_c}(S)$ and $L_\bullet(S)$ are balanced. We establish the characterization of signed graphs S for which $S \sim L_{\bullet_c}(S)$, $S \sim L_{\bullet}(S)$, $\eta(S) \sim L_{\bullet_c}(S)$ and $\eta(S) \sim L_{\bullet}(S)$, here $\eta(S)$ is negation of S and $\sim$ stands for switching equivalence. Similarly, we characterize signed graphs on K_p, $p \geq 2$, on cycle C_n and on $K_{m,n}$ which are lict signed graphs or line signed graphs and we also characterize signed graphs S so that $L_c(S)$ and $L(S)$ are balanced. We establish the characterization of signed graphs S for which $S \sim L_c(S)$, $S \sim L(S)$, $\eta(S) \sim L_c(S)$ and $\eta(S) \sim L(S)$.

Motivated from the paper entitled canonical consistency of signed line structures (see [60]), in **Chapter 5**, we establish structural characterizations of signed graph S so that $L_{\bullet}(S)$ is $\mathcal{C}$-consistent and $\mathcal{C}$-cycle compatible. We extend the notion of cycle compatible signed graph to $\mathcal{C}$-cycle compatible signed graph (*see* **Figure 1.10**) as follows:

A signed graph S is $\mathcal{C}$-cycle compatible if for every cycle Z in S,

$$\prod_{e \in E(Z)} \sigma(e) = \prod_{v \in V(Z)} \mu_{\sigma}(v).$$

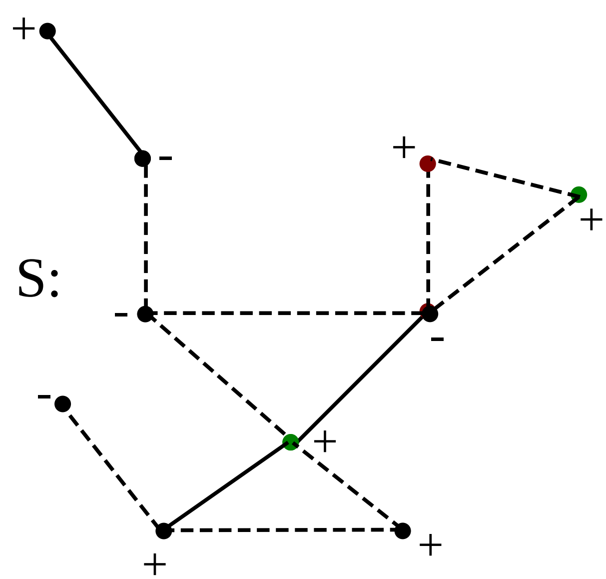

Figure 1.10: A $\mathcal{C}$-cycle compatible signed graph S

Sampathkumar and Walikar [56] introduced the concept of splitting graph of a graph. The splitting graph of a graph G, denoted here $\mathfrak{S}(G)$, is formed as follows:

Take a copy of G and for each vertex v of G, take a new vertex v'. Join v' to all adjacent vertices of v. Sinha et al. introduced splitting signed graph $\Gamma(S)$ of a signed graph S. In **Chapter 6**, we introduced splitting signed graph $\mathfrak{S}(S)$ of a signed graph S and we establish structural characterizations of signed graph S for which $\mathfrak{S}(S)$ is balanced, $\mathcal{C}$-consistent, $\mathfrak{S}(S)$ and $\Gamma(S)$ are isomorphic and $\mathcal{C}$-cycle compatible. We also establish a characterization of $\mathfrak{S}$-splitting signed graphs.

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Seenivasan and Lourdusamy [58] introduced a new type of graph labeling known as vertex equitable labeling. They studied the properties of this labeling and investigated vertex equitable behaviors of some standard graphs. In **chapter 7**, we initiate vertex equitable labeling of signed graphs and study a vertex equitable behavior of signed paths, signed stars, signed bistars and signed complete bipartite graphs $K_{2,n}$.

In **Chapter 8**, we present some of the problems which we encountered during the entire course of investigation reported in the foregoing chapters and some how or the other remained unsolved or fully investigated; their inclusion here, we hope, will provide impetus for powering our future research endeavors. However, any reader of this thesis is most welcome to tackle them.

Lastly, we must add here, the fact that all the major results contained in this thesis are either published or communicated.

1.4 Key to Cross-References and Citations in the Thesis

Books, articles and other reading material referred in each of the eight chapters of the thesis, are given at the end of each chapter under References section. Bibliography given at the end of the thesis, contains almost all the relevant research or review material in addition to the reference material. None of the references included in the bibliography is referred directly in the text of the thesis.

For cross-referencing and citations to results of the thesis, we encounter a reference in the form '$PX.Y.Z$', where 'P' is the title of the statement such as 'Theorem', 'Corollary', 'Problem', 'Lemma', etc., 'X' is a number varying between 1 to 8 representing chapters, 'Y', 'Z' are natural numbers.

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Chapter 2

LICT GRAPHS

The line-cut (or, in short, lict) graph of a graph $G = (V, E)$ is the intersection graph $\Omega(E(G) \cup C(G))$, where $C(G)$ is the set of cut-vertices of G. In this chapter, we establish a structural characterization of lict graphs.

2.1 Introduction

In communications and electronics, especially in telecommunications, interference is anything which modifies, or disrupts a signal as it travels along a channel between a source and a receiver. The term typically refers to the addition of unwanted signals to a useful signal. The line graph of a graph $G = (V, E)$ is the intersection graph $\Omega(E(G))$. A wireless adhoc network can be modeled as a graph $G = (V, E)$. Two links $e, f \in E(G)$ cannot be simultaneously active if they are incident to the same node. This type of interference is called primary interference. Primary interference occurs if two nodes use the same channel and one is inside the transmission region of the other. The graph that models this interference is the line graph of G. This interference model is well studied in [4, 7] and more details about this application can be found in [3]. In this chapter, we characterize lict graphs and hope that this concept will facilitate to analyze wireless networks.

2.2 Line graph and Lict graph

The *line graph* of a graph G, denoted $L(G)$, is the graph in which edges of G are represented as vertices, two of these vertices are adjacent if the corresponding edges are adjacent in G, i.e., the line graph $L(G)$ of a graph $G = (V, E)$ without isolates may be considered as the intersection graph $\Omega(E(G))$ [5].

A graph G is said to be a line graph if there exists a graph H whose line graph $L(H)$ is isomorphic to G. Following theorem is the well-known characterization of line graphs originally due to Beineke [2].

Theorem 2.2.1. [5] *The following statements are equivalent:*

(1) *G is a line graph.*

(2) *The lines of G can be partitioned into complete subgraphs in such a way that no point lies in more than two of these subgraphs.*

(3) *G does not have $K_{1,3}$ as an induced subgraph and if two odd triangles have a common edge then the subgraph induced by their vertices is K_4.*

(4) *None of the nine subgraphs shown in* **Figure 2.1** *is an induced subgraph of G.*

A triangle in a graph is said to be odd if there is a vertex in the graph adjacent to an odd number of vertices of the triangle.

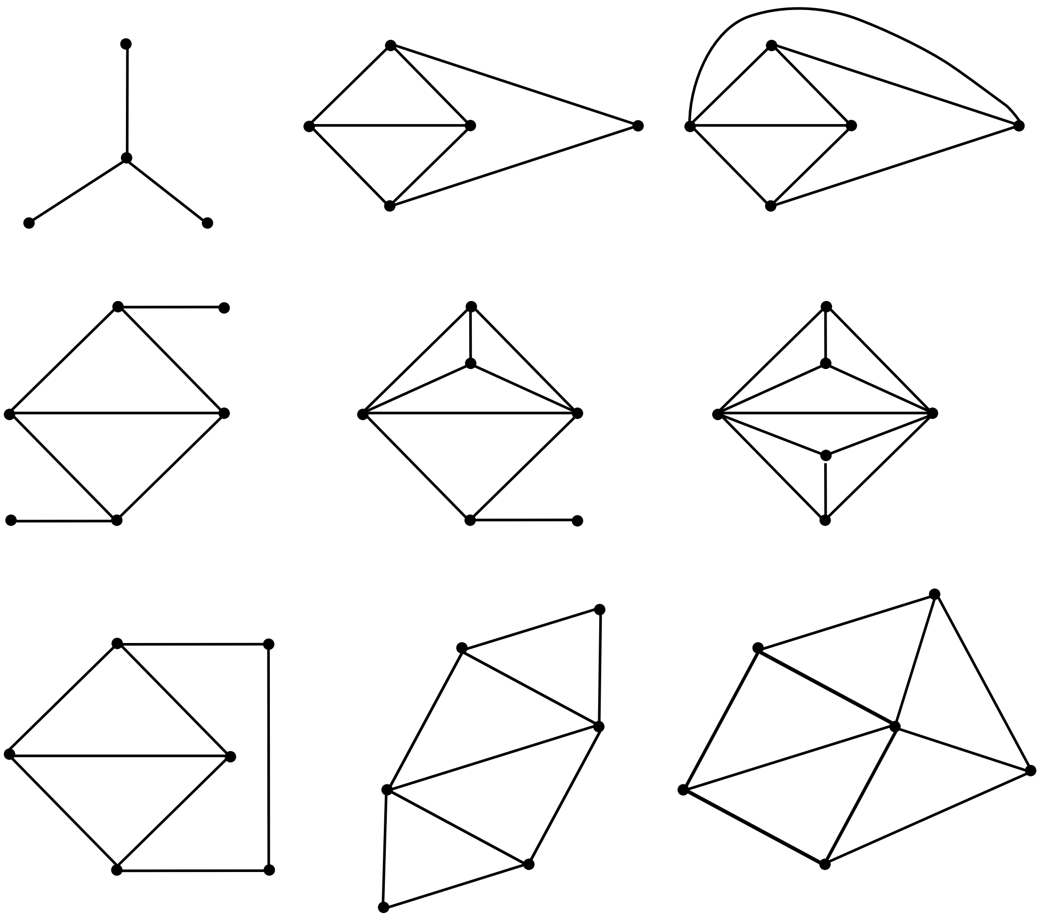

Figure 2.1: The nine forbidden subgraphs for line graphs

Recall that few years back Kulli and Muddebihal [6] introduced the idea of line-cut (or, in short, lict) graph of a graph:
The *line-cut (or, in short, lict) graph* of a graph G, denoted by $L_c(G)$, is a graph whose vertex set is $E(G) \cup C(G)$, where $C(G)$ is the set of cut vertices of G, in which its two vertices are adjacent if they correspond to adjacent edges of G or one corresponds to an edge e_i of G and the other corresponds to a cut-vertex c_j of G such that e_i is incident with c_j, that is, $L_c(G)$ is the intersection graph $\Omega(E(G) \cup C(G))$, where $C(G)$ is the set of cut-vertices of G.

A graph G is said to be a lict graph if there exists a graph H whose lict graph $L_c(H)$ is isomorphic to G, i.e., $L_c(H) \cong G$. Clearly, $L(G) \cong$

$L_c(G)$ if $C(G) = \phi$.

Line graph and lict graph of a graph G are shown in **Figure 2.2**.

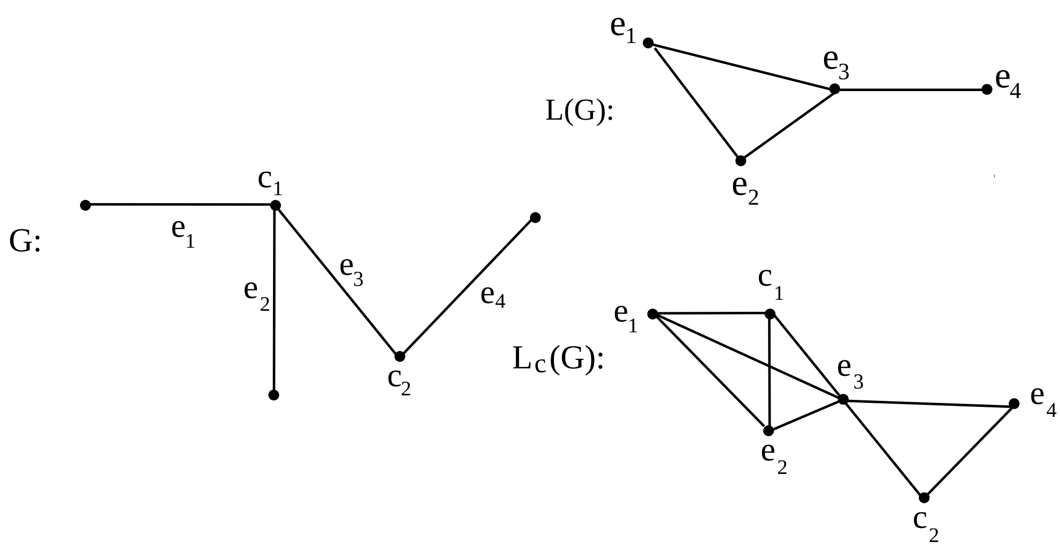

Figure 2.2: A graph G, its line graph $L(G)$ and lict graph $L_c(G)$

2.3 Characterization of Lict graphs

Motivated from the result on characterization of line graphs, we have given the characterization of lict graphs. Recall that a clique of a graph is its maximal complete subgraph. The following theorem is the main result of this chapter as reported in [1].

Theorem 2.3.1. *The following statements are equivalent:*

(1) *$G = (V, E)$ is a lict graph.*

(2) *The edges of G can be partitioned into cliques in such a way that no vertex lies in more than two of these cliques and for each clique G',*

(i) *if each vertex of G' lies in two cliques of the partition, then $G - E(G')$ is connected; and* *(ii)* *if atleast one vertex v of G' does not lie in another clique of the partition then $G - E(G') - v$ is disconnected.*

Proof. **(1)** $\Rightarrow$ **(2)** Let G be a lict graph. Therefore, $G \cong L_c(H)$ for some graph H. Without loss of generality we assume that G has no isolated vertex. By the definition of lict graph, the edges incident on a vertex v of H with $d(v) = p$, that is not a cut-vertex, induces a clique of G of order p. The edges incident on a cut-vertex c of H with $d(c) = p$ induce a clique of G of order $p+1$ having c as one of its vertices. Since every edge of G either results from two adjacent edges of H or from a cut vertex of H and an edge of H that is incident with that cut vertex, every edge of G is contained in precisely one such clique. This is illustrated in **Figure 2.3**.

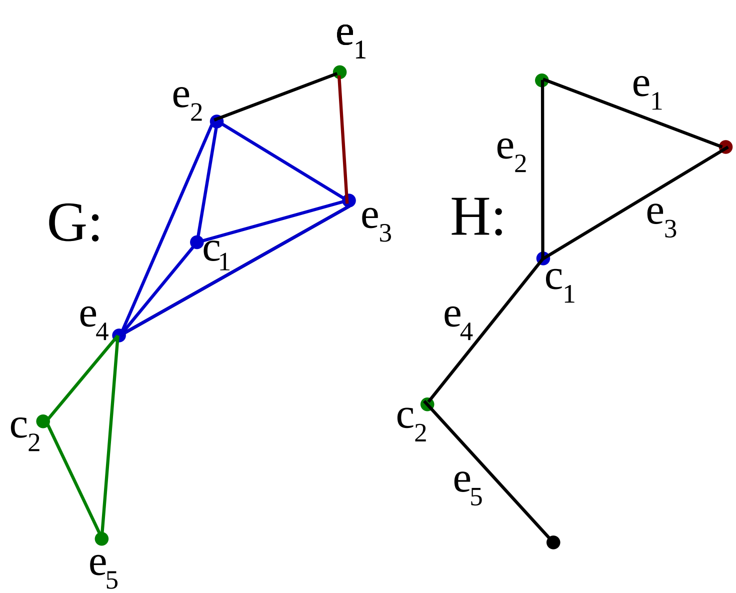

Figure 2.3: A graph G and a graph H such that $G \cong L_c(H)$

Note that $V(G) = E(H) \cup C(H)$, where $C(H)$ is the set of cut vertices of H. Clearly, if e_i is a pendant edge of H then the corresponding vertex in G is contained in only one clique. If e_i is a non-pendant edge of H then the corresponding vertex in G is contained in precisely two cliques. No vertex of G can be contained in more than two of the cliques. Thus, the edges of G can be partitioned among cliques of G in such a way that no vertex of G lies in more than two cliques.

For any clique G_i under the edge partition of G, the following two statements are true.

Statement 1: If each vertex of G_i lies in two cliques, then G_i is induced by the edges incident with a non cut vertex of H.

Statement 2: If some vertices (say m) of G_i are also not contained in another clique then one of these vertices corresponds to a cut-vertex c of H and the remaining $m - 1$ vertices correspond to pendant edges of H that are incident on c.

If each vertex of G_i is contained in two cliques, then by Statement 1, G_i is induced by the edges incident with a non cut-vertex v of H. Suppose vertex v_j of G_i is also contained in clique G_j. Then the clique G_j must result from a vertex of H belonging to $N(v)$, the neighborhood of v in H. Because $H - v$ is connected, then $G - E(G_i)$ is connected, which is Condition 2(i) of the theorem.

In **Figure 2.4** we give a graph G that does not satisfy condition 2(i) and is not a lict graph.

G:

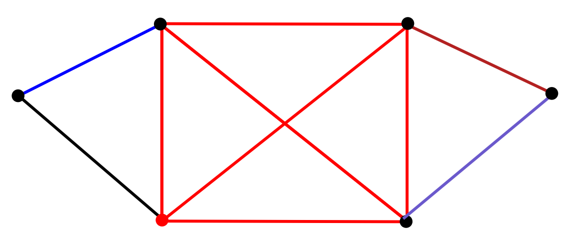

Figure 2.4: A graph G that is not a lict graph

If precisely one vertex v of G_i does not lie in two of the cliques, then by Statement 2, G_i is induced by the edges incident with a cut-vertex v

of H. Suppose any other vertex v_j of G_i is also contained in the clique G_j. Then G_j must result from a vertex v of H belonging to $N(v)$. Because $H - v$ is disconnected, then $G - E(G_i) - v$ is disconnected. Hence, G_i cannot be a pendant edge and G cannot contain a pendant vertex. This is Condition 2(ii) of the theorem.

In **Figure 2.5** we give graphs that do not satisfy Condition 2(ii) and are not lict graphs.

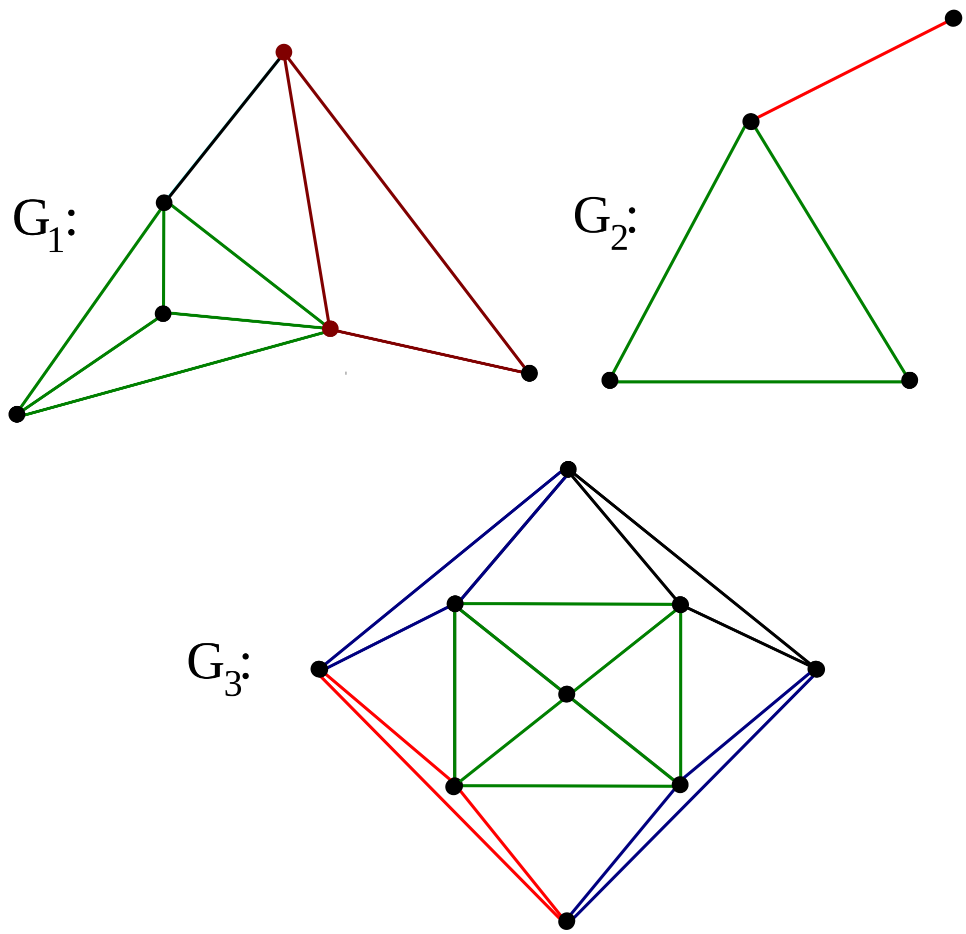

Figure 2.5: Graphs that are not lict graphs

(2) ⇒ (1) Let $\mathcal{P}(G) = \{G_1, G_2, ..., G_n\}$ be a partition of $E(G)$ that satisfies the Condition **(2)**. Note that the ordered pair $(G, \mathcal{P}(G))$ is a hypergraph. We provide the construction of a graph H whose lict graph is G. The vertices of H correspond to the set $\mathcal{P}(G)$ together with the set U of vertices of G that belong to only one of the cliques G_i except one such vertex for each G_i. Thus, $V(H) = \mathcal{P}(G) \cup U$ and two of these vertices are adjacent whenever they have a nonempty intersection. That is, H is the intersection graph $\Omega(\mathcal{P}(G) \cup U)$. For this graph H, $G \cong L_c(H)$. Hence, G is a lict graph.

Figure 2.6 illustrates the construction of a graph H such that $G \cong L_c(H)$ for a graph G that satisfies condition **(2)** of the Theorem. □

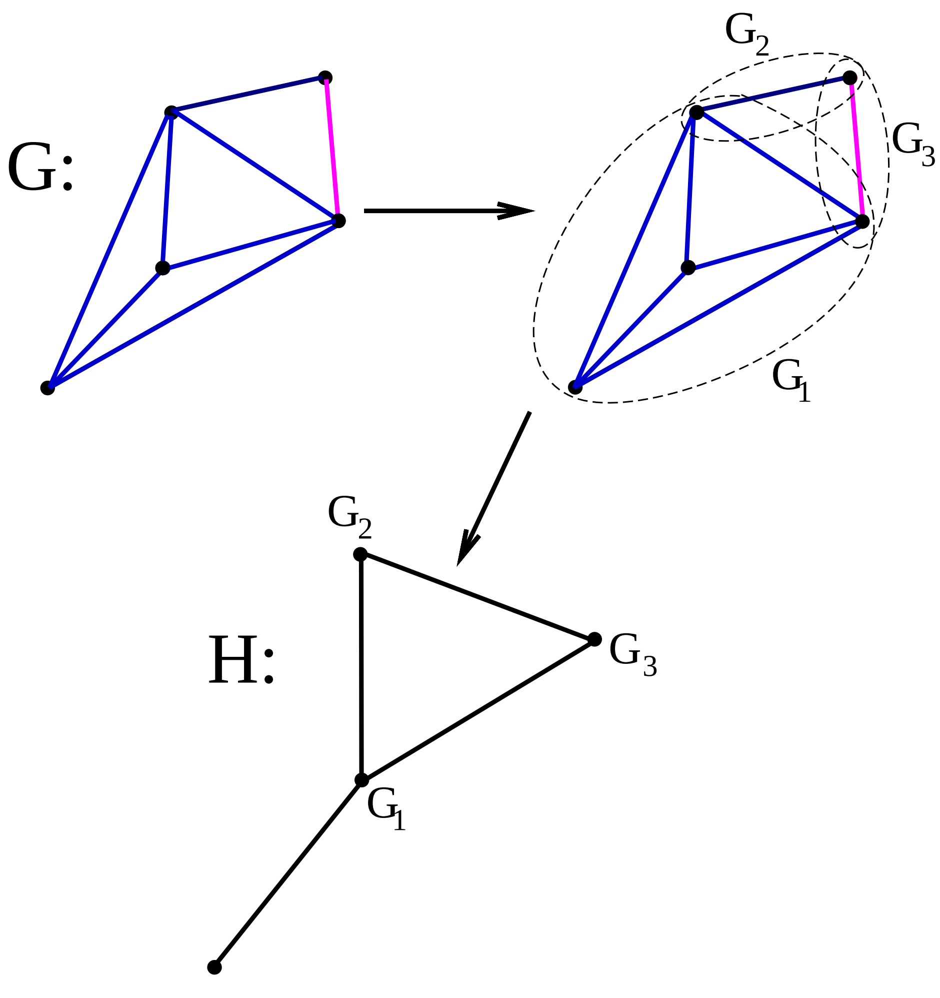

Figure 2.6: A graph G and a graph H such that $G \cong L_c(H)$

2.4 Conclusion and Scope

In this chapter, we have established a characterization of lict graphs. To the best of our knowledge, the characterization of lict graphs was long awaited since the publication of [6]. In the same paper they have defined the litact graph of a graph G as follows:

The litact graph of a graph $G = (V, E)$ is the graph having vertex set $E(G) \cup C(G)$, in which two vertices are adjacent if they correspond to adjacent edges of G or one corresponds to an edge e_i of G and the other corresponds to a cut-vertex c_j of G such that e_i is incident with c_j or two adjacent cut-vertices of G. Clearly, litact graph of a graph G is isomorphic to lict graph of G if G has no two adjacent cut-vertices.

Thus the problem of characterizing litact graph is still an open problem.

* * * * *

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Chapter 3

LICT SIGNED GRAPHS AND THEIR CHARACTERIZATIONS

In [10], Mathad and Narayankar extended the definition of lict graph to product-lict signed graph. We define two types of lict signed graphs of a signed graph S that are $L_c(S)$ and $L_{\bullet_c}(S)$. In the previous chapter we have established characterization for lict graphs. In this chapter structural characterizations of lict signed graphs $L_c(S)$, $L_{\times_c}(S)$, $L_{\bullet_c}(S)$ and also for line signed graphs $L_\times(S)$ and $L_\bullet(S)$ have been established.

3.1 Introduction

Recall that there are three notions of a *line signed graph* of a signed graph $S = (S^u, \sigma)$ in the literature, viz., $L(S)$, $L_\times(S)$ and $L_\bullet(S)$, all of which have $L(S^u)$ as their underlying graph; only the rule to assign signs to the edges of $L(S^u)$ differ. Every edge ee' in $L(S)$ is negative whenever both the adjacent edges e and e' in S are negative [7], an edge ee' in $L_\times(S)$ has the product $\sigma(e)\sigma(e')$ as its sign [3] and an edge ee' in $L_\bullet(S)$ has $\mu_\sigma(v)$ as its sign, where $v \in V(S)$ is a common vertex of edges e and e' [1] and $\mu_\sigma(v)$ is the product of the signs of the edges incident to v. Note that for a graph G, $L(G) \cong L_\times(G) \cong L_\bullet(G)$ as G is an all-positive signed graph. For an all-negative signed graph S without pendent edges in which every vertex is positive, $L_\times(S) \cong L_\bullet(S) \cong \eta(L(S))$ and all are all-positive.

Proposition 3.1.1. [3] *For any signed graph S, its $\times$-line signed graph $L_\times(S)$ is balanced.*

Figure 3.1 illustrates a signed graph and its line, $\times$-line and $\bullet$-line signed graphs.

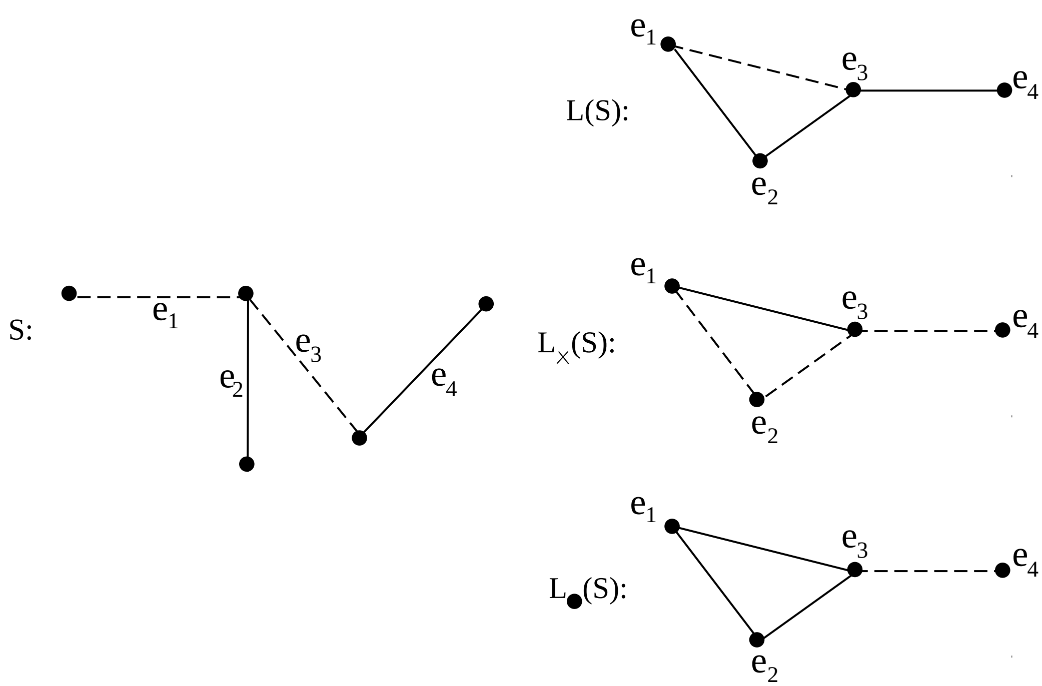

Figure 3.1: A signed graph S and its $L(S)$, $L_\times(S)$ and $L_\bullet(S)$

A signed graph S is said to be a line ($\times$-line or $\bullet$-line) signed graph if there exists a signed graph T whose line ($\times$-line or $\bullet$-line) signed graph $L(T)$ ($L_\times(T)$ or $L_\bullet(T)$) is isomorphic to S and this signed graph T is called the line ($\times$-line or $\bullet$-line) root of S.

In the previous chapter we have discussed lict graph of a graph. Recall that Mathad and Narayankar [10] extended the definition of lict graph to lict signed graph as follows:

The *lict signed graph* (for convenient, we call this signed graph product-lict signed graph or $\times$-lict signed graph) of a signed graph $S = (S^u, \sigma)$, is the signed graph $L_{\times_c}(S) = (L_c(S^u), \sigma')$, where for every edge uv of $L_c(S^u)$

$$\sigma'(uv) = \begin{cases} \sigma(u)\sigma(v), & \text{if } u, v \in E(S); \\ \sigma(u), & \text{if } u \in E(S) \text{ and } v \in C(S) \end{cases}$$

Proposition 3.1.2. [10] *For a signed graph S, its $\times$-lict signed graph $L_{\times_c}(S)$ is balanced.*

Acharya et al. in [4, 5] introduced the lict signed graph $L_c(S)$ and the $\bullet$-lict signed graph $L_{\bullet_c}(S)$ for a signed graph S as follows:
The *lict signed graph* of a signed graph $S = (S^u, \sigma)$, is a signed graph $L_c(S) = (L_c(S^u), \sigma')$, where for every edge uv of $L_c(S^u)$

$$\sigma'(uv) = \begin{cases} -, & \text{if } u \in E^{-}(S) \text{ and } v \in E^{-}(S) \text{ or negative cut-vertex}; \\ +, & \text{otherwise.} \end{cases}$$

The dot-lict signed graph (or $\bullet$-*lict signed graph*) of a signed graph $S = (S^u, \sigma)$, is a signed graph $L_{\bullet c}(S) = (L_c(S^u), \sigma')$, where for every edge uv of $L_c(S^u)$

$$\sigma'(uv) = \begin{cases} \mu_\sigma(p), & \text{if } u, v \in E(S) \text{ and } p \text{ is their common vertex}; \\ \mu_\sigma(v), & \text{if } u \in E(S) \text{ and } v \in C(S) \end{cases}$$

Figure 3.2 illustrates a signed graph and its lict, $\times$-lict and $\bullet$-lict signed graphs.

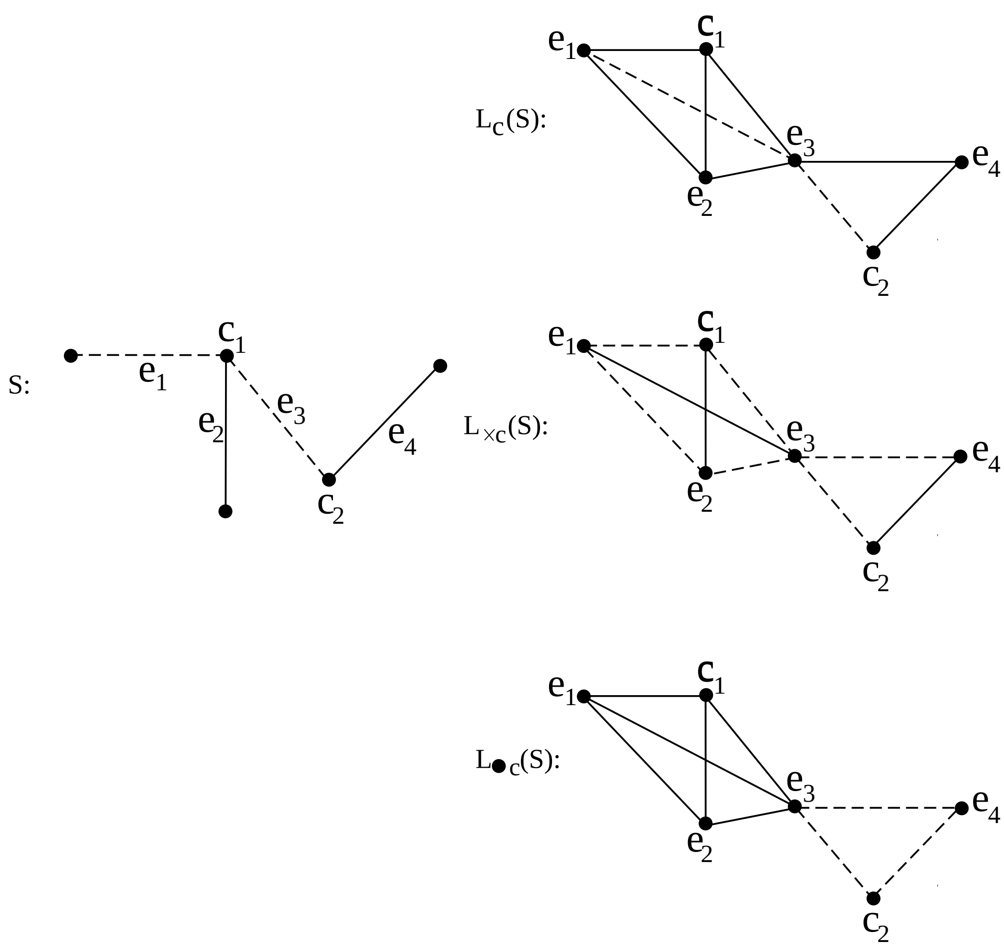

Figure 3.2: A signed graph S and its $L_c(S)$, $L_{\times_c}(S)$ and $L_{\bullet c}(S)$

Note that for a graph G, $L_c(G) \cong L_{\times_c}(G) \cong L_{\bullet c}(G)$ as G is an all-positive signed graph. For an all-negative signed graph S in which every vertex is positive, $\eta(L_c(S)) \cong L_{\times_c}(S)$.

A signed graph S is said to be a lict ($\times$-lict or $\bullet$-lict) signed graph if there exists a signed graph T whose lict ($\times$-lict or $\bullet$-lict) signed graph $L_c(T)$ ($L_{\times_c}(T)$ or $L_{\bullet_c}(T)$) is isomorphic to S and this signed graph T is called the lict ($\times$-lict or $\bullet$-lict) root of S.

3.2 Characterization of lict signed graphs and line signed graphs

In this section, we establish structure characterization of lict signed graphs. This result has been reported in [8].

Theorem 3.2.1. *S is a lict signed graph if and only if the following conditions hold:*

(i) *S^u is a lict graph;*

(ii) *S does not contain a path $P_4 = (u, v, w, x)$ or a triangle (u, v, w) in which exactly one edge vw is positive and;*

(iii) *for any clique S_i, if atleast one vertex c of S_i does not lie in another clique then if $d^-(c) = 0$ ($d^-(c) \neq 0$) then an even (odd) number of vertices adjacent to c have non-zero negative degree and these negative vertices must be adjacent to each other (adjacent to each other and to c also) by negative edges.*

Proof. Necessity:

Let S be a lict signed graph. Therefore, $S \cong L_c(T)$ for some signed graph T. This implies that $S^u \cong L_c(T^u)$, i.e., S^u is a lict graph. Thus **(i)** follows. Now, we prove the necessity of **(ii)** by contradiction:

Assume that S contains a path $P_4 = (u, v, w, x)$ in which exactly one edge vw is positive, i.e., $uv, wx \in E^-(S)$. Since no two adjacent vertices of S correspond to two cut-vertices of T and $S \cong L_c(T)$, we have following possible cases:

Case I: $u, v, w, x \in E(T)$. Then $uv, wx \in E^-(S) \Rightarrow u, v, w, x \in E^-(T) \Rightarrow vw \in E^-(S)$.

Case II: $u \in C(T)$ and $v, w, x \in E(T)$. Then $uv, wx \in E^-(S) \Rightarrow u$ is negative cut-vertex of T and $v, w, x \in E^-(T) \Rightarrow vw \in E^-(S)$.

Case III: $u, w \in C(T)$ and $v, x \in E(T)$. Then $uv, wx \in E^-(S) \Rightarrow u, w$ are negative cut-vertices of T and $v, x \in E^-(T) \Rightarrow vw \in E^-(S)$.

Case IV: $u, x \in C(T)$ and $v, w \in E(T)$. Then $uv, wx \in E^-(S) \Rightarrow u, x$ are negative cut-vertices of T and $v, w \in E^-(T) \Rightarrow vw \in E^-(S)$.

Similarly, if S contains a triangle (u, v, w) in which exactly one edge vw is positive, i.e., $uv, wu \in E^-(S)$. Then we have following two possible cases:

Case I: $u, v, w \in E(T)$. Then $uv, wu \in E^-(S) \Rightarrow u, v, w \in E^-(T) \Rightarrow vw \in E^-(S)$.

Case II: $u \in C(T)$ and $v, w \in E(T)$. Then $uv, wu \in E^-(S) \Rightarrow u$ is negative cut-vertex of T and $v, w \in E^-(T) \Rightarrow vw \in E^-(S)$.

Since we get $vw \in E^-(S)$ that contradicts our assumption. Thus **(ii)** follows.

Next, suppose for any clique S_i of S, atleast one vertex c of S_i does not lie in another clique then obviously, c corresponds to a cut-vertex or a pendant edge of T. Since lict root of S is not unique (as shown in **Figure 3.3**), doesn't matter if we assume c as a cut-vertex or a pendant edge of T. Let c be a cut-vertex of T.

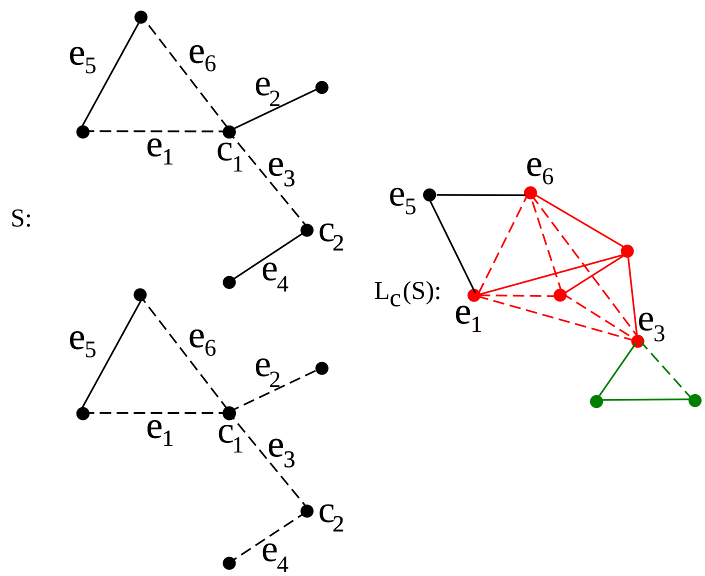

Figure 3.3: Signed graphs S and its $L_c(S)$

Now, we have following two possible cases:

Case I: If $d^{-}(c) = 0$ then by the definition of lict signed graphs, an even number of edges incident to c must be negative in S, hence an even number of vertices of $\mathcal{N}(c)$ in S, have non-zero negative degree and adjacent to each other by negative edges, where $\mathcal{N}(c)$ is the set of all adjacent vertices of c.

Case II: If $d^{-}(c) \neq 0$ then an odd number of edges incident to c must be negative in S, hence an odd number of vertices of $\mathcal{N}(c)$ in S

have non-zero negative degree and adjacent to each other and with c by negative edges.

Thus **(iii)** follows.

Hence, the necessity follows.

Sufficiency:

Suppose conditions hold. We give the construction of a signed graph T whose lict signed graph is S. Since S^u is a lict graph, by Theorem 2.3.1, the edges of S^u can be partitioned into cliques in such a way that no vertex lies in more than two of these cliques. Let $\mathcal{P}(S) = \{S_1, S_2, ..., S_n\}$ be such a partition of $E(S)$. The vertices of T^u correspond to the set $\mathcal{P}(S)$ together with the set U of vertices of S belonging only to one of the cliques S_i leaving one such vertex for each S_i. Thus $V(T^u) = \mathcal{P}(S) \cup U$, two of these vertices are adjacent whenever they have a nonempty intersection, that is, T^u is the intersection graph $\Omega(\mathcal{P}(S) \cup U)$. Now, we assign the labels to the edges and cut-vertices of T^u and then give labels to vertices of S. Further, we construct a signed graph T on T^u such that an edge e_i of T is positive if negative degree of its corresponding vertex in S is 0 and negative otherwise. For this signed graph T, $S \cong L_c(T)$. This completes the proof. □

Figure 3.4 illustrates the construction of a signed graph T from a signed graph S such that $S \cong L_c(T)$ and S satisfies sufficiency conditions of Theorem 3.2.1:

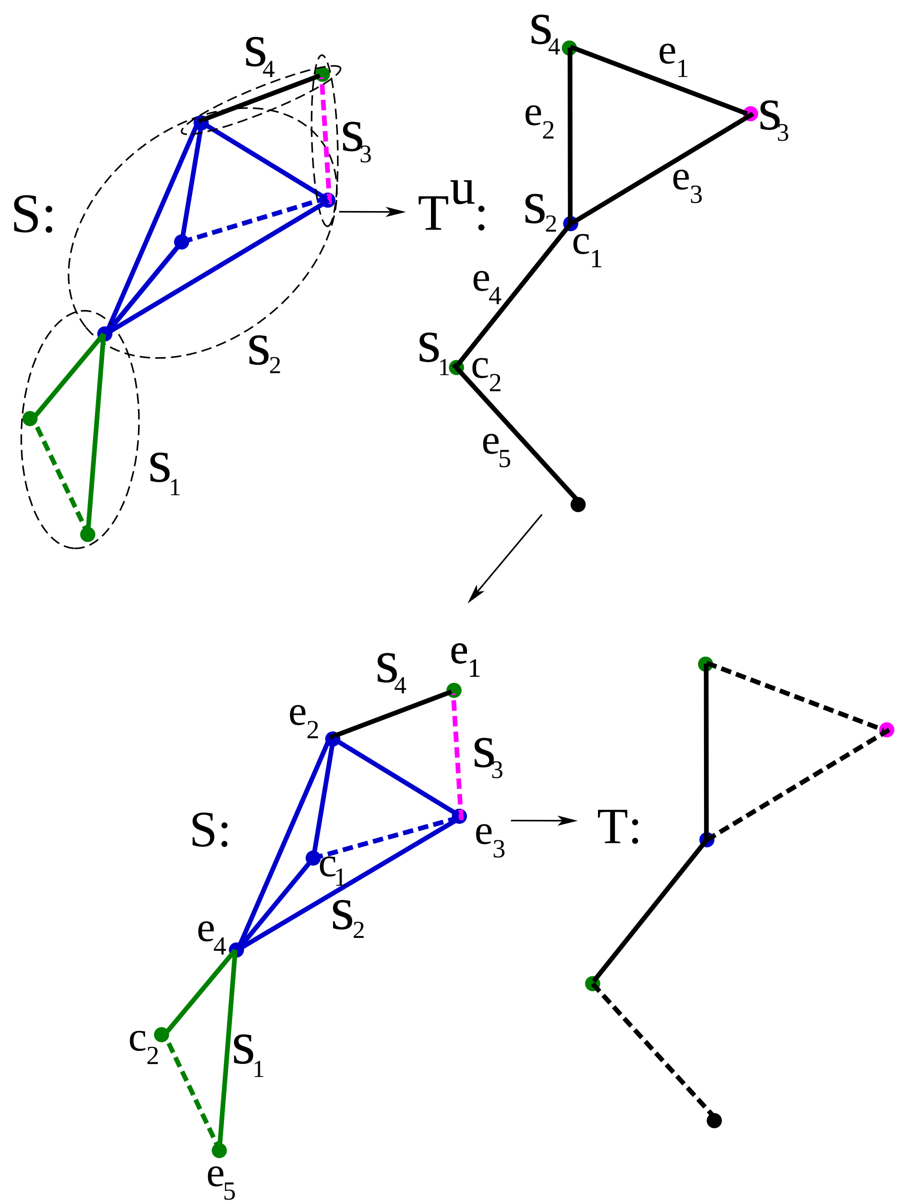

Figure 3.4: The construction of signed graph T from S such that $S \cong L_c(T)$

Proposition 3.2.1. *S is a line signed graph if and only if the following conditions hold:*

(i) *S^u is a line graph and;*

(ii) *S does not contain a path $P_4 = (u, v, w, x)$ or a triangle (u, v, w) in which exactly one edge vw is positive.*

Proposition 3.2.1 has been proved by Acharya and Sinha in [6].

3.3 Characterization of $\times$-lict signed graphs and $\times$-line signed graphs

In the previous section we have characterized lict signed graphs. Now, we move towards $\times$-lict and $\times$-line signed graphs. The following theorem is about the characterization of $\times$-lict signed graph as reported in [9].

Theorem 3.3.1. *S is a $\times$-lict signed graph if and only if S^u is a lict graph and S is balanced.*

Proof. Necessity:

Let S be a $\times$-lict signed graph. Therefore, $S \cong L_{\times_c}(T)$ for some signed graph T. This implies that $S^u \cong L_{\times_c}(T^u)$, i.e., S^u is a lict graph. Further, by Proposition 3.1.2, S is balanced. Thus, the necessity follows.

Sufficiency:

Suppose S^u is a lict graph and S is balanced. Therefore, $S^u \cong L_c(T^u)$ for some graph T^u. We construct T^u and assign labels to the edges of S by the procedure as discussed in the sufficiency of Theorem 3.2.1. Now, by Theorem 1.2.3, a signed graph S is balanced if and only if there exits a marking μ of S such that every edge uv of S satisfies $\sigma(uv) = \mu(u)\mu(v)$. We assign marking μ to S such that vertices of S which correspond to cut-vertices of T^u are '+' and $\sigma(uv) = \mu(u)\mu(v)$. Further, we construct a signed graph T on T^u such that an edge e_i

of T is positive (negative) if its corresponding vertex in S is positive (negative). For this signed graph T, $S \cong L_{\times_c}(T)$. This completes the proof. $\square$

Figure 3.5 illustrates the construction of a signed graph T from a signed graph S such that $S \cong L_{\times_c}(T)$ and S satisfies sufficiency conditions of Theorem 3.3.1.

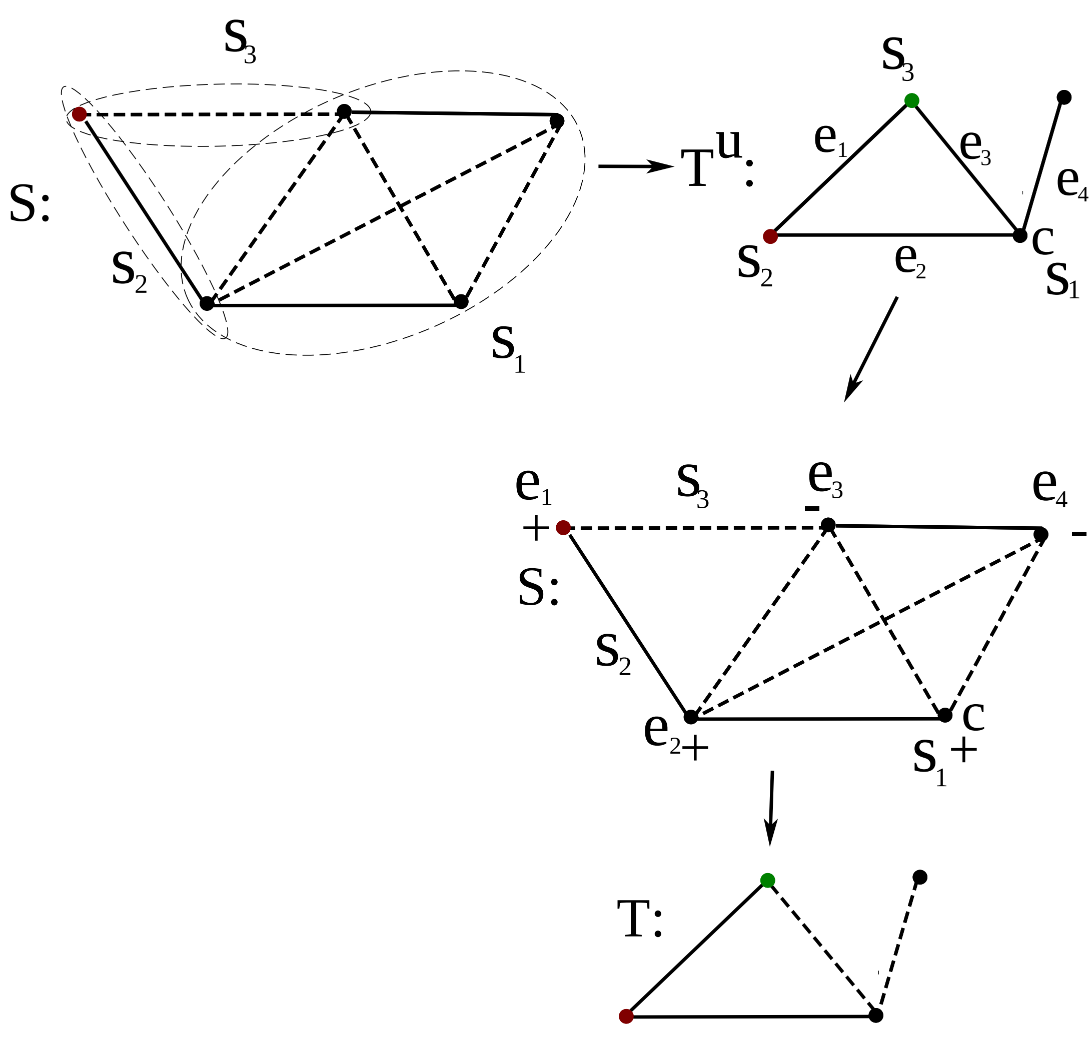

Figure 3.5: The construction of signed graph T from S such that $S \cong L_{\times_c}(T)$

Observation 3.3.1. *If a signed graph T is $\times$-lict root of a signed* *graph S then $\eta(T)$ is also $\times$-lict root of S.*

Theorem 3.3.2. *S is a $\times$-line signed graph if and only if S^u is a line graph and S is balanced.*

Proof. Necessity:

Let S be a $\times$-line signed graph. Therefore, $S \cong L_\times(T)$ for some signed graph T. This implies that $S^u \cong L_\times(T^u)$, i.e., S^u is a line graph. Further, by Proposition 3.1.1, S is balanced. Thus, the necessity follows.

Sufficiency:

Suppose conditions hold. We give the construction of a signed graph T whose $\times$-line signed graph is S. Since S^u is a line graph, by Theorem 2.2.1, the edges of S^u can be partitioned into complete subgraphs in such a way that no vertex lies in more than two of these complete subgraphs. Let $\mathcal{P}(S) = \{S_1, S_2, ..., S_n\}$ be such a partition of $E(S)$. The vertices of T^u correspond to the set $\mathcal{P}(S)$ together with the set U of vertices of S belonging only to one of the complete subgraphs S_i. Thus $V(T^u) = \mathcal{P}(S) \cup U$, two of these vertices are adjacent whenever they have a nonempty intersection, that is, T^u is the intersection graph $\Omega(\mathcal{P}(S) \cup U)$. Now, we assign the labels to the edges of T^u and then give labels to vertices of S. Further, By Theorem 1.2.3, a signed graph S is balanced if and only if there exits a marking μ of S such that every edge uv of S satisfies $\sigma(uv) = \mu(u)\mu(v)$. We assign marking μ to S such that $\sigma(uv) = \mu(u)\mu(v)$ and construct a signed graph T on T^u such that an edge e_i of T is positive (negative) if its corresponding vertex in S is positive (negative). For this signed graph T, $S \cong L_{\times}(T)$. This completes the proof. □

Figure 3.6 illustrates the construction of a signed graph T from a signed graph S such that $S \cong L_{\times}(T)$ and S satisfies sufficiency conditions of Theorem 3.3.2.

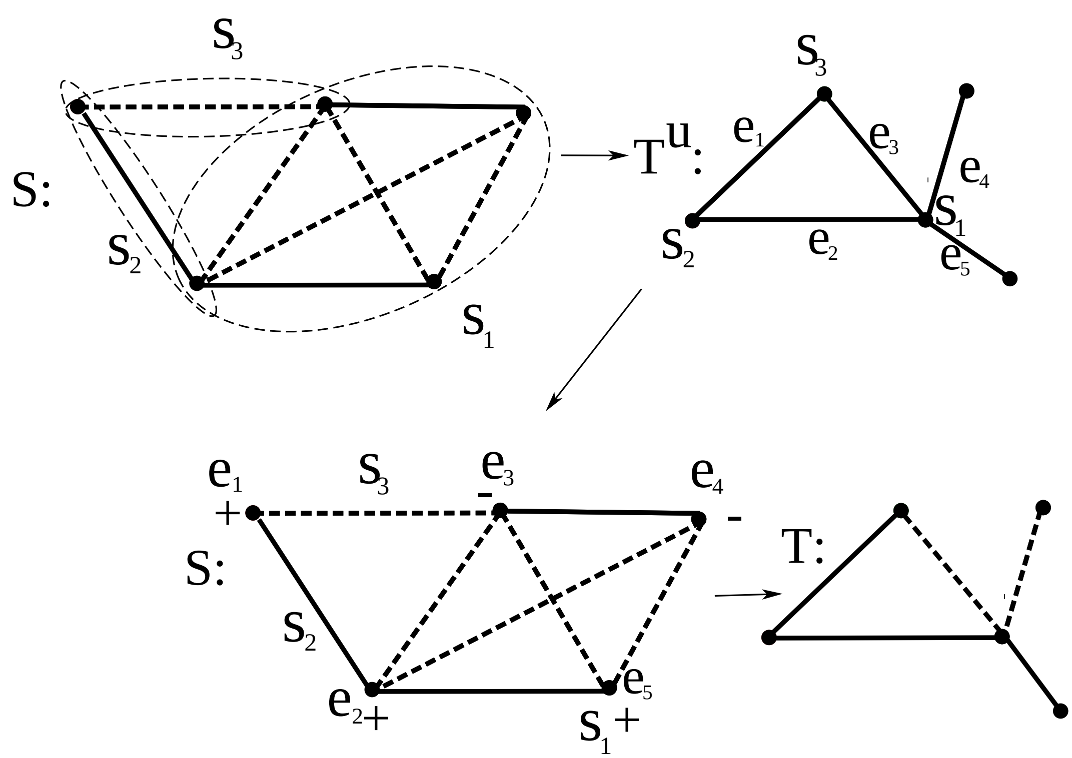

Figure 3.6: The construction of signed graph T from S such that $S \cong L_{\times}(T)$

Observation 3.3.2. *If a signed graph T is $\times$-line root of a signed graph S then $\eta(T)$ is also $\times$-line root of S.*

3.4 Characterization of $\bullet$-lict signed graphs and $\bullet$-line signed graphs

In the following theorem, we give the characterization of $\bullet$-lict signed graph as reported in [9].

Theorem 3.4.1. *S is a $\bullet$-lict signed graph if and only if the following conditions hold in S:*

(i) *S^u is a lict graph and the edges of S can be partitioned into homogeneous cliques in such a way that no vertex lies in more than two of these cliques;*

(ii) *if for every such clique S_i, each vertex of S_i lies in two of these cliques except at most one vertex of S_i then the number of all-negative cliques is even.*

Proof. Necessity:

Let S be a $\bullet$-lict signed graph. Therefore, $S \cong L_{\bullet c}(T)$ for some signed graph $T = (T^u, \sigma')$. This implies that $S^u \cong L_{\bullet c}(T^u)$, i.e., S^u is a lict graph. By Theorem 2.3.1, the edges of S^u can be partitioned into cliques in such a way that no vertex lies in more than two of these cliques and by the definition of $L_{\bullet c}(T)$, the edges incident on a vertex v of T of degree p, that is not a cut-vertex, induces a homogeneous clique of order p in S and the edges incident on a cut-vertex c of T of degree p induce a homogeneous clique of order $p+1$ in S having c as one of its vertices. This is illustrated in **Figure 3.7**. Thus, **(i)** follows.

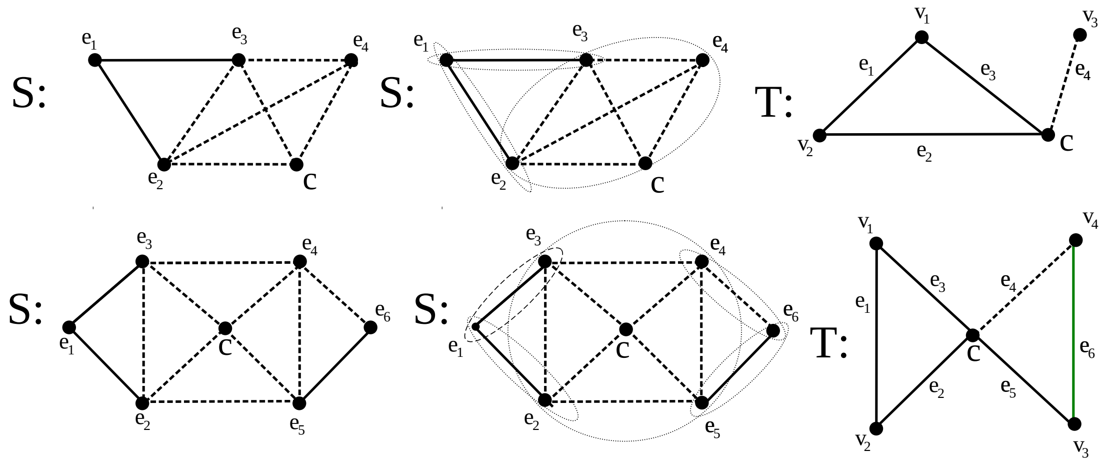

Figure 3.7: Signed graph S and signed graph T such that $S \cong L_{\bullet c}(T)$

By Lemma 1.2.2, in any canonically marked signed graph there are an even number of vertices marked negative. Clearly, if the number of negative pendant vertices in T are even (odd) then the number of all-negative cliques in S will be even (odd). This is illustrated in **Figure 3.6**. Hence, if T does not contain any pendant edge, that is, it has zero pendant edges then the number of all-negative cliques in S will be even and by the definition of $L_{\bullet c}(T)$, for this T, in S, each vertex of a clique S_i lies in two of these cliques except at most one vertex of S_i for each clique of S. Thus, the necessity follows.

<u>Sufficiency</u>:

Suppose conditions hold. We give the construction of a signed graph T whose $\bullet$-lict signed graph is S. Let $\mathcal{P}(S) = \{S_1, S_2, ..., S_n\}$ be the partition of $E(S)$ into homogeneous cliques. The vertices of T^u correspond to the set $\mathcal{P}(S)$ together with the set U of vertices of S belonging only to one of the homogeneous cliques S_i leaving one such vertex for each S_i. Thus $V(T^u) = \mathcal{P}(S) \cup U$, two of these vertices are adjacent whenever they have a nonempty intersection. We construct T^u and assign '+'

('$-$') sign to each non-pendant vertex $S_i \in V(T^u)$ if it corresponds to an all-positive (all-negative) S_i in S and take signature of T in such a way that signs assigned to vertices of T^u are preserved under canonical marking of T. For this signed graph T, $S \cong L_{\bullet c}(T)$, that is, S is a $\bullet$-lict signed graph. This completes the proof.

$\square$

Figure 3.8 illustrates the construction of a signed graph T from a signed graph S such that $S \cong L_{\bullet c}(T)$ and S satisfies sufficiency conditions of Theorem 3.4.1.

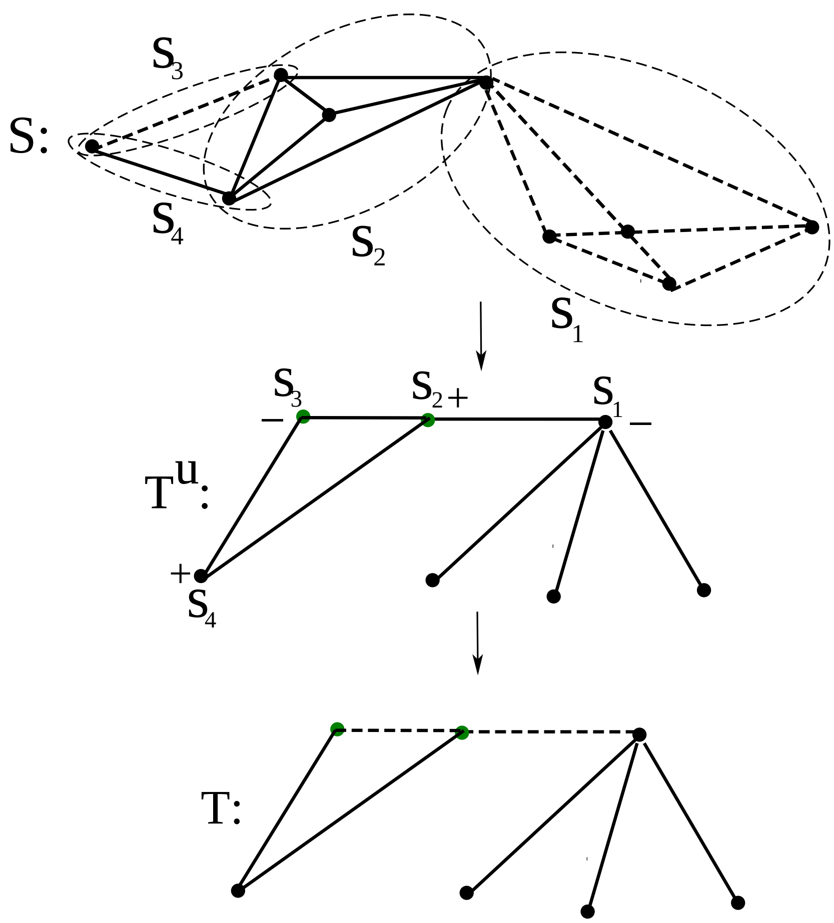

Figure 3.8: The construction of signed graph T from S such that $S \cong L_{\bullet c}(T)$

Proposition 3.4.1. *S is a $\bullet$-line signed graph if and only if the following conditions hold in S:*

(i) *S^u is a line graph and the edges of S can be partitioned into homogeneous cliques in such a way that no vertex lies in more than two of these cliques;*

(ii) *if each vertex of S belongs to exactly two of these cliques then the number of all-negative cliques is even.*

Proposition 3.4.1 has been proved by Sinha and Dhama in [11]. For more study on dot-line signed graph, the reader is referred to [2].

3.5 Conclusion and Scope

In this chapter, we have initiated the study on lict signed graphs $L_c(S)$ and $L_{\bullet_c}(S)$, and we established characterizations of $L_c(S)$, $L_{\times_c}(S)$, $L_{\bullet_c}(S)$ and also of line signed graphs $L_\times(S)$ and $L_\bullet(S)$. Litact signed graphs are yet to be defined and study on this concept is still open.

* * * * *

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Chapter 4

FURTHER RESULTS ON DOT-LICT AND LICT SIGNED GRAPHS

In the previous chapter we have established characterization for lict, $\times$-lict and $\bullet$-lict signed graphs. In this chapter, we characterize signed graphs on K_p, $p \geq 2$, on cycle C_n and on $K_{m,n}$ which are $\bullet$-lict signed graphs or $\bullet$-line signed graphs and we also characterize signed graphs S so that $L_{\bullet_c}(S)$ and $L_\bullet(S)$ are balanced. We also establish the characterization of signed graphs S for which $S \sim L_{\bullet_c}(S)$, $S \sim L_\bullet(S)$, $\eta(S) \sim L_{\bullet_c}(S)$ and $\eta(S) \sim L_\bullet(S)$, here $\eta(S)$ is the negation of S and $\sim$ stands for switching equivalence. Similarly, we characterize signed graphs on K_p, $p \geq 2$, on cycle C_n and on $K_{m,n}$ which are lict signed graphs or line signed graphs and we also characterize signed graphs S so that $L_c(S)$ and $L(S)$ are balanced. Further, we also establish the characterization of signed graphs S for which $S \sim L_c(S)$, $S \sim L(S)$, $\eta(S) \sim L_c(S)$ and $\eta(S) \sim L(S)$.

4.1 Introduction

Though we have defined $\bullet$-line, $\bullet$-lict, line and lict signed graphs in the previous chapter but here we again give definitions in order to make the chapter easily readable.

There are three notions of a *line signed graph* of a signed graph

$S = (S^u, \sigma)$ in the literature, viz., $L(S)$, $L_\times(S)$ and $L_\bullet(S)$, all of which have $L(S^u)$ as their underlying graph; only the rule to assign signs to the edges of $L(S^u)$ differ. Every edge ee' in $L(S)$ is negative whenever both the adjacent edges e and e' in S are negative, an edge ee' in $L_\times(S)$ has the product $\sigma(e)\sigma(e')$ as its sign and an edge ee' in $L_\bullet(S)$ has $\mu_\sigma(v)$ as its sign, where $v \in V(S)$ is a common vertex of edges e and e' and $\mu_\sigma(v)$ is the product of the signs of the edges incident to v.

The dot-lict signed graph (or $\bullet$-lict signed graph) of a signed graph $S = (S^u, \sigma)$, is a signed graph $L_{\bullet c}(S) = (L_c(S^u), \sigma')$, where for every edge uv of $L_c(S^u)$

$$\sigma'(uv) = \begin{cases} \mu_\sigma(p), & \text{if } u, v \in E(S) \text{ and } p \text{ is their common vertex;} \\ \mu_\sigma(v), & \text{if } u \in E(S) \text{ and } v \in C(S) \end{cases}$$

The *lict signed graph* of a signed graph $S = (S^u, \sigma)$, is a signed graph $L_c(S) = (L_c(S^u), \sigma')$, where for every edge uv in $L_c(S^u)$

$$\sigma'(uv) = \begin{cases} - & \text{if } u, v \in E^-(S) \text{ or } u \in E^-(S) \text{ and } v \text{ is negative cut-vertex;} \\ + & \text{otherwise.} \end{cases}$$

Some important concepts and theorems which are useful to prove the results of this chapter are given below.

Theorem 4.1.1. [5] *For a connected graph G, $G \cong L(G)$ if and only if G is a cycle.*

Theorem 4.1.2. [6] *For a connected graph G, $G \cong L_c(G)$ if and only* *if G is a cycle.*

In following sections, we give results on •-lict and •-line signed graphs. These results have been reported in [1].

4.2 •-lict and •-line signed graphs on K_p and C_n

Theorem 4.2.1. *A signed graph $S = (S^u, \sigma)$, on a complete graph $S^u := K_p$, $p \geq 3$, is a •-lict signed graph if and only if S is homogeneous or a triangle having two negative edges.*

Proof. Necessity: Let $S = (S^u, \sigma)$, on a complete graph K_p, $p \geq 3$, be a •-lict signed graph. Therefore, $S \cong L_{\bullet c}(T)$ for some signed graph $T = (T^u, \sigma')$. This implies that $S^u \cong L_{\bullet c}(T^u)$, i.e., $K_p \cong L_c(T^u)$. By the definition of lict graph it is clear that

$$T^u = \begin{cases} K_3 \text{ } or \text{ } K_{1,2} & \text{if p=3;} \\ K_{1,p-1} & \text{if } p \geq 4. \end{cases}$$

- If $T^u := K_3$, then for homogeneous T on T^u, S is all-positive triangle and for heterogeneous T, S is a triangle having two negative edges, as shown in **Figure 4.1**.

- If $T^u := K_{1,p-1}$, $p \geq 3$, then whether non-pendant vertex is positive or negative, S is homogeneous. Thus, the necessity follows.

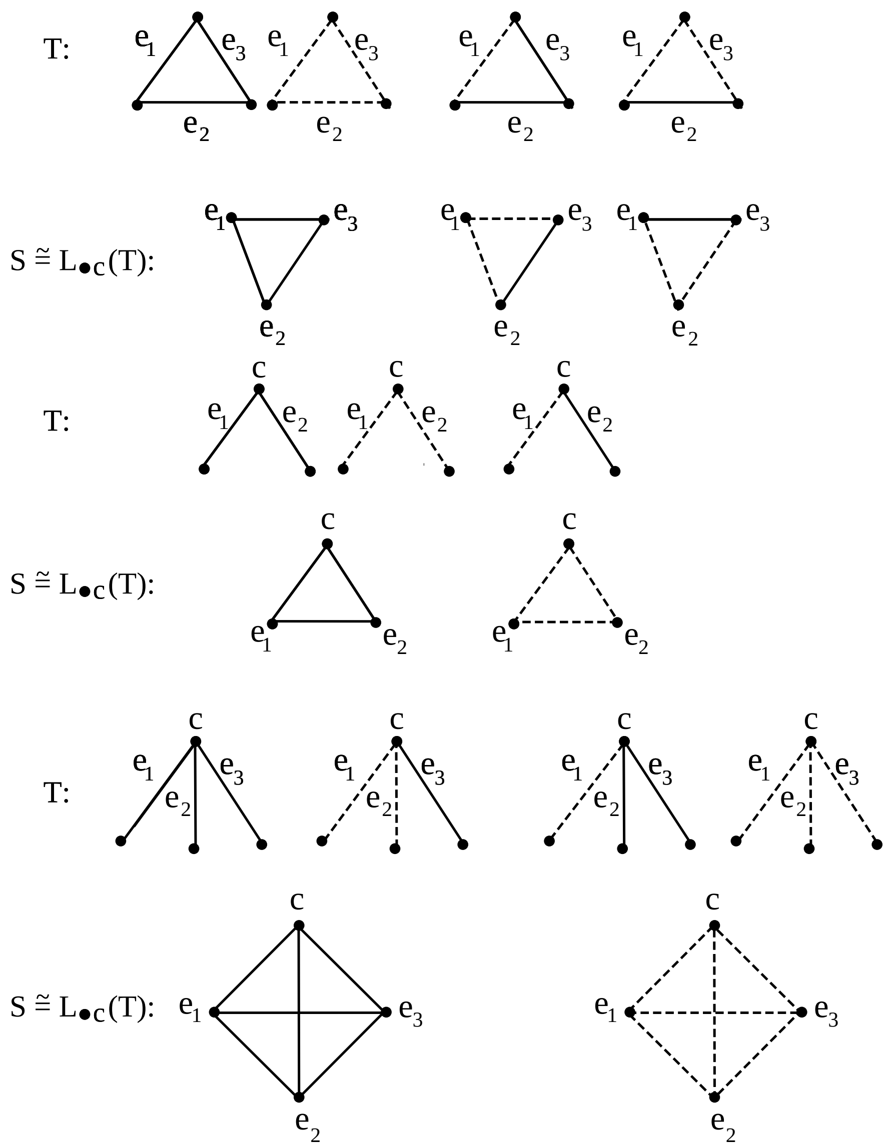

Figure 4.1: Signed graphs S and T such that $S \cong L_{\bullet c}(T)$

Sufficiency:

Suppose conditions hold. We give the construction of a signed graph T whose $\bullet$-lict signed graph is S. Let $\mathcal{P}(S) = \{S_1, S_2, ..., S_n\}$ be the partition of $E(S)$ into homogeneous cliques. Note that the ordered pair $(S, \mathcal{P}(S))$ is a hypergraph. The vertices of T^u correspond to the set $\mathcal{P}(S)$ together with the set U of vertices of S belonging only to one of the homogeneous cliques S_i leaving one such vertex for each S_i. Thus $V(T^u) = \mathcal{P}(S) \cup U$, two of these vertices are adjacent whenever they have a nonempty intersection in S. Now, assign '+' ('−') sign to each non-pendant vertex $S_i \in V(T^u)$ if it corresponds to an all-positive (all-negative) S_i in S and take signature of T in such a way that signs assigned to vertices of T^u are preserved under canonical marking of T. For this signed graph T, $S \cong L_{\bullet c}(T)$; that is, S is a $\bullet$-lict signed graph. This completes the proof.

□

Figure 4.2 illustrates construction of a signed graph T such that $S \cong L_{\bullet c}(T)$ for a signed graph S that satisfies sufficiency condition of Theorem 4.2.1. Note that T need not be unique.

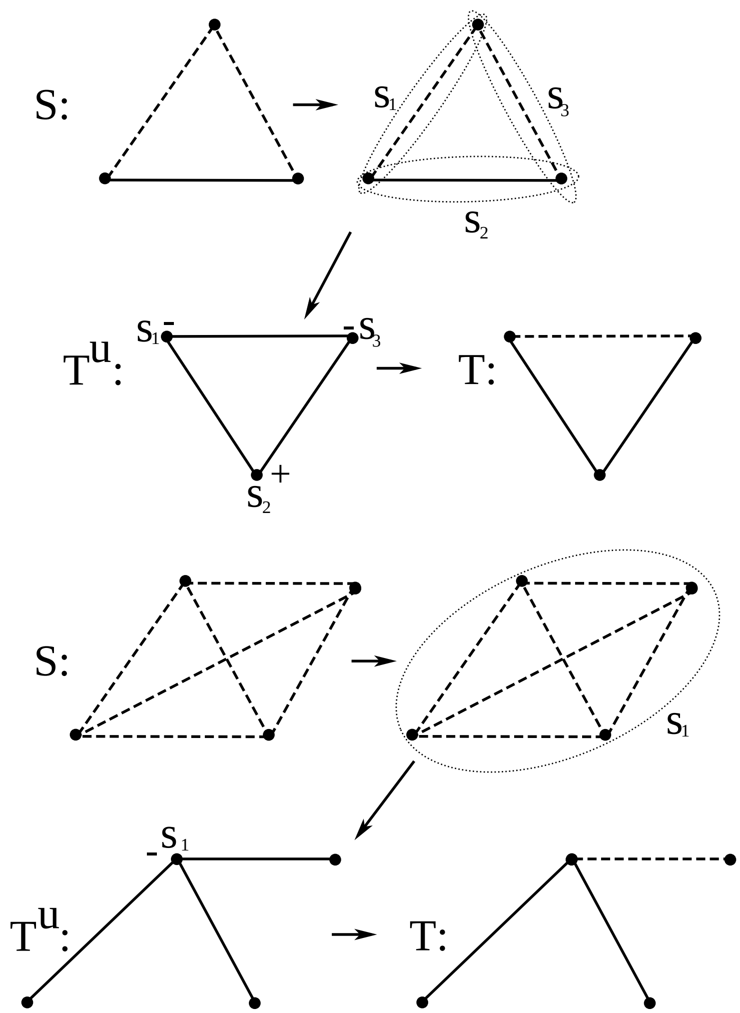

Figure 4.2: The construction of signed graph T from S such that $S \cong L_{\bullet c}(T)$

Theorem 4.2.2. *A signed graph $S = (S^u, \sigma)$, on a complete graph $S^u := K_p$, $p \geq 2$, is a $\bullet$-line signed graph if and only if S is homogeneous or a triangle having two negative edges.*

Proof. Necessity:
Let $S = (S^u, \sigma)$, on a complete graph K_p, $p \geq 2$, be a $\bullet$-line signed graph. Therefore, $S \cong L_\bullet(T)$ for some signed graph $T = (T^u, \sigma')$. This implies that $S^u \cong L_\bullet(T^u)$, i.e., $K_p \cong L(T^u)$. Clearly,

$$T^u = \begin{cases} K_3 \ or \ K_{1,3} & \text{if p=3;} \\ K_{1,p} & \text{if } p = 2 \text{ or } p \geq 4. \end{cases}$$

- If $T^u := K_3$, then for homogeneous T on T^u, S is all-positive triangle and for heterogeneous T, S is a triangle having two negative edges, as shown in **Figure 4.3**.

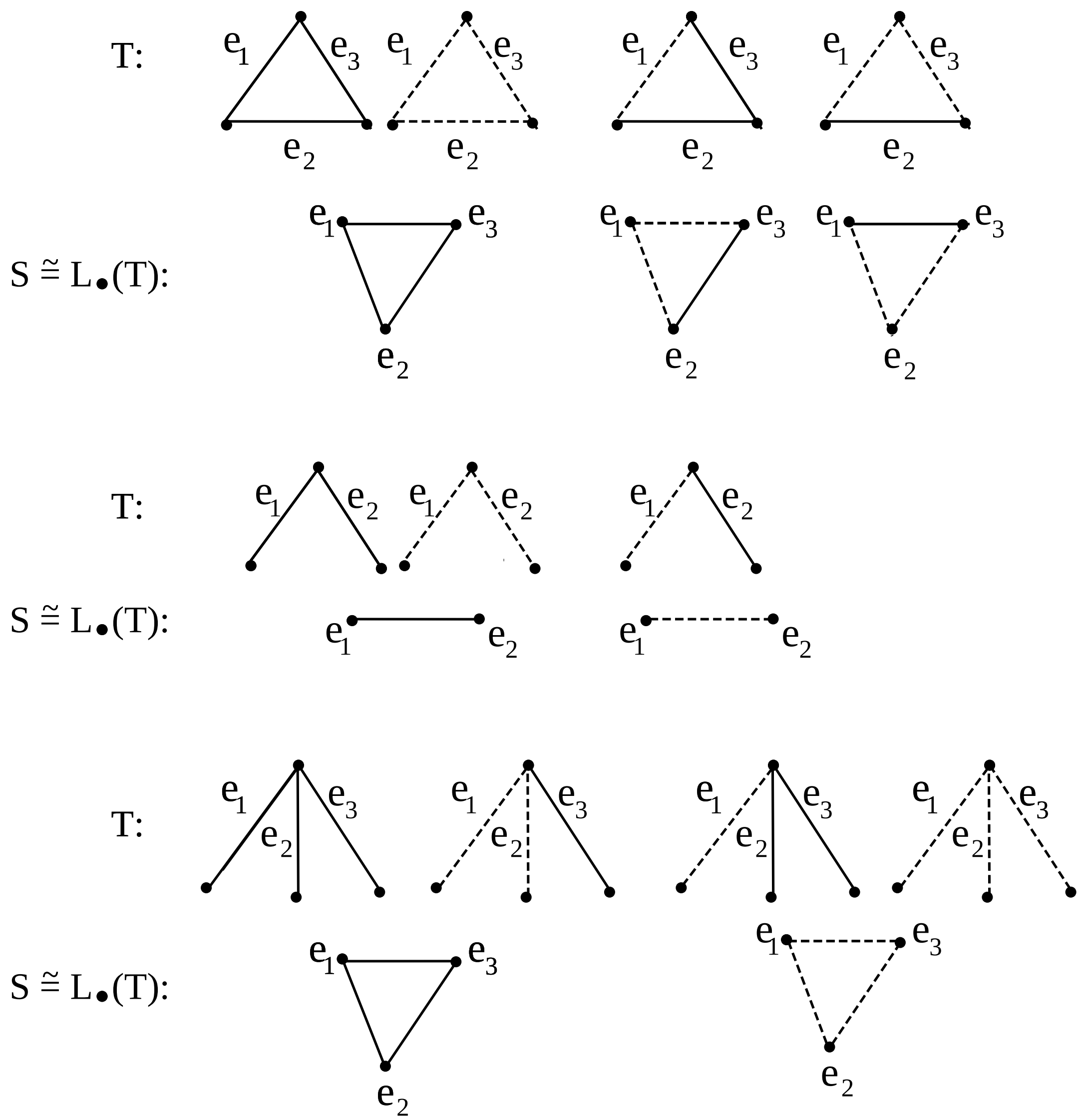

Figure 4.3: Signed graph T such that $S \cong L_\bullet(T)$

- If $T^u := K_{1,p}$, $p \geq 2$, then whether non-pendant vertex is positive or negative, S is always homogeneous. This is illustrated in **Figure 4.3**. Thus, the necessity follows.

<u>Sufficiency</u>:

Suppose conditions hold. We give the construction of a signed graph T whose $\bullet$-line signed graph is S. Let $\mathcal{P}(S) = \{S_1, S_2, ..., S_n\}$ be the partition of $E(S)$ into homogeneous complete subgraphs. The vertices of T^u correspond to the set $\mathcal{P}(S)$ together with the set U of vertices of S belonging only to one of the homogeneous complete subgraphs S_i. Thus $V(T^u) = \mathcal{P}(S) \cup U$, two of these vertices are adjacent whenever they have a nonempty intersection. Now, assign '+' ('−') sign to each non-pendant vertex $S_i \in V(T^u)$ if it corresponds to an all-positive (all-negative) S_i in S and take signature of T in such a way that signs assigned to vertices of T^u are preserved under canonical marking of T. For this signed graph T, $S \cong L_\bullet(T)$; that is, S is a $\bullet$-line signed graph. This completes the proof.

□

Figure 4.4 illustrates construction of a signed graph T such that $S \cong L_\bullet(T)$ for a signed graph S that satisfies sufficiency condition of Theorem 4.2.2.

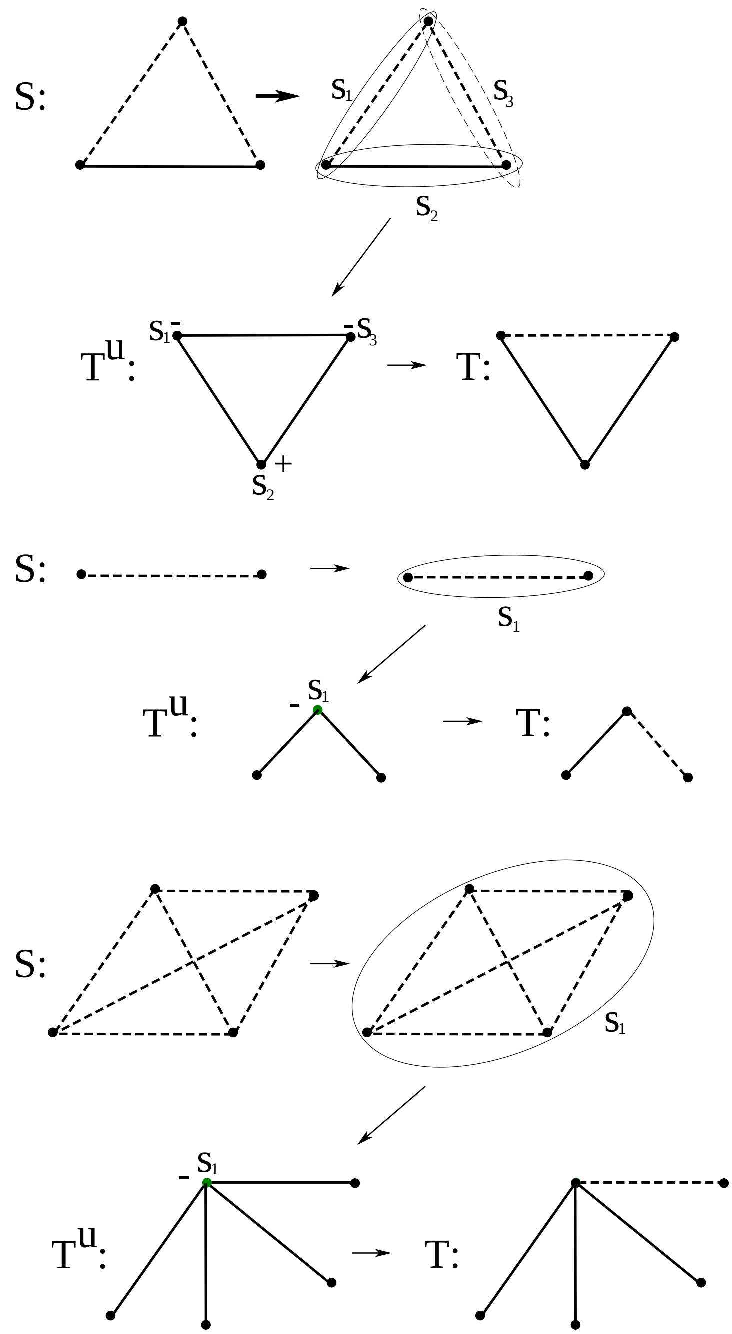

Figure 4.4: The construction of signed graph T from S such that $S \cong L_{\bullet}(T)$

Theorem 4.2.3. *A signed graph $S = (S^u, \sigma)$, on a cycle $S^u := C_n$, is $\bullet$-lict signed graph if and only if S is an all-negative triangle or a positive cycle.*

Proof. Necessity:

Let $S = (S^u, \sigma)$, on cycle C_n, be a $\bullet$-lict signed graph. Therefore, $S \cong L_{\bullet c}(T)$ for some signed graph $T = (T^u, \sigma')$. This implies that $S^u \cong L_{\bullet c}(T^u)$, i.e., $C_n \cong L_c(T^u)$. By the definition of lict graph it is clear that

$$T^u = \begin{cases} C_3 \ or \ K_{1,2} & \text{if n=3;} \\ C_n & \text{if } n \geq 4. \end{cases}$$

- If $T^u := K_{1,2}$, then for T on T^u containing one negative edge, S is an all-negative triangle, as shown in **Figure 4.1** and for other $T's$, S is an all-positive triangle.

- If $T^u := C_n$, then by the definition of $L_{\bullet c}(T)$, $|E^-(L_{\bullet c}(T))| =$ the number of negative vertices in T. By Lemma 1.2.2, in any canonically marked signed graph there are an even number of vertices marked negative. Hence $|E^-(L_{\bullet c}(T))| =$ even, i.e., S is a positive cycle.

  Thus the necessity follows.

Sufficiency:

Suppose conditions hold. We give the construction of a signed graph T by the procedure as discussed in the sufficiency of Theorem 4.2.1. For this signed graph T, $S \cong L_{\bullet c}(T)$; that is, S is a $\bullet$-lict signed graph. This completes the proof. □

Figure 4.5 illustrates construction of a signed graph T such that $S \cong L_{\bullet c}(T)$ for a signed graph S that satisfies sufficiency condition of

Theorem 4.2.3.

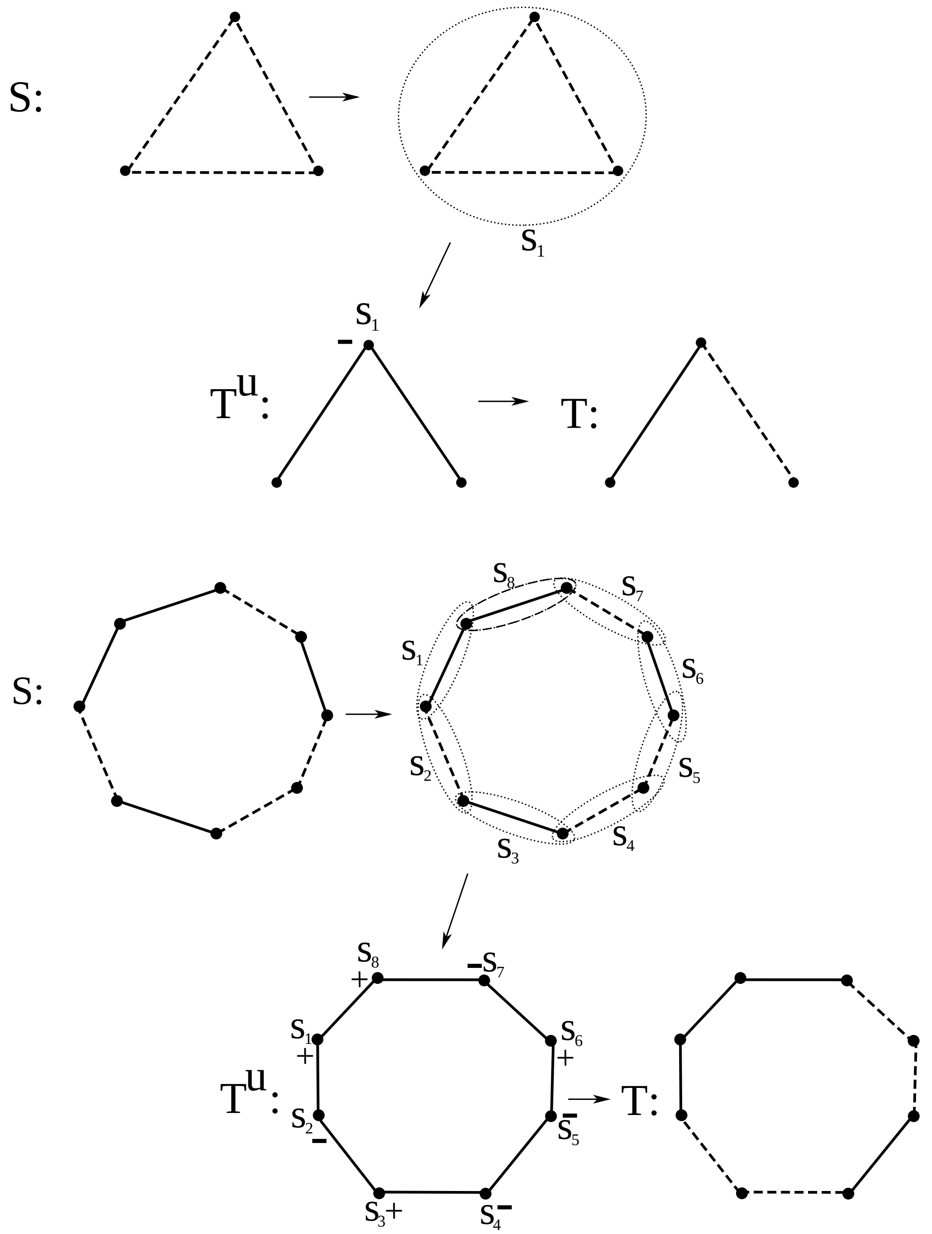

Figure 4.5: The construction of signed graph T from S such that $S \cong L_{\bullet c}(T)$

Corollary 4.2.1. *A signed graph $S = (S^u, \sigma)$, on a cycle $S^u := C_n$, is $\bullet$-line signed graph if and only if S is a homogeneous triangle or a positive cycle.*

Proposition 4.2.1. *For a signed graph S on cycle C_n, $L_{\bullet c}(S) \cong L_{\bullet}(S)$ and it is a positive cycle.*

Proof. For a signed graph S on cycle C_n, $L_{\bullet c}(S) \cong L_{\bullet}(S)$, since $C(S) = \phi$. By the definition of $L_{\bullet c}(S)$, $|E^{-}(L_{\bullet c}(S))|$ = the number of negative vertices in S. By Lemma 1.2.2, in any canonically marked signed graph there are an even number of vertices marked negative. Hence $|E^{-}(L_{\bullet c}(S))|$ = even, i.e., $L_{\bullet c}(S)$ is a positive cycle. □

Corollary 4.2.2. *For a signed graph $S = (S^u, \sigma)$, on a cycle $S^u := C_n$, $L_{\bullet c}(S) \sim S$ and $L_{\bullet}(S) \sim S$ if and only if S is a positive cycle.*

4.3 •-lict and •-line signed graphs on $K_{m,n}$

Theorem 4.3.1. *A signed graph $S = (S^u, \sigma)$ on a complete bipartite graph $S^u := K_{m,n}$, is a •-lict signed graph if and only if S is a positive cycle of order 4.*

Proof. Necessity:

Let $S = (S^u, \sigma)$ on a complete bipartite graph $K_{m,n}$, be a •-lict signed graph. Therefore, $S \cong L_{\bullet c}(T)$ for some signed graph T. This implies that $S^u \cong L_{\bullet c}(T^u)$ or $K_{m,n} \cong L_c(T^u)$, i.e., $K_{m,n}$ is a lict graph.

Let v be a cut-vertex of T^u then clearly $d(v) \geq 2$ and by the definition of lict graph, the edges incident with cut-vertex v in T^u induce a complete subgraph $K_{d(v)+1}$, i.e., K_p, $p \geq 3$ in $S^u := K_{m,n}$. Since a complete bipartite graph does not contain any odd cycle, K_p, $p \geq 3$

can not be a subgraph of $K_{m,n}$. Hence, $C(T^u) = \phi$. Therefore $K_{m,n}$ is also a line graph. Since $K_{1,3}$ is a forbidden induced subgraph of a line graph, $m \leq 2$ and $n \leq 2$. Furthermore by Theorem 2.3.1, a lict graph does not contain a pendant vertex, $K_{m,n} \ncong K_{1,1}$ and $K_{1,2}$. Thus $K_{m,n} \cong C_4$ and by Theorem 4.2.3, S is a positive cycle of order 4.

Sufficiency:

Suppose S is a positive cycle of order 4 then by Theorem 4.2.3, S is a $\bullet$-lict signed graph. This completes the proof. □

Corollary 4.3.1. *A signed graph $S = (S^u, \sigma)$ on a complete bipartite graph $S^u := K_{m,n}$, is a $\bullet$-line signed graph if and only if S is any one of the following:*

(i) *any signed graph on $K_{1,1}$ or $K_{1,2}$*

(ii) *a positive cycle of order 4.*

4.4 Balanced $\bullet$-lict and $\bullet$-line signed graphs

Theorem 4.4.1. *For a signed graph S, $L_{\bullet c}(S)$ is balanced if and only if the following conditions hold in S:*

(i) *S is $\mathcal{C}$-consistent and;*

(ii) *each vertex v of $d(v) \geq 3$ and cut-vertex of degree 2 are positive vertices.*

Proof. Necessity:

Suppose for a signed graph S, $L_{\bullet c}(S)$ is balanced, i.e., every cycle in

$L_{\bullet c}(S)$ is a positive cycle. By the definition of $L_{\bullet c}(S)$, a cycle Z in S induces a cycle Z' in $L_{\bullet c}(S)$ and $|E^{-}(Z')|$ = the number of negative vertices in Z. Since Z' is a positive cycle, every cycle Z in S contains an even number of negative vertices; that is, S is $\mathcal{C}$-consistent. Thus, **(i)** follows.

Next, we prove the necessity of condition **(ii)** by contrapositive:

Assume that a vertex $v \in V(S)$ which is of degree≥ 3 or a cut-vertex of degree 2, is a negative vertex, i.e., $d^{-}(v)$ is odd or $\mu_{\sigma}(v) = -$. Then by the definition of $L_{\bullet c}(S)$, the edges incident with v will induce an all-negative complete subsignedgraph of order ≥ 3 in $L_{\bullet c}(S)$ that makes $L_{\bullet c}(S)$ unbalanced. Thus, **(ii)** follows.

Sufficiency:

A cycle in $L_{\bullet c}(S)$ is induced due to a cycle or a vertex of degree≥ 2 or their combinations in S. Suppose conditions **(i)** and **(ii)** hold in S then every chordless cycle in $L_{\bullet c}(S)$ will be positive. By Lemma 1.2.1, a signed graph in which every chordless cycle is positive, is balanced. Hence $L_{\bullet c}(S)$ is balanced. This completes the proof. □

Corollary 4.4.1. $L_{\bullet}(S)$ *is balanced if and only if the following conditions hold in signed graph* S*:*

(i) S *is* $\mathcal{C}$*-consistent and;*

(ii) *each vertex* v *of* $d(v) \geq 3$ *is a positive vertex.*

4.5 Switching equivalence of $L_{\bullet c}(S)$ and $L_{\bullet}(S)$ to S

Theorem 4.5.1. *For a signed graph S, $S \sim L_{\bullet c}(S)$ if and only if S is a positive cycle.*

Proof. Suppose for a signed graph S, $S \sim L_{\bullet c}(S)$. This implies that $S^u \cong L_{\bullet c}(S^u)$, i.e., $S^u \cong L_c(S^u)$. By Theorem 4.1.2, S^u is a cycle and by Proposition 4.2.1, $L_{\bullet c}(S)$ is a positive cycle. By Theorem 1.2.4, two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic. Hence S is a positive cycle.

Conversely, Suppose S is a positive cycle then by Proposition 4.2.1, $L_{\bullet c}(S)$ is also a positive cycle. Hence by Theorem 1.2.4, $S \sim L_{\bullet c}(S)$. This is illustrated in **Figure 4.6**. Thus the result follows. □

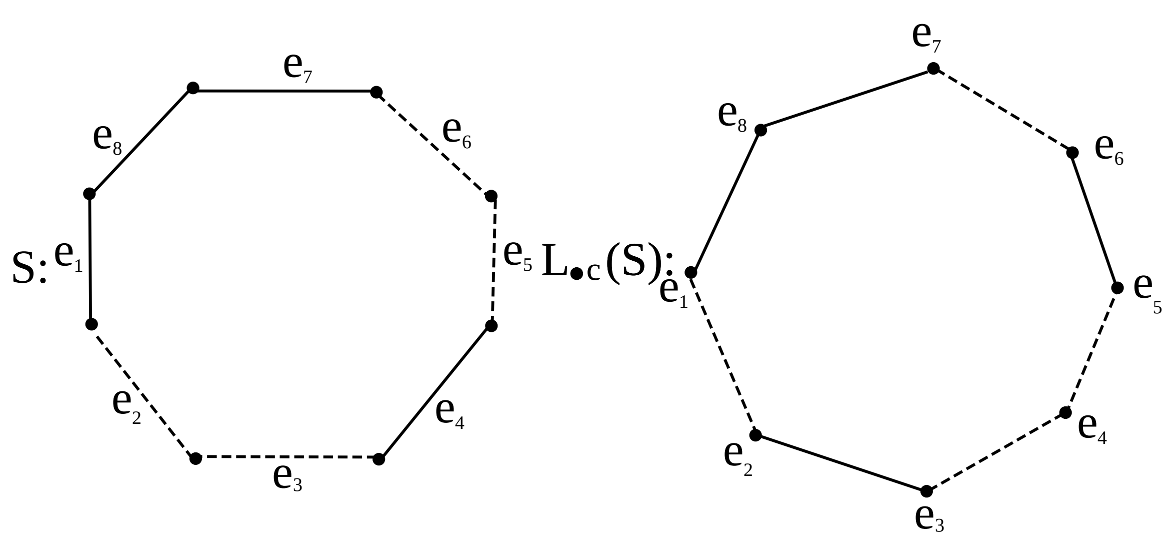

Figure 4.6: A signed graph S such that $S \sim L_{\bullet c}(S)$

Proposition 4.5.1. *For a signed graph S, $S \sim L_{\bullet}(S)$ if and only if S is a positive cycle.*

4.6 Switching equivalence of $L_{\bullet c}(S)$ and $L_{\bullet}(S)$ to $\eta(S)$

Theorem 4.6.1. *For a signed graph S, $\eta(S) \sim L_{\bullet c}(S)$ if and only if S is a positive even cycle or a negative odd cycle.*

Proof. Necessity:

Suppose for a signed graph S, $\eta(S) \sim L_{\bullet c}(S)$. This implies that $S^u \cong L_{\bullet c}(S^u)$, i.e., $S^u \cong L_c(S^u)$. By Theorem 4.1.2, S^u is a cycle and by Proposition 4.2.1, $L_{\bullet c}(S)$ is a positive cycle. By Theorem 1.2.4, two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic. Hence $\eta(S)$ is also a positive cycle; that is, S is a positive even cycle or a negative odd cycle.

Sufficiency:

Suppose S is a positive even cycle or a negative odd cycle. Then clearly $\eta(S)$ is a positive cycle. This is illustrated in **Figure 4.7** and by Proposition 4.2.1, $L_{\bullet c}(S)$ is a also positive cycle. Hence, by Theorem 1.2.4, $\eta(S) \sim L_{\bullet c}(S)$. This completes the proof. □

Proposition 4.6.1. *For a signed graph S, $\eta(S) \sim L_{\bullet}(S)$ if and only if S is a positive even cycle or a negative odd cycle.*

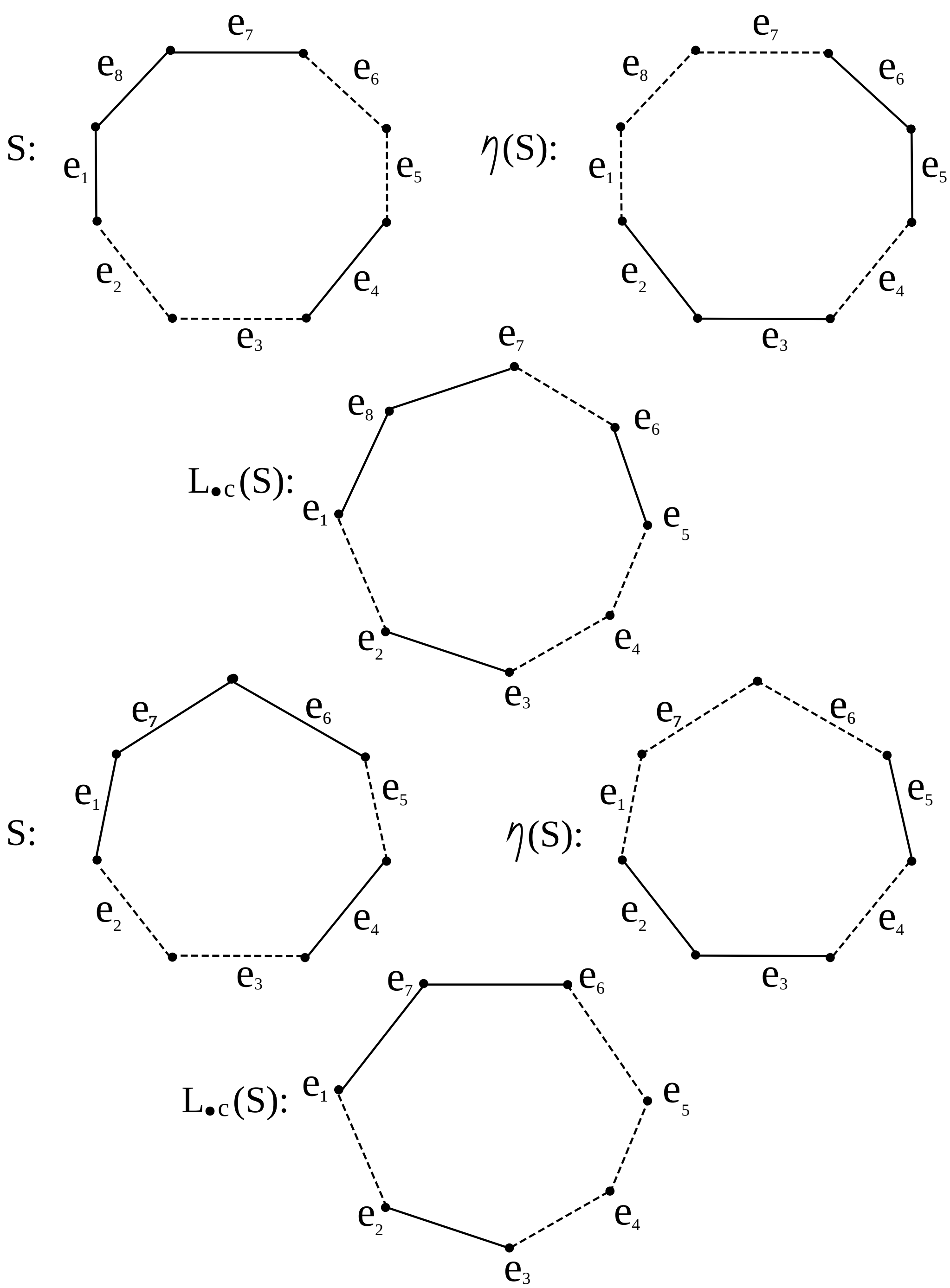

Figure 4.7: A signed graph S such that $\eta(S) \sim L_{\bullet c}(S)$

In following sections, we give results on lict and line signed graphs and these results have been reported in [2].

4.7 Lict and line signed graphs on K_p and C_n

Theorem 4.7.1. *A signed graph $S = (S^u, \sigma)$, on a complete graph $S^u := K_p$, $p \geq 3$, is a lict signed graph if and only if S is any one of the following:*

1. *all-positive*

2. *all-negative for $p = 3$ or even*

3. *heterogeneous, in which all-negative maximal subsignedgraph is an even order complete signed graph.*

Proof. Necessity:

Let $S = (S^u, \sigma)$, on a complete graph K_p, $p \geq 3$, be a lict signed graph. Therefore, $S \cong L_c(T)$ for some signed graph $T = (T^u, \sigma')$. This implies that $S^u \cong L_c(T^u)$, i.e., $K_p \cong L_c(T^u)$. By the definition of lict graph it is clear that

$$T^u = \begin{cases} K_3 \ or \ K_{1,2} & \text{if } p = 3; \\ K_{1,p-1} & \text{if } p \geq 4. \end{cases}$$

Since $S \cong L_c(T)$, we have following cases:

(1) If $T^u := K_3$, then

- for T containing at most one negative edge, S is an all-positive triangle.
- for T containing two negative edges, S is a triangle containing one negative edge.

- for all-negative T, S is an all-negative triangle, as shown in **Figure 4.8**.

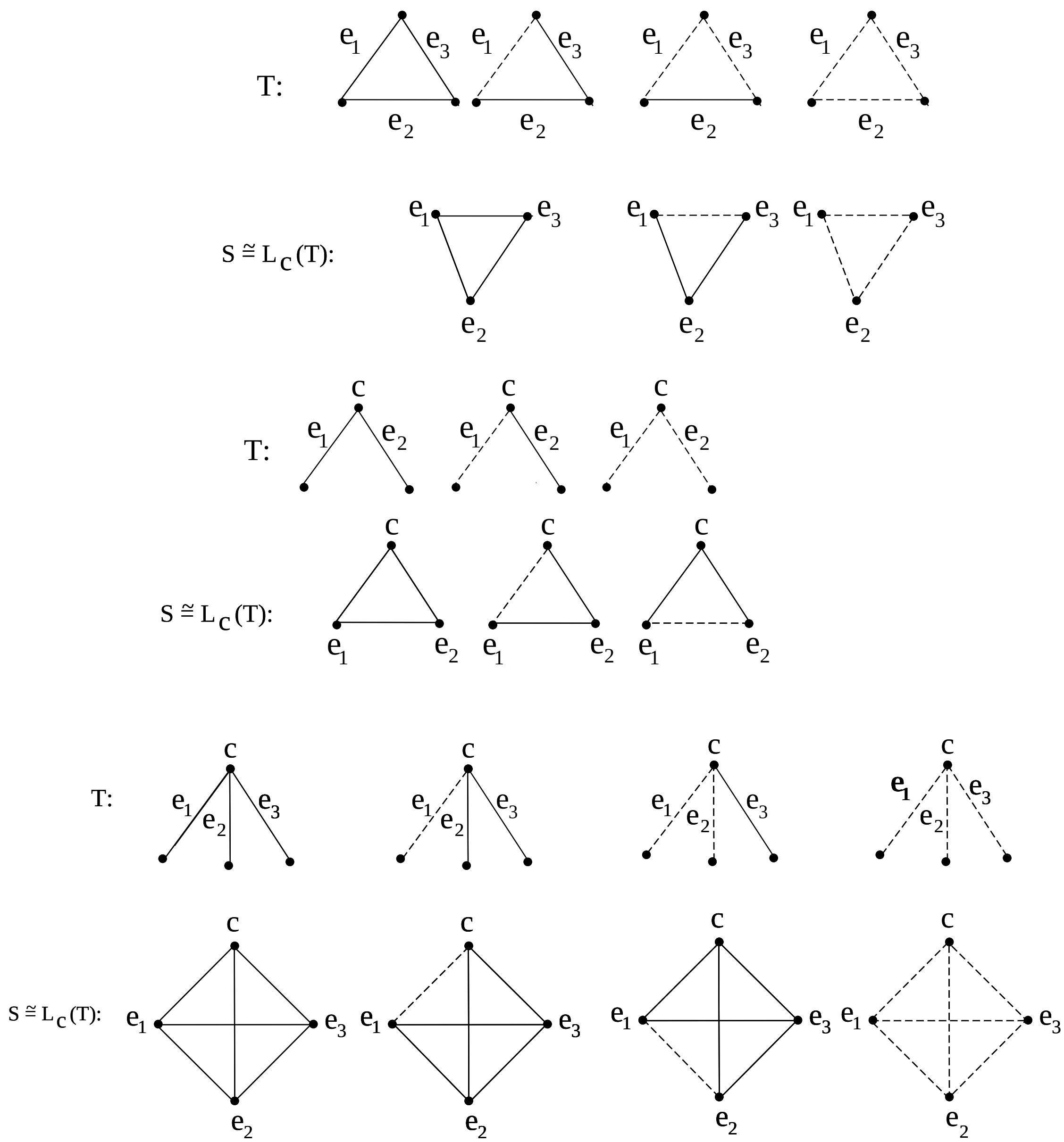

Figure 4.8: Signed graph T such that $S \cong L_c(T)$

(2) If $T^u := K_{1,p-1}$, $p \geq 3$, then

- for all-positive T, S is all-positive K_p .
- for all-negative T of odd order or heterogeneous T, S is heterogeneous, in which all-negative maximal subsignedgraph is

of an even order complete signed graph.

- for all-negative T of even order, S is an even order all-negative complete signed graph. This is illustrated in **Figure 4.8**.

Thus, the necessity follows.

Sufficiency:

Suppose conditions hold. We give the construction of a signed graph T whose lict signed graph is S. Let $\mathcal{P}(S^u) = \{S_1^u, S_2^u, ..., S_n^u\}$ be the partition of $E(S^u)$ into complete subgraphs in such a way that no vertex lies in more than two of these complete subgraphs. The vertices of T^u correspond to the set $\mathcal{P}(S^u)$ together with the set U of vertices of S^u belonging to only one of the complete subgraphs S_i^u leaving one such vertex for each S_i^u. Thus $V(T^u) = \mathcal{P}(S^u) \cup U$, two of these vertices are adjacent whenever they have a nonempty intersection; that is, T^u is the intersection graph $\Omega(\mathcal{P}(S^u) \cup U)$. Now, we construct a signed graph T on T^u such that an edge e_i in T is negative whenever its corresponding vertex in S has negative degree $\neq 0$. For this signed graph T, $S \cong L_c(T)$. This completes the proof.

□

In **Figure 4.9**, we construct signed graph T such that $S \cong L_c(T)$. Note that in this construction, the lict root T is unique.

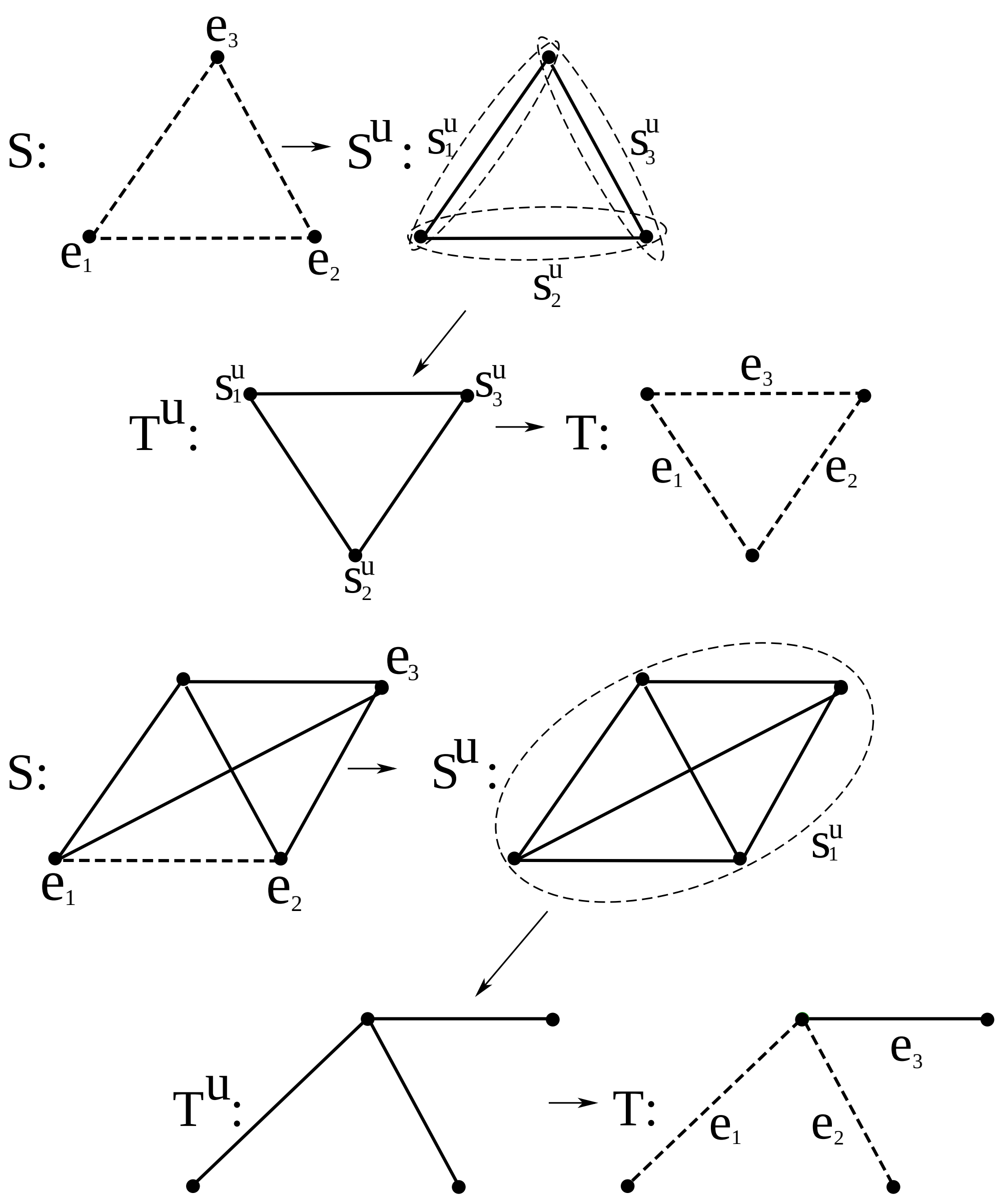

Figure 4.9: The construction of signed graph T from S such that $S \cong L_c(T)$

Theorem 4.7.2. *A signed graph* $S = (S^u, \sigma)$*, on a complete graph* $S^u := K_p$*,* $p \geq 2$*, is a line signed graph if and only if* S *is any one of the following:*

1. *homogeneous*

2. *heterogeneous, in which all-negative maximal subsignedgraph is complete.*

Proof. Necessity:

Let $S = (S^u, \sigma)$, on a complete graph K_p, $p \geq 2$, be a line signed graph. Therefore, $S \cong L(T)$ for some signed graph $T = (T^u, \sigma')$. This implies that $S^u \cong L(T^u)$, i.e., $K_p \cong L(T^u)$. Clearly,

$$T^u = \begin{cases} K_3 \; or \; K_{1,3} & \text{if } p = 3; \\ K_{1,p} & \text{if } p = 2 \text{ or } p \geq 4. \end{cases}$$

Since $S \cong L(T)$, we have following cases:

(1) If $T^u := K_3$, then

- for T containing at most one negative edge, S is an all-positive triangle.
- for T containing two negative edges, S is a triangle containing one negative edge.
- for all-negative T, S is an all-negative triangle, as shown in **Figure 4.10**.

(2) If $T^u := K_{1,p}$, $p \geq 2$, then

- for T containing at most one negative edge, S is all-positive K_p.
- for heterogeneous T containing at least two negative edges, S is heterogeneous, in which all-negative maximal subsignedgraph is complete.
- for all-negative T, S is an all-negative complete signed graph, as shown in **Figure 4.10**.

Thus, the necessity follows.

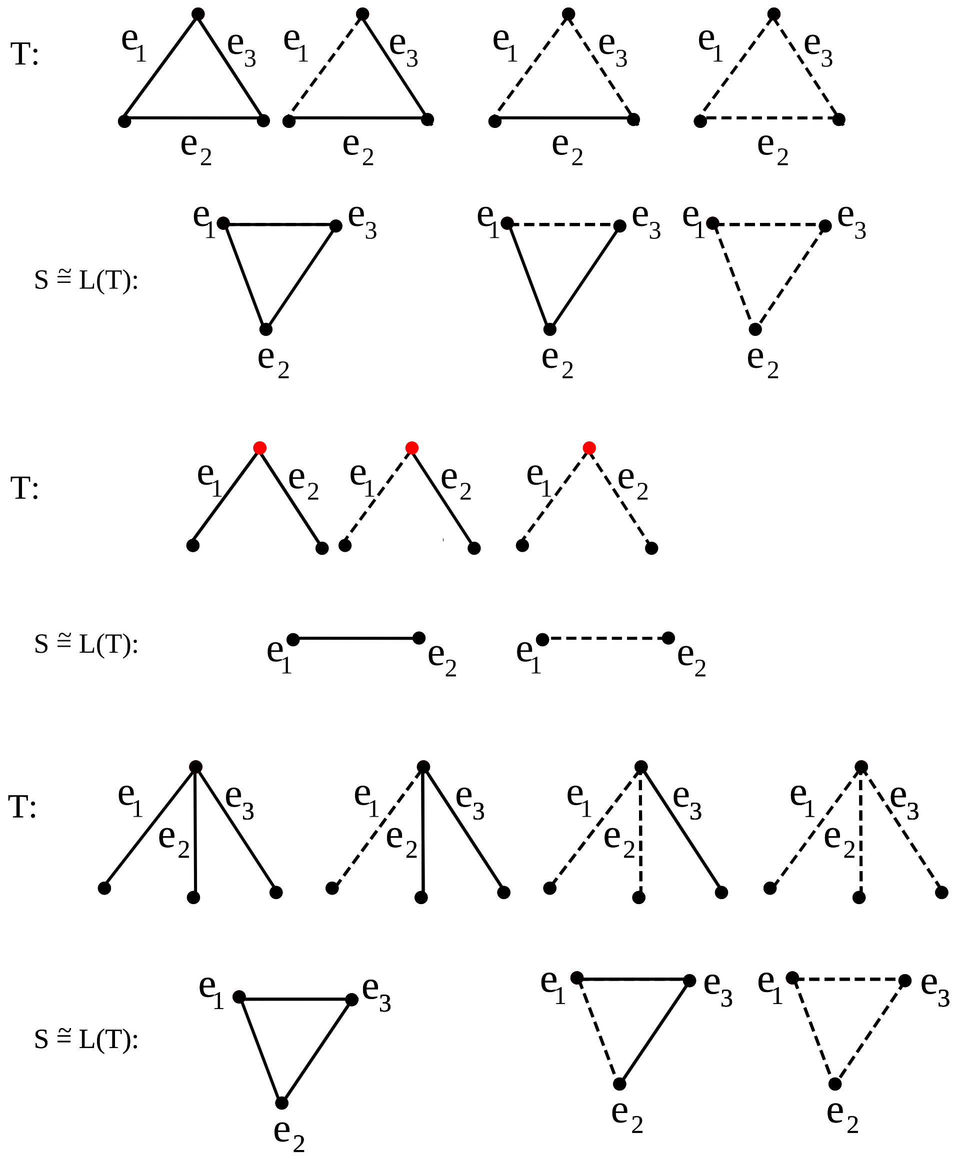

Figure 4.10: Signed graph T such that $S \cong L(T)$

Sufficiency:

Suppose conditions hold. We give the construction of a signed graph T whose line signed graph is S. By Theorem 2.2.1, S^u is a line graph. Let $\mathcal{P}(S^u) = \{S_1^u, S_2^u, ..., S_n^u\}$ be the partition of $E(S^u)$ into complete subgraphs in such a way that no vertex lies in more than two of these complete subgraphs. The vertices of T^u correspond to the set $\mathcal{P}(S^u)$ together with the set U of vertices of S^u belonging to only one of the complete subgraphs S_i^u. Thus $V(T^u) = \mathcal{P}(S^u) \cup U$, two of these vertices are adjacent whenever they have a nonempty intersection; that is, T^u is the intersection graph $\Omega(\mathcal{P}(S^u) \cup U)$. Now, we construct a signed graph T on T^u such that an edge e_i in T is negative whenever its corresponding vertex in S has negative degree $\neq 0$, as shown in **Figure 4.11**. For this signed graph T, $S \cong L(T)$; that is, S is a line signed graph. This completes the proof.

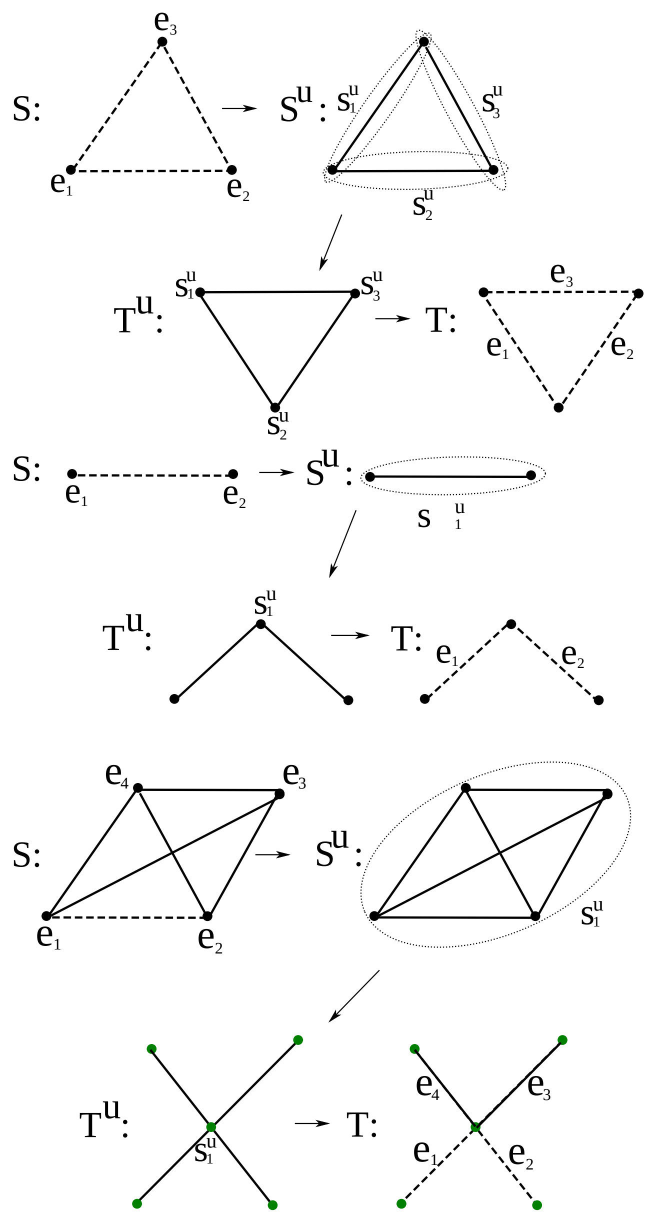

Figure 4.11: The construction of signed graph T from S such that $S \cong L(T)$

Note that by this construction, the line-root T of S is unique.

□

Theorem 4.7.3. *A signed graph $S = (S^u, \sigma)$, on a cycle $S^u := C_n$, is lict signed graph if and only if S does not contain a path $P_4 = (u, v, w, x)$ and a triangle (u, v, w) in which exactly one edge vw is positive.*

Proof. Necessity:
Let $S = (S^u, \sigma)$, on cycle C_n, be a lict signed graph. Therefore $S \cong L_c(T)$ for some signed graph T. We prove the necessity by contradiction:

Assume that S contains a path $P_4 = (u, v, w, x)$ in which exactly one edge vw is positive, i.e., $uv, wx \in E^{-}(S)$. By the definition of $L_c(T)$, two adjacent vertices of S can not correspond to two cut-vertices of T. Since $S \cong L_c(T)$, we have following possible cases:

Case I: $u, v, w, x \in E(T)$. Then $uv, wx \in E^{-}(S) \Rightarrow u, v, w, x \in E^{-}(T) \Rightarrow vw \in E^{-}(S)$.

Case II: $u \in C(T)$ and $v, w, x \in E(T)$. Then $uv, wx \in E^{-}(S) \Rightarrow u$ is negative cut-vertex of T and $v, w, x \in E^{-}(T) \Rightarrow vw \in E^{-}(S)$.

Case III: $u, w \in C(T)$ and $v, x \in E(T)$. Then $uv, wx \in E^{-}(S) \Rightarrow u, w$ are negative cut-vertices of T and $v, x \in E^{-}(T) \Rightarrow vw \in E^{-}(S)$.

Similarly, if S contains a triangle (u, v, w) in which exactly one edge vw is positive, i.e., $uv, wu \in E^{-}(S)$. Then we have following two possible cases:

Case I: $u, v, w \in E(T)$. Then $uv, wu \in E^{-}(S) \Rightarrow u, v, w \in E^{-}(T) \Rightarrow vw \in E^{-}(S)$.

Case II: $u \in C(T)$ and $v, w \in E(T)$. Then $uv, wu \in E^{-}(S) \Rightarrow u$ is negative cut-vertex of T and $v, w \in E^{-}(T) \Rightarrow vw \in E^{-}(S)$.

Thus, in all possible cases, we get $vw \in E^{-}(S)$ that contradicts our assumption. Hence, the necessity follows.

Sufficiency:

Suppose conditions hold. We can construct a signed graph T whose lict signed graph is S by the procedure as discussed in the sufficiency of Theorem 4.7.1, as shown in **Figure 4.12**. For this signed graph T, $S \cong L_c(T)$; that is, S is a lict signed graph. This completes the proof. □

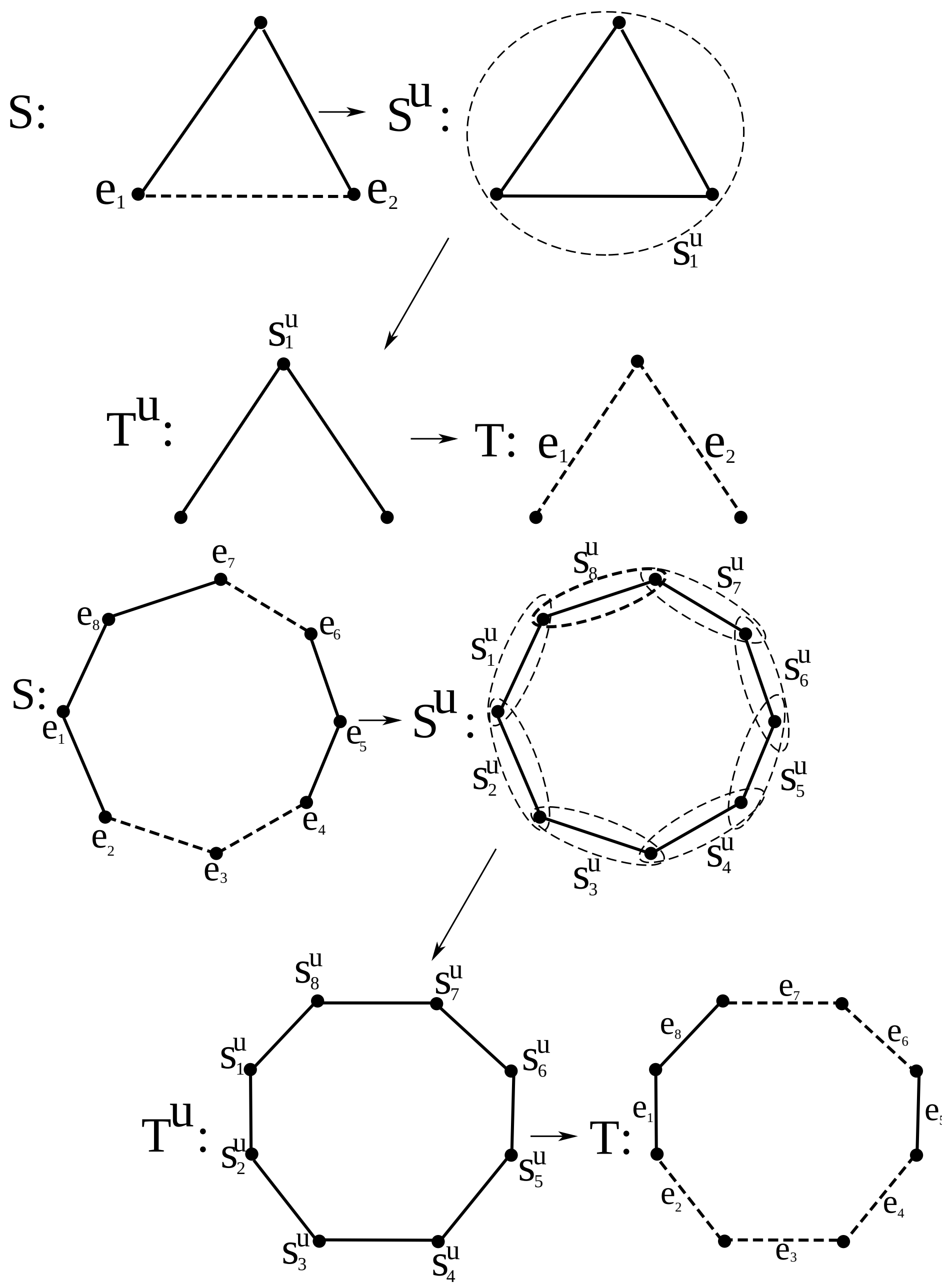

Figure 4.12: The construction of signed graph T from S such that $S \cong L_c(T)$

Corollary 4.7.1. *A signed graph $S = (S^u, \sigma)$, on a cycle $S^u := C_n$, is line signed graph if and only if S does not contain a path $P_4 = (u, v, w, x)$ and a triangle (u, v, w) in which exactly one edge vw is positive.*

Corollary 4.7.1 has been proved in [3].

4.8 Lict and line signed graphs on $K_{m,n}$

Theorem 4.8.1. *A signed graph $S = (S^u, \sigma)$ on a complete bipartite graph $S^u := K_{m,n}$, is a lict signed graph if and only if S is any one of the following:*

1. *homogeneous cycle C_4.*

2. *heterogeneous cycle C_4 containing exactly one negative section of length one or two.*

Proof. Necessity:
Let $S = (S^u, \sigma)$, on a complete bipartite graph $K_{m,n}$ be a lict signed graph. Therefore, $S \cong L_c(T)$ for some signed graph T. This implies that $S^u \cong L_c(T^u)$ or $K_{m,n} \cong L_c(T^u)$, i.e., $K_{m,n}$ is a lict graph.

Let v be a cut-vertex of T^u then clearly $d(v) \geq 2$ and by the definition of lict graph, the edges incident with cut-vertex v in T^u induce a homogeneous complete subgraph $K_{d(v)+1}$, i.e., K_p, $p \geq 3$ in $S^u := K_{m,n}$. Since a complete bipartite graph does not contain any odd cycle and thus K_p, $p \geq 3$, can not be a subgraph of $K_{m,n}$. Hence, $C(T^u) = \phi$. Therefore, $K_{m,n}$ is also a line graph. Since $K_{1,3}$ is a forbidden induced subgraph of a line graph, therefore, $m \leq 2$ and $n \leq 2$. Also, $K_{m,n} \ncong K_{1,1}$ and $K_{1,2}$, since they are not lict graph of any graph. Thus, $K_{m,n} \cong C_4$. Since $K_{m,n} \cong L_c(T^u)$, by the definition of lict graph, $T^u \cong C_4$.
Since $S \cong L_c(T)$, we have following cases:

- for T containing at most two negative sections each of length one, S is an all-positive cycle C_4.

- for T containing a negative sections of length 2 or 3, S is a heterogeneous cycle C_4 containing a negative sections of length 1 or 2 respectively.
- for all-negative T, S is an all-negative cycle C_4, as shown in **Figure 4.13**.

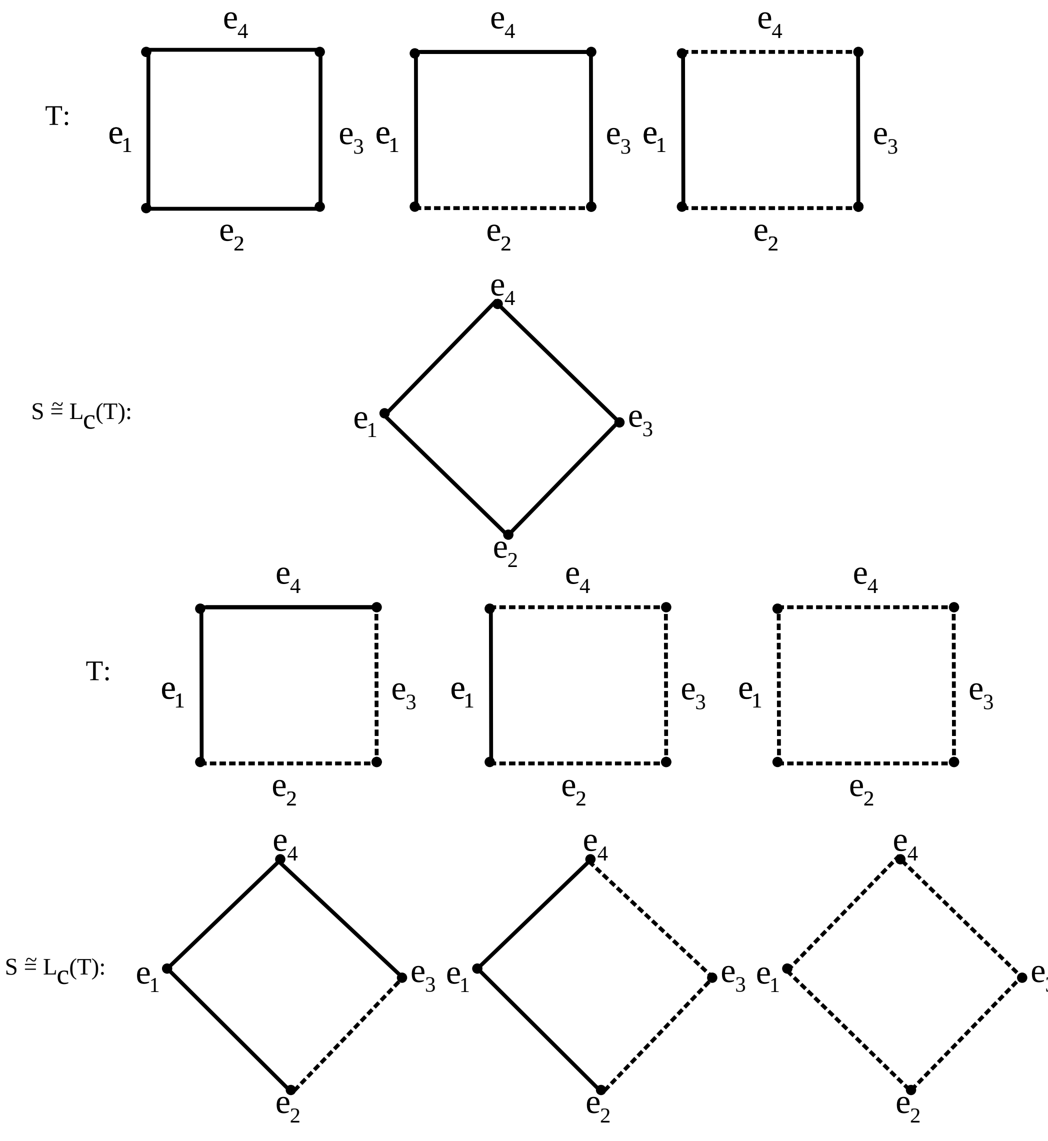

Figure 4.13: Signed graph T such that $S \cong L_c(T)$

Thus, the necessity follows.
Sufficiency:
Suppose conditions hold. Then by Theorem 4.7.3, S is a lict signed graph. Hence the result follows. □

Corollary 4.8.1. *A signed graph $S = (S^u, \sigma)$ on a complete bipartite graph $S^u := K_{m,n}$, is a line signed graph if and only if S is any one of the following:*

1. *homogeneous cycle C_4.*

2. *heterogeneous cycle C_4 containing exactly one negative section of length one or two.*

4.9 Balanced lict and line signed graphs

One important theorem which is useful to prove a result of this section is given below.

Theorem 4.9.1. [4] *For a signed graph S, $L(S)$ is balanced if and only if the following conditions hold:*

(i) *For any cycle Z in S,*

- **(a)** *if Z is all-negative, then Z has even length;*
- **(b)** *if Z is heterogeneous, then Z has an even number of negative sections with even length;*

(ii) *for $v \in V(S)$, if $d(v) \geq 3$, then there is at most one negative edge incident at v in S.*

Theorem 4.9.2. *For a signed graph S, $L_c(S)$ is balanced if and only if the following conditions hold in S:*

(i) *For any cycle Z in S,*

- **(a)** *if Z is all-negative, then Z has even length;*
- **(b)** *if Z is heterogeneous, then Z has an even number of negative sections with even length;*

(ii) *for $v \in V(S)$,*

- **(a)** *if $v \notin C(S)$ and $d(v) \geq 3$, then $d^-(v) \leq 1$;*
- **(b)** *if $v \in C(S)$, then $d^-(v) = 0$.*

Proof. Necessity:
The necessity of **(i)** and **(ii)(a)** follow from Theorem 4.9.1. We prove the necessity of **(ii)(b)** by contrapositive:

Assume that for a cut-vertex c of S, $d^-(c) \neq 0$. By the definition of $L_c(S)$, if $d^-(c) = 1$, 2 or $d^-(c) \geq 3$, then the edges incident with c will induce a triangle containing one negative edge or an all-negative triangle respectively in $L_c(S)$ that makes $L_c(S)$ unbalanced. Thus, the necessity of **(ii)(b)** follows.
Sufficiency:
A cycle in $L_c(S)$ is induced due to a cycle or a cut-vertex of degree 2 or a vertex of degree ≥ 3 or their combinations in S. Suppose conditions **(i)**and **(ii)** hold in S then every chordless cycle in $L_c(S)$ will be positive. By Lemma 1.2.1, a signed graph in which every chordless cycle is positive, is balanced. Hence $L_c(S)$ is balanced. This completes the proof. □

4.10 Switching equivalence of $L_c(S)$ and $L(S)$ to S

Theorem 4.10.1. *For a signed graph S, $S \sim L_c(S)$ if and only if S is any one of the following:*

1. *homogeneous cycle.*

2. *heterogeneous cycle having an even number of negative sections.*

Proof. Suppose for a signed graph S, $S \sim L_c(S)$. This implies that $S^u \cong L_c(S^u)$. By Theorem 4.1.2, S^u is a cycle. By Theorem 1.2.4, two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic. Hence, if S is homogeneous cycle then there is nothing to prove as $S \cong L_c(S)$ and if S is heterogeneous cycle having k negative sections of even lengths (say $n_{e1}, n_{e2}, ..., n_{ek}$) and m negative sections of odd lengths (say $n_{o1}, n_{o2}, ..., n_{om}$) then,

$$S \sim L_c(S) \Leftrightarrow S \text{ and } L_c(S) \text{ are cycle isomorphic}$$
$$\Leftrightarrow |E^{-}(S)| + |E^{-}(L_c(S))| \equiv 0 \pmod 2$$

$$\Leftrightarrow [\sum_{i=1}^{k} n_{ei} + \sum_{i=1}^{m} n_{oi}] + [\sum_{i=1}^{k}(n_{ei} - 1) + \sum_{i=1}^{m}(n_{oi} - 1)] \equiv 0 \pmod 2$$

$$\Leftrightarrow -(m+k) \equiv 0 \pmod 2 \text{ or } m+k \equiv 0 \pmod 2$$

Thus, the result follows. □

Proposition 4.10.1. *For a signed graph S, $S \sim L(S)$ if and only if S is any one of the following:*

1. *homogeneous cycle.*

2. *heterogeneous cycle having an even number of negative sections.*

4.11 Switching equivalence $L_c(S)$ and $L(S)$ to $\eta(S)$

Let a and b are two integers and n is a positive integer then notation $a \equiv b \pmod n$ means that n divides $a-b$ and we say that a is congruent to b modulo n. Clearly, if $a \equiv b \pmod n$ then $a \pmod n = b \pmod n$.

Theorem 4.11.1. *For a signed graph S of order n, $\eta(S) \sim L_c(S)$ if and only if S is any one of the following:*

1. *homogeneous even cycle.*

2. *heterogeneous cycle having number of negative sections $\equiv n \pmod 2$.*

Proof. Suppose for a signed graph S of order n, $\eta(S) \sim L_c(S)$. This implies that $S^u \cong L_c(S^u)$. By Theorem 4.1.2, S^u is a cycle. By Theorem 1.2.4, two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic. Hence,

$$\eta(S) \sim L_c(S) \Leftrightarrow |E^{-}(\eta(S))| + |E^{-}(L_c(S))| \equiv 0 \pmod 2$$

Therefore, we have following three possible cases:

Case I: If S is an all-positive cycle then

$$\begin{aligned} \eta(S) \sim L_c(S) &\Leftrightarrow n + 0 \equiv 0 \pmod 2 \\ &\Leftrightarrow n \equiv 0 \pmod 2 \end{aligned}$$

Case II: If S is an all-negative cycle (as shown in **Figure 4.14**) then

$$\begin{aligned} \eta(S) \sim L_c(S) &\Leftrightarrow 0 + n \equiv 0 \pmod 2 \\ &\Leftrightarrow n \equiv 0 \pmod 2 \end{aligned}$$

Thus, in I and II cases, S must be homogeneous even cycle.

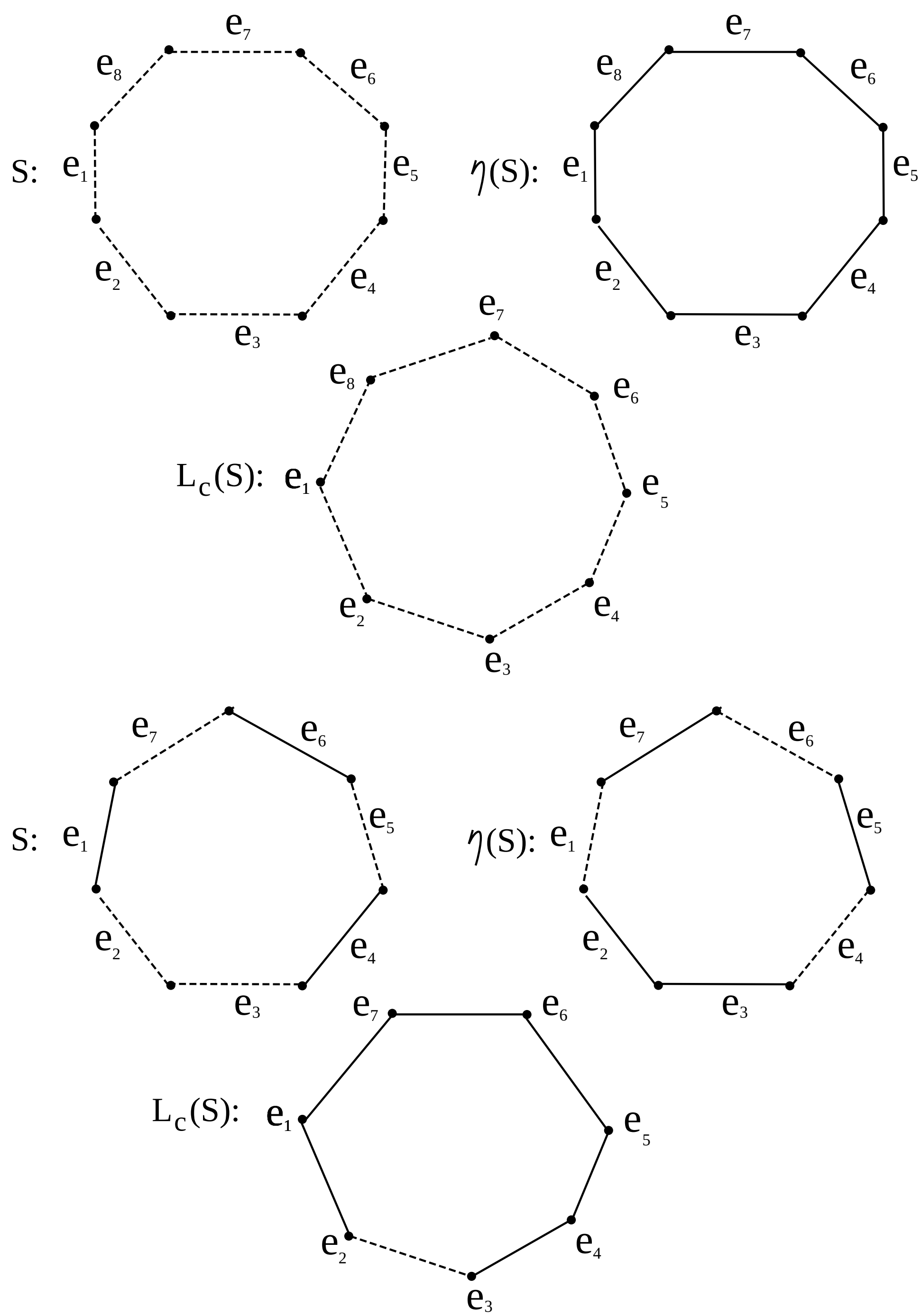

Figure 4.14: A Signed graph S such that $\eta(S) \sim L_c(S)$

Case III: If S is heterogeneous cycle having k negative sections of even lengths (say $n_{e1}, n_{e2}, ..., n_{ek}$) and m negative sections of odd lengths (say $n_{o1}, n_{o2}, ..., n_{om}$) then
$\eta(S) \sim L_c(S) \Leftrightarrow [n - (\sum_{i=1}^{k} n_{ei} + \sum_{i=1}^{m} n_{oi})] + [\sum_{i=1}^{k}(n_{ei} - 1) + \sum_{i=1}^{m}(n_{oi} - 1)] \equiv 0 \pmod 2$

$\Leftrightarrow n - (m + k) \equiv 0 \pmod 2$ or $m + k \equiv n \pmod 2$
Therefore, S must be heterogeneous cycle having number of negative sections $\equiv n \pmod 2$.

Thus, the result follows. □

Proposition 4.11.1. *For a signed graph S of order n, $\eta(S) \sim L(S)$ if and only if S is any one of the following:*

1. *homogeneous even cycle.*

2. *heterogeneous cycle having number of negative sections $\equiv n \mod 2$.*

4.12 Conclusion and Scope

In this chapter, we have established many results on $\bullet$-lict signed graphs, lict signed graphs and also for $\bullet$-line signed graphs and line signed graphs. Study on litact signed graphs is yet to be taken up. After defining litact signed graphs, we propose that study done in this chapter for lict signed graphs can be taken up for litact signed graphs. Mathad and Narayanker in [7], characterized signed graphs S and S' for which $L_\times(S) \sim L_{\times c}(S')$, $J(S) \sim L_{\times c}(S')$ and $T_1(S) \sim L_{\times c}(S')$, where $J(S)$ and $T_1(S)$ are jump signed graph and semitotal signed graph of S respectively. This study related to dot-lict signed graphs and lict signed graphs as well as for litact signed graphs is yet open.

* * * * *

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Chapter 5

$\mathcal{C}$-CONSISTENT AND $\mathcal{C}$-CYCLE COMPATIBLE $\bullet$-LINE SIGNED GRAPHS

Sinha and Garg [1] have established structural characterizations of signed graph S so that its line signed graphs $L(S)$ and $\times$-line signed graphs $L_\times(S)$ are $\mathcal{C}$-consistent. In this chapter, we establish structural characterizations of signed graph S so that $L_\bullet(S)$ is $\mathcal{C}$-consistent and $\mathcal{C}$-cycle compatible.

5.1 Introduction

Recall that a cycle in a signed graph is called *canonically consistent (or $\mathcal{C}$-consistent)* with respect to canonical marking if it contains an even number of negative vertices and a signed graph is said to be $\mathcal{C}$-consistent if every cycle in it is $\mathcal{C}$-consistent. $\bullet$-line signed graph of a signed graph has been defined in previous chapter.

A vertex v of even (odd) degree is called *even* (*odd*) vertex. The *edge degree* of an edge uv, denoted by $d_e(uv)$, is the total number of edges adjacent to uv. Clearly, $d_e(uv) = d(u) + d(v) - 2$.

5.2 $\mathcal{C}$-consistent $\bullet$-line signed graphs

In this section, we give results on $\bullet$-line signed graphs and establish structural characterization of signed graphs S so that $L_\bullet(S)$ is

$\mathcal{C}$-consistent. These results have been reported in [2].

For a signed graph S, we define a set

$\mathcal{E}(S) = \{uv \in E(S)$:*u is a negative even vertex and v is positive or* u, v *are negative and* $d_e(uv)$ *is odd*$\}$

Observation 5.2.1. *For a signed graph S, $e \in \mathcal{E}(S)$ if and only if e is a negative vertex of $L_\bullet(S)$.*

Proof. Necessity:

Let $e = uv$ be any edge of a signed graph S. By the definition of $L_\bullet(S)$, $e = uv$ is a vertex of $L_\bullet(S)$ whose negative degree depends on $d^-(u)$, $d^-(v)$ and $d_e(uv)$ in S. If $e \in \mathcal{E}(S)$ then we have following two possible cases:

Case I: u is a negative even vertex and v is a positive vertex then $d_S(u) - 1$ (that is odd) negative edges will be incident to vertex $e = uv$ in $L_\bullet(S)$, here $d_S(u)$ denotes the degree of a vertex u in S. Hence e is a negative vertex of $L_\bullet(S)$.

Case II: If both u and v are negative vertices and $d_e(uv)$ is odd then $d_e(uv)$ (that is odd) negative edges will be incident to vertex $e = uv$ in $L_\bullet(S)$. Hence e is a negative vertex of $L_\bullet(S)$.

Thus, the necessity follows.

Sufficiency:

We prove the sufficiency by contrapositive, i.e., we prove that if $e \notin \mathcal{E}(S)$ then e is a positive vertex of $L_\bullet(S)$.

If $e \notin \mathcal{E}(S)$ then there are following possibilities for edge $e = uv$ of S:

Case I: u and v are positive vertices. Then all the edges incident to vertex $e = uv$ in $L_\bullet(S)$ will be positive. Hence e is a positive vertex of $L_\bullet(S)$.

Case II: u is a negative odd vertex and v is a positive vertex. Then, $d_S(u) - 1$ (that is even) negative edges will be incident to vertex $e = uv$ in $L_\bullet(S)$. Hence e is a positive vertex of $L_\bullet(S)$.

Case III: u and v both are negative vertices and $d_e(uv)$ is even. Then, $d_e(uv)$ (that is even) negative edges will be incident to vertex $e = uv$ in $L_\bullet(S)$. Hence e is a positive vertex of $L_\bullet(S)$.

This completes the proof. □

Observation 5.2.2. *In a signed graph S, an even number of edges of any cycle belong to $\mathcal{E}(S)$.*

Proof. We prove this Observation by mathematical induction:
It is easy to observe that an even number of edges of a cycle C_3 belong to $\mathcal{E}(S)$.
Let an even number of edges of cycle C_k, $k > 3$, belong to $\mathcal{E}(S)$. To obtain cycle C_{k+1}, we divide an edge uv of the cycle by the vertex w, thus we have two edges uw and wv. In order to show that C_{k+1} has an even number of edges belonging to $\mathcal{E}(S)$, we have the following two cases:

Case I: $uv \in \mathcal{E}(S)$ then it can be easily observed that uw or $wv \in \mathcal{E}(S)$, as w is a positive vertex as shown in **Figure 5.1**. In case $uv \in \mathcal{E}(S)$ due to the fact that u and v both are negative and $d_e(uv)$ is odd then it can also be seen that uw or $wv \in \mathcal{E}(S)$.

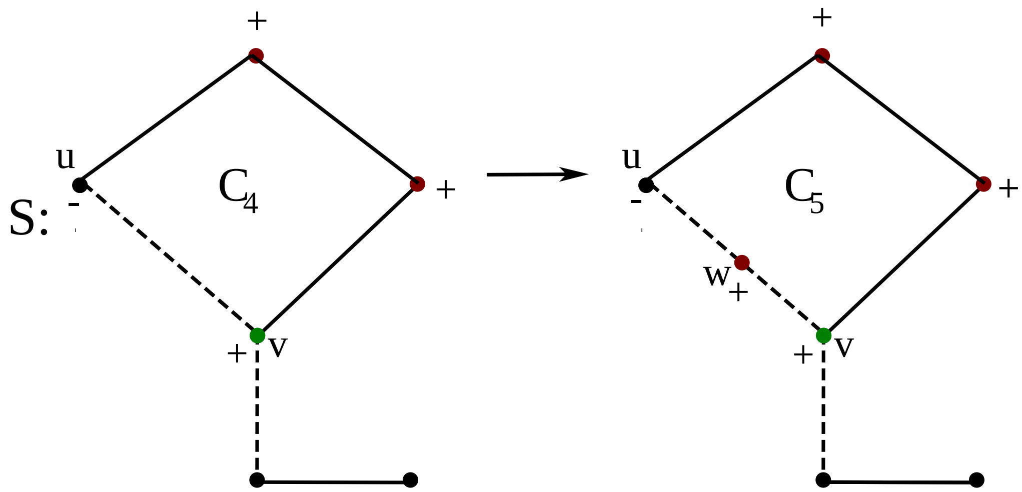

Figure 5.1: A signed graph S

Case II: $uv \notin \mathcal{E}(S)$ then $uw, wv \in \mathcal{E}(S)$ or $uw, wv \notin \mathcal{E}(S)$. As shown in **Figure 5.2**, $uw, wv \in \mathcal{E}(S)$.

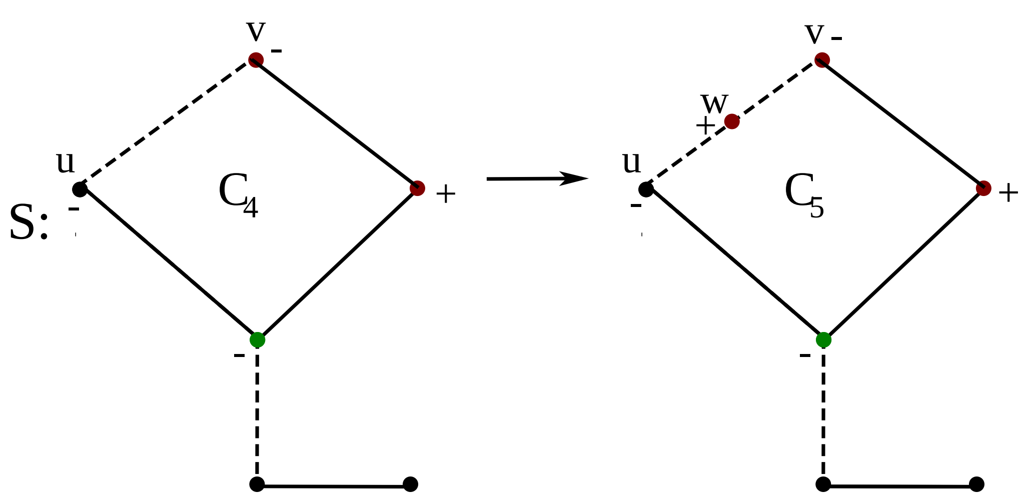

Figure 5.2: A signed graph S

Thus, an even number of edges of cycle C_{k+1} belong to $\mathcal{E}(S)$. This completes the proof. □

Lemma 5.2.1. *Every cycle of a signed graph S induces a $\mathcal{C}$-consistent cycle in $L_{\bullet}(S)$.*

Proof. By the definition of $L_{\bullet}(S)$, a cycle $Z = v_1e_1v_2e_2...v_ne_nv_1$ of a signed graph S induces a cycle $Z' = e_1e_2...e_ne_1$ in $L_{\bullet}(S)$. By Observation 5.2.2, an even number of edges of cycle Z of S belong to $\mathcal{E}(S)$ and by Observation 5.2.1, if $e \in \mathcal{E}(S)$ then e is a negative vertex of $L_{\bullet}(S)$. Hence an even number of vertices of cycle Z' are negative. Thus, Z' is $\mathcal{C}$-consistent. This completes the proof. □

Theorem 5.2.1. *For a signed graph S, $L_{\bullet}(S)$ is $\mathcal{C}$-consistent if and only if the following conditions hold for every vertex v of S:*

(1) *If $d(v) = 3$ then an even number of edges incident to v belong to $\mathcal{E}(S)$ and also if two edges e_i, e_j incident to v lye on a cycle then both of the edges e_i and e_j are of the same parity (i.e., $e_i, e_j \in \mathcal{E}(S)$ or $e_i, e_j \notin \mathcal{E}(S)$) and*

(2) *if $d(v) \geq 4$ then no edge incident to v belongs to $\mathcal{E}(S)$.*

Proof. Necessity:

Suppose for a signed graph S, $L_{\bullet}(S)$ is $\mathcal{C}$-consistent, i.e., an even number of vertices of every cycle of $L_{\bullet}(S)$ are negative, as shown in **Figure 5.3**.

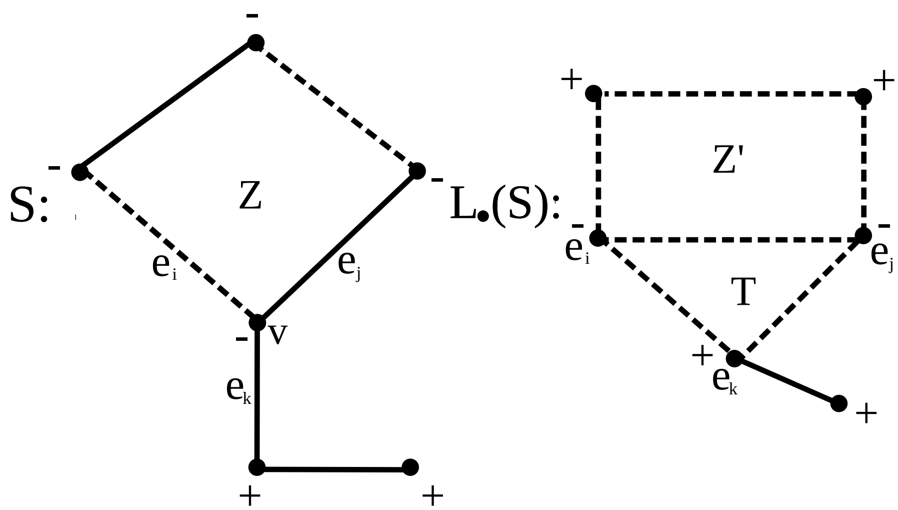

Figure 5.3: A signed graph S and its $L_\bullet(S)$

If v is a vertex of S of degree 3 then the edges incident to vertex v (say e_i, e_j and e_k) induce a triangle (say T) (e_i, e_j, e_k) in $L_\bullet(S)$. Since $L_\bullet(S)$ is $\mathcal{C}$-consistent, an even number of vertices of triangle T are negative. By Observation 5.2.1, $e \in \mathcal{E}(S)$ if and only if e is a negative vertex of $L_\bullet(S)$. Hence, an even number of edges incident to v must belong to $\mathcal{E}(S)$.

Further, if two edges e_i, e_j incident to v lye on a cycle Z then in $L_\bullet(S)$ there will a cycle Z' and a triangle T having one common edge e_ie_j. Since $L_\bullet(S)$ is $\mathcal{C}$-consistent, an even number of vertices of cycle Z', triangle T and a cycle due to the symmetric sum of cycles of Z' and T are negative.

Now, since by Lemma 5.2.1, Z' is $\mathcal{C}$-consistent and by above discussion T is also $\mathcal{C}$-consistent, if only one edge e_i or e_j belongs to $\mathcal{E}(S)$ then only one vertex e_i or e_j in $L_\bullet(S)$ will be negative and then combined cycle of Z' and T will contain an odd number of negative vertices that makes $L_\bullet(S)$ $\mathcal{C}$-inconsistent, as shown in **Figure 5.4**.

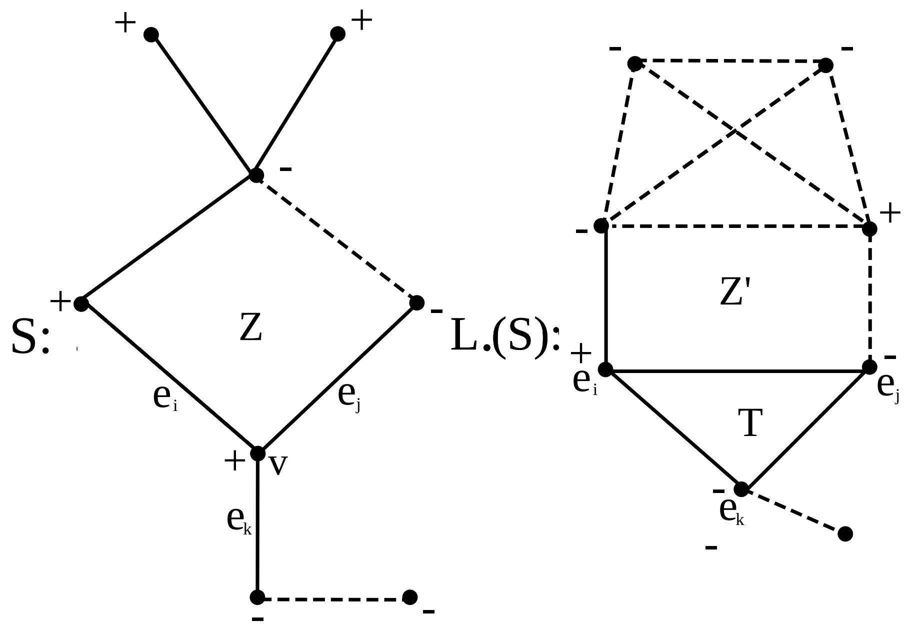

Figure 5.4: A signed graph S and its $\mathcal{C}$-inconsistent $L_\bullet(S)$

Therefore, both of the edges e_i and e_j must be of the same parity (i.e., $e_i, e_j \in \mathcal{E}(S)$ or $e_i, e_j \notin \mathcal{E}(S)$).

Next, we prove the necessity of condition **(2)** by contrapositive, i.e., we prove that if for a vertex $v \in V(S)$ of $d(v) \geq 4$, at least one edge incident to v belongs to $\mathcal{E}(S)$ then $L_\bullet(S)$ is $\mathcal{C}$-inconsistent. Suppose that for a vertex $v \in V(S)$ of $d(v) \geq 4$, at least one edge incident to v belongs to $\mathcal{E}(S)$. By the definition of $L_\bullet(S)$, the edges incident to v induce a complete subsignedgraph K_p, $p \geq 4$ in $L_\bullet(S)$. Since by Observation 5.2.1, $e \in \mathcal{E}(S)$ if and only if e is a negative vertex of $L_\bullet(S)$, at least one vertex of K_p will be negative, that is, K_p will always contain a $\mathcal{C}$-inconsistent triangle that makes $L_\bullet(S)$ $\mathcal{C}$-inconsistent, as shown in **Figure 5.4**. Thus, the necessity of **(2)** follows.

Sufficiency:

A cycle in $L_\bullet(S)$ is induced due to a cycle or a vertex of degree≥ 3 or their combinations in S. By Lemma 5.2.1, every cycle in $L_\bullet(S)$ which is induced due to a cycle of S, is $\mathcal{C}$-consistent. Further, suppose conditions **(1)** and **(2)** hold in S then obviously every cycle in $L_\bullet(S)$ will be $\mathcal{C}$-consistent. Hence $L_\bullet(S)$ is $\mathcal{C}$-consistent. This completes the proof. □

5.3 $\mathcal{C}$-cycle compatible $\bullet$-line signed graphs

Theorem 5.3.1. *For a signed graph S, $L_\bullet(S)$ is $\mathcal{C}$-cycle compatible if and only if the following conditions hold in S:*

(i) *S is $\mathcal{C}$-consistent.*

(ii) *for a positive (negative) vertex v of $d(v) = 3$, an even (odd) number of edges incident to v must belong to $\mathcal{E}(S)$ and also if two edges e_i, e_j incident to v lye on a cycle then both of the edges e_i and e_j must (must not) be of the same parity (i.e., $e_i, e_j \in \mathcal{E}(S)$ or $e_i, e_j \notin \mathcal{E}(S)$).*

(iii) *for a positive (negative) vertex v of $d(v) \geq 4$, no edge (atleast one edge) incident to v belongs to $\mathcal{E}(S)$.*

Proof. Necessity:

Let for a signed graph S, $L_\bullet(S)$ be $\mathcal{C}$-cycle compatible, i.e., every cycle in $L_\bullet(S)$ is either positive and $\mathcal{C}$-consistent or negative and $\mathcal{C}$-inconsistent. By Lemma 5.2.1, every cycle Z' in $L_\bullet(S)$ which is induced due to a cycle Z of signed graph S, is $\mathcal{C}$-consistent. Hence, Z' must also be positive. By the definition of $L_\bullet(S)$, $|E^-(Z')|$ = the number of negative vertices in Z. Therefore, every cycle in S contains an even number of negative vertices, i.e., S is $\mathcal{C}$-consistent. Thus, **(i)** follows.

We prove the necessity of condition **(ii)** by contra positive, i.e., we prove that if condition **(ii)** does not hold then $L_\bullet(S)$ is not $\mathcal{C}$-cycle compatible.

Suppose for a positive vertex $v \in V(S)$ of $d(v) = 3$, an odd number of edges incident to v belong to $\mathcal{E}(S)$ (as shown in **Figure 5.5**, for vertices v_1 and v_2).

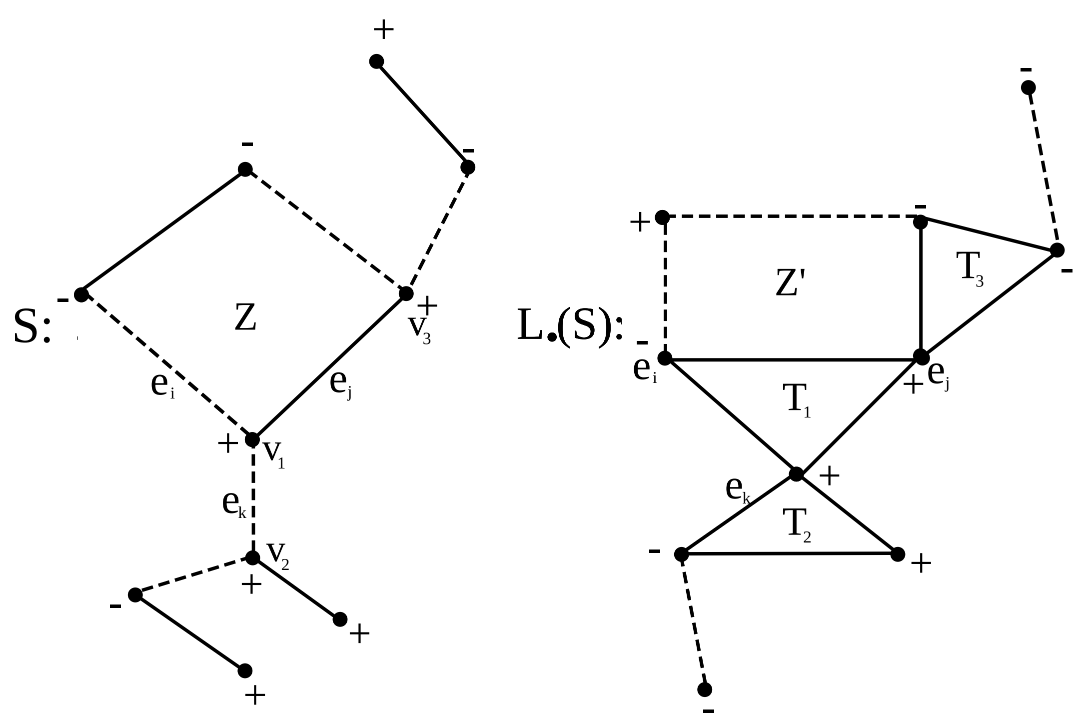

Figure 5.5: S and its $\mathcal{C}$-cycle incompatible $L_\bullet(S)$

By the definition of $\bullet$-line signed graph, the edges incident to vertex v induce a positive triangle (say T) in $L_\bullet(S)$. Since by Observation 5.2.1, $e \in \mathcal{E}(S)$ if and only if e is a negative vertex of $L_\bullet(S)$, an odd number of vertices of T will be negative, i.e., T is $\mathcal{C}$-inconsistent and positive. Thus, $L_\bullet(S)$ is not $\mathcal{C}$-cycle compatible. Further, suppose an even number of edges incident to positive vertex v belong to $\mathcal{E}(S)$ with two edges e_i, e_j incident to v lye on a cycle Z and $e_i \in \mathcal{E}(S)$, $e_j \notin \mathcal{E}(S)$ (as shown in **Figure 5.5**, for vertex v_3). Then in $L_\bullet(S)$, there will a cycle Z' and a triangle T having one common edge e_ie_j and a cycle due to the symmetric sum of cycles of Z' and T.

Now, By Lemma 5.2.1, Z' is $\mathcal{C}$-consistent and by Observation 5.2.1, T is also $\mathcal{C}$-consistent. If only one edge e_i or e_j belongs to $\mathcal{E}(S)$ then only one vertex e_i or e_j in $L_\bullet(S)$ will be negative and then combined cycle of Z' and T will contain an odd number of negative vertices that makes $L_\bullet(S)$ $\mathcal{C}$-inconsistent. By condition **(i)**, Z' is positive and due to the fact that v is a positive vertex, T is also positive. Hence combined cycle of Z' and T will also positive but this is $\mathcal{C}$-inconsistent. Therefore, $L_\bullet(S)$ is not $\mathcal{C}$-cycle compatible.

Similarly, we can show that for a vertex v of $d(v) = 3$, if v is a negative vertex and an even number of edges incident to v belong to $\mathcal{E}(S)$ (as shown in **Figure 5.6**, for vertices v_1 and v_2) then triangle T in $L_\bullet(S)$ is $\mathcal{C}$-consistent and negative. Further, suppose an odd number of edges incident to negative vertex v belong to $\mathcal{E}(S)$ with two edges e_i, e_j incident to v lye on a cycle Z and both of the edges e_i and e_j are of the same parity (i.e., $e_i, e_j \in \mathcal{E}(S)$ or $e_i, e_j \notin \mathcal{E}(S)$) (as shown in **Figure 5.6**, for vertex v_3) then combined cycle of Z' and T will be positive and $\mathcal{C}$-inconsistent. Therefore $L_\bullet(S)$ will not be $\mathcal{C}$-cycle compatible.
Thus, the necessity of **(ii)** follows.

Moreover, we also prove the necessity of condition **(iii)** by contra positive, i.e., we prove that if condition **(iii)** does not hold then $L_\bullet(S)$

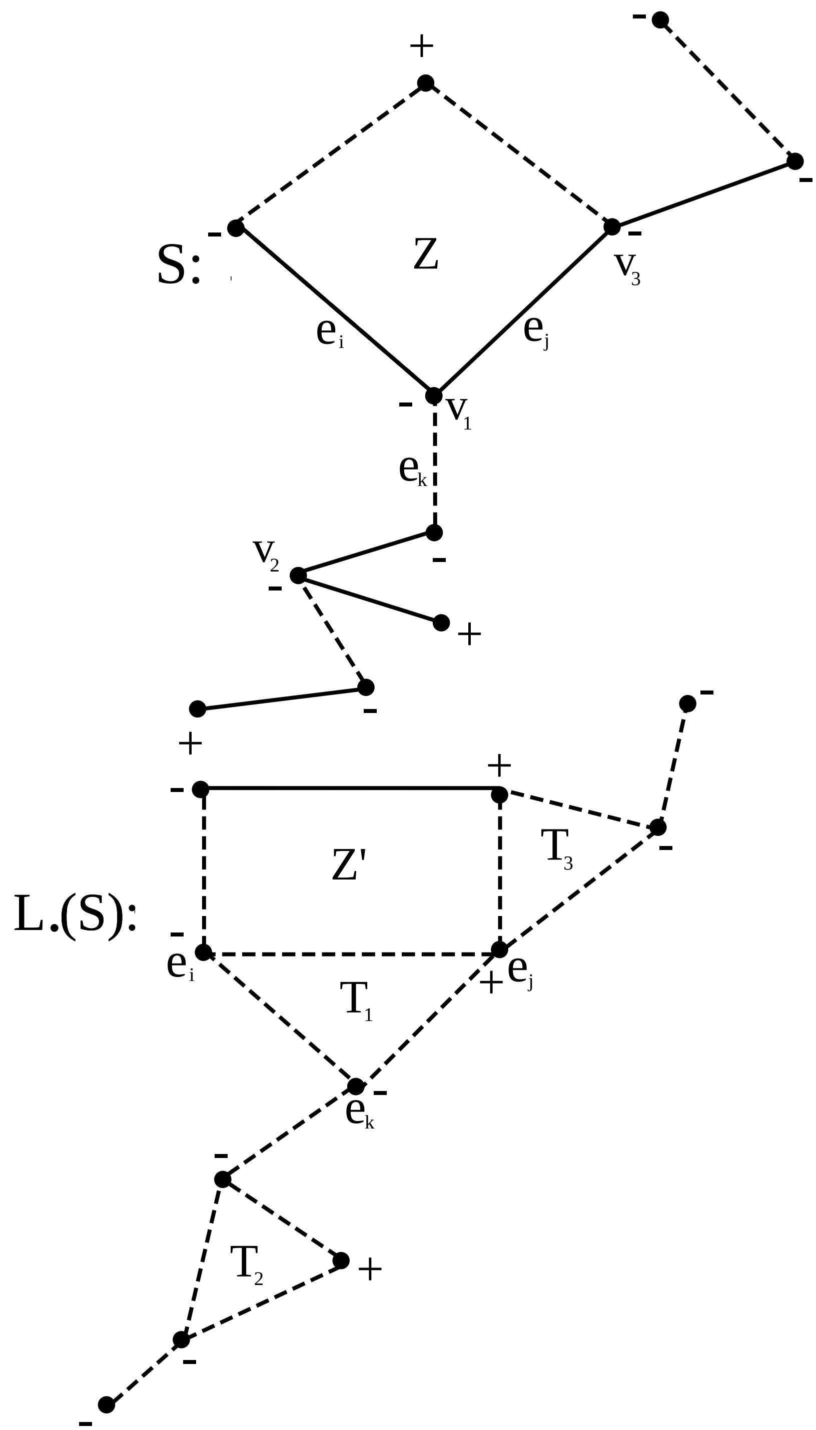

Figure 5.6: A signed graph S and its $\mathcal{C}$-cycle incompatible $L_\bullet(S)$

is not $\mathcal{C}$-cycle compatible. Suppose that for a positive vertex $v \in V(S)$ of $d(v) \geq 4$, at least one edge incident to v belongs to $\mathcal{E}(S)$ (as shown in **Figure 5.7**, for vertex v_1) then by the definition of $L_\bullet(S)$, the edges incident to v induce an all-positive complete subsignedgraph K_p, $p \geq 4$, in $L_\bullet(S)$ and by Observation 5.2.1, at least one vertex of K_p will be negative, that is, K_p will always contain a positive and $\mathcal{C}$-inconsistent triangle.

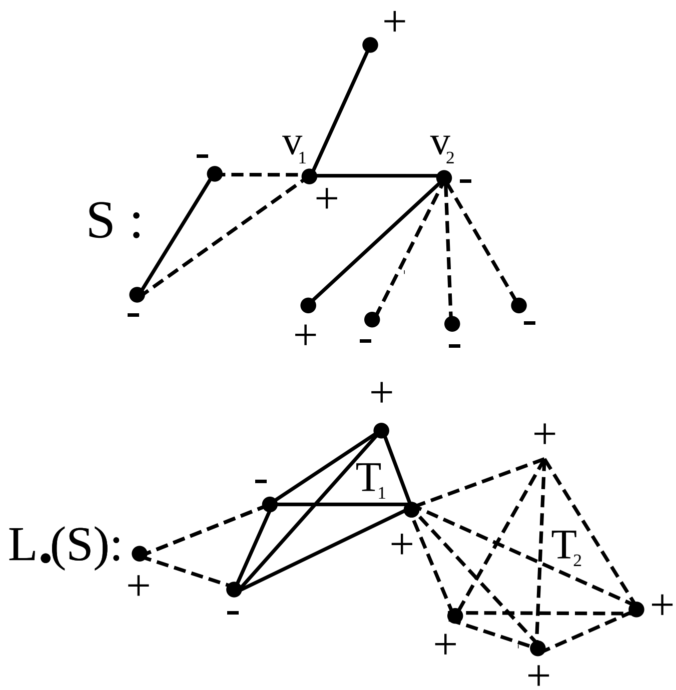

Figure 5.7: A signed graph S and its $\mathcal{C}$-cycle incompatible $L_\bullet(S)$

Hence $L_\bullet(S)$ is not $\mathcal{C}$-cycle compatible. Similarly, we can show that for a negative vertex v of $d(v) \geq 4$, if no edge incident to v belongs to $\mathcal{E}(S)$ (as shown in **Figure 5.7**, for vertex v_2) then by the definition of $L_\bullet(S)$, the edges incident to v induce an all-negative complete subsignedgraph K_p, $p \geq 4$ in $L_\bullet(S)$ and all vertices of K_p will be positive, that is, K_p will always contain a negative and $\mathcal{C}$-consistent triangle. Hence $L_\bullet(S)$ is not $\mathcal{C}$-cycle compatible. Thus, the necessity of **(iii)** follows.

Sufficiency:

Suppose conditions hold in S then it can be easily seen that every cycle in $L_\bullet(S)$ is positive and $\mathcal{C}$-consistent or negative and $\mathcal{C}$-inconsistent, i.e, $L_\bullet(S)$ is $\mathcal{C}$-cycle compatible. This completes the proof. $\square$

5.4 Conclusion and Scope

In this chapter, we have studied $\bullet$-line signed graphs and characterized signed graphs S for which $\bullet$-line signed graphs are $\mathcal{C}$-consistent and $\mathcal{C}$-cycle compatible. As reported earlier that the study on litact signed graphs yet to be taken up. Therefore, the characterization of signed graphs S whose litact signed graphs are $\mathcal{C}$-consistent and $\mathcal{C}$-cycle compatible is an open area.

* * * * *

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[1] Sinha D. and Garg P., *Canonical consistency of signed line structures*, Graph Theory Notes N.Y., 59 (2010), 22-27.

[2] Jain R., Acharya M. and Kansal S., *$\mathcal{C}$-consistent and $\mathcal{C}$-cycle compatible $\bullet$-line signed graphs*, Accepted, 2016.

Chapter 6

SPLITTING SIGNED GRAPHS

Sampathkumar and Walikar [3] introduced the concept of splitting graph of a graph. The splitting graph of a graph G, denoted here as $\mathfrak{S}(G)$, is formed as follows:
Take a copy of G and for each vertex v of G, take a new vertex v'. Join v' to all adjacent vertices of v. Sinha et al. [5] introduce splitting signed graph $\Gamma(S)$ of a signed graph S. In this chapter, we introduce splitting signed graph $\mathfrak{S}(S)$ of a signed graph S and establish structural characterizations of signed graph S for which $\mathfrak{S}(S)$ is balanced, $\mathcal{C}$-consistent, $\mathfrak{S}(S)$ and $\Gamma(S)$ are isomorphic and $\mathcal{C}$-cycle compatible. We also establish a characterization of $\mathfrak{S}$-splitting signed graphs.

6.1 Introduction

Sampathkumar and Walikar introduced the concept of *splitting graph* of a graph in [3]. The splitting graph of a graph G, denoted here $\mathfrak{S}(G)$, is formed as follows:
Take a copy of G and for each vertex v of G, take a new vertex v'. Join v' to all adjacent vertices of v.

There are two notions of *splitting signed graphs* of a signed graph $S = (S^u, \sigma)$ in the literature, viz., $\mathfrak{S}(S)$ and $\Gamma(S)$, both of which have $\mathfrak{S}(S^u)$ as their underlying graph; only the rule to assign signs to the edges of $\mathfrak{S}(S^u)$ differ. An edge uv' in $\mathfrak{S}(S)$ is negative if u and v are negative vertices of S under canonical marking of S and an edge uv' in $\Gamma(S)$ is negative whenever uv is a negative edge of S and positive otherwise as reported in [1] and [5] respectively.

A signed graph S is called a $\mathfrak{S}$-splitting (Γ-splitting) signed graph if there exists a signed graph T such that S is isomorphic to $\mathfrak{S}(T)$ ($\Gamma(T)$).

Figure 6.1 illustrates a signed graph S and its splitting signed graphs $\mathfrak{S}(S)$ and $\Gamma(S)$.

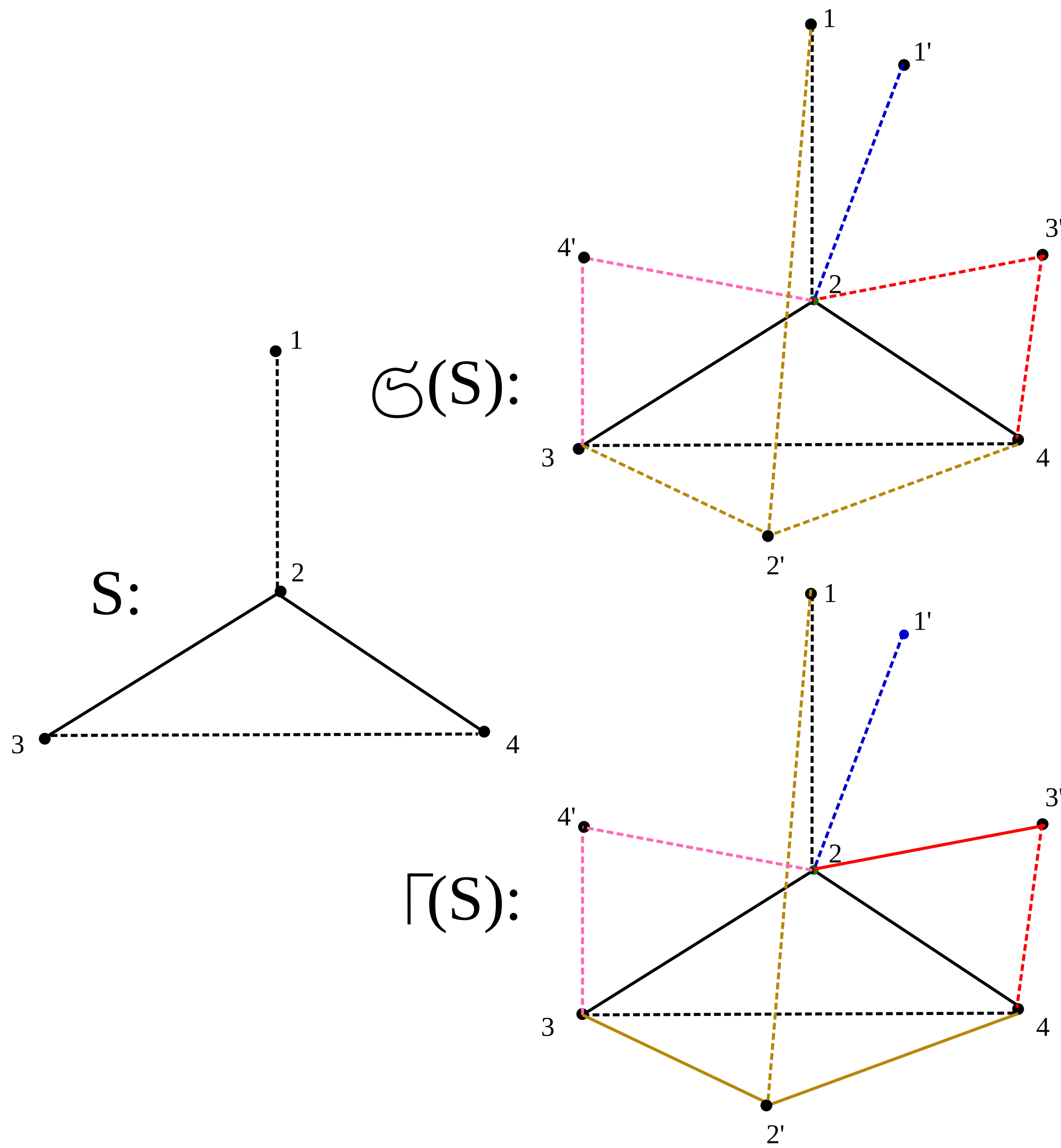

Figure 6.1: A signed graph S and its splitting signed graphs $\mathfrak{S}(S)$ and $\Gamma(S)$

A signed graph $S = (S^u, \sigma)$ is said to be sign-compatible [4] if it has a vertex marking μ such that every edge $e = uv$ has $\sigma(e) = -$ if and only if $\mu(u) = \mu(v) = -$. If the canonical marking μ_σ has this property, then S is said to be canonically sign-compatible (or $\mathcal{C}$-sign-compatible).

In the following sections, we give structural characterization of signed graph S for which $\mathfrak{S}(S)$ is balanced and $\mathcal{C}$-consistent. We also establish a characterization of splitting signed graphs $\mathfrak{S}(S)$. These results have been reported in [1].

6.2 Some results on $\mathfrak{S}(S)$

Observation 6.2.1. *For a signed graph S, $\mathfrak{S}(S)$ is homogeneous if and only if S is an all-positive signed graph or S is an all-negative signed graph in which degree of each vertex is odd.*

Theorem 6.2.1. *In $\mathfrak{S}(S)$ of a signed graph S, the following conditions hold:*

(i) *if $v \in V(S)$ is positive then $v, v' \in V(\mathfrak{S}(S))$ are positive.*

(ii) *if $v \in V(S)$ is negative having even (odd) negative vertices in $N(v)$ then $v \in V(\mathfrak{S}(S))$ is negative (positive) and v' is of opposite sign to v.*

Here v' is the vertex as defined above.

Proof. Let v be a positive vertex of signed graph S then by the definition of $\mathfrak{S}(S)$, $|N(v)|$ new positive edges will be incident with v and v' in $\mathfrak{S}(S)$. Hence, v and v' are positive vertices in $\mathfrak{S}(S)$.

Further, let $v \in V(S)$ be negative having even (odd) negative vertices in $N(v)$ then by the definition of $\mathfrak{S}(S)$, new even (odd) negative edges will be incident with v and v' in $\mathfrak{S}(S)$. Hence, $v \in V(\mathfrak{S}(S))$ is negative (positive) and v' is of opposite sign to v. This completes the proof. □

Corollary 6.2.1. *In $\mathfrak{S}(S)$ of a signed graph S,*

(a) *if $v \in V(S)$ is positive then*

(i) $d^-_{\mathfrak{S}(S)}(v) = d^-_S(v)$ *and* $d^+_{\mathfrak{S}(S)}(v) = d^+_S(v) + d_S(v)$,

(ii) $d^-_{\mathfrak{S}(S)}(v') = 0$ *and* $d^+_{\mathfrak{S}(S)}(v') = d_S(v)$.

(b) *if $v \in V(S)$ is negative having m negative vertices in $N(v)$ then*

(i) $d^-_{\mathfrak{S}(S)}(v) = d^-_S(v) + m$ *and* $d^+_{\mathfrak{S}(S)}(v) = d^+_S(v) + d_S(v) - m$,

(ii) $d^-_{\mathfrak{S}(S)}(v') = m$ *and* $d^+_{\mathfrak{S}(S)}(v') = d_S(v) - m$,

Proposition 6.2.1. *In a signed graph S, if a cycle contains a negative vertex v then in $\mathfrak{S}(S)$, cycle containing v but not v' and cycle containing v' but not v such that remaining vertices of cycles are common, are of opposite parity; that is, if one cycle is $\mathcal{C}$-consistent then other cycle is $\mathcal{C}$-inconsistent.*

Proof. In a signed graph S, if a cycle contains a negative vertex v then by Theorem 6.2.1, $v, v' \in V(\mathfrak{S}(S))$ are of opposite signs. Hence, in $\mathfrak{S}(S)$, if a cycle containing v but not v' is $\mathcal{C}$-consistent then the cycle containing v' but not v and remaining vertices common will be $\mathcal{C}$-inconsistent and vice-versa. Thus, the result follows. □

In a signed graph S shown in **Figure 6.2**, vertex 4 is negative.

In $\mathfrak{S}(S)$, cycle $(2,3,4,2)$ is $\mathcal{C}$-inconsistent but cycle $(2,3,4',2)$ is $\mathcal{C}$-consistent.

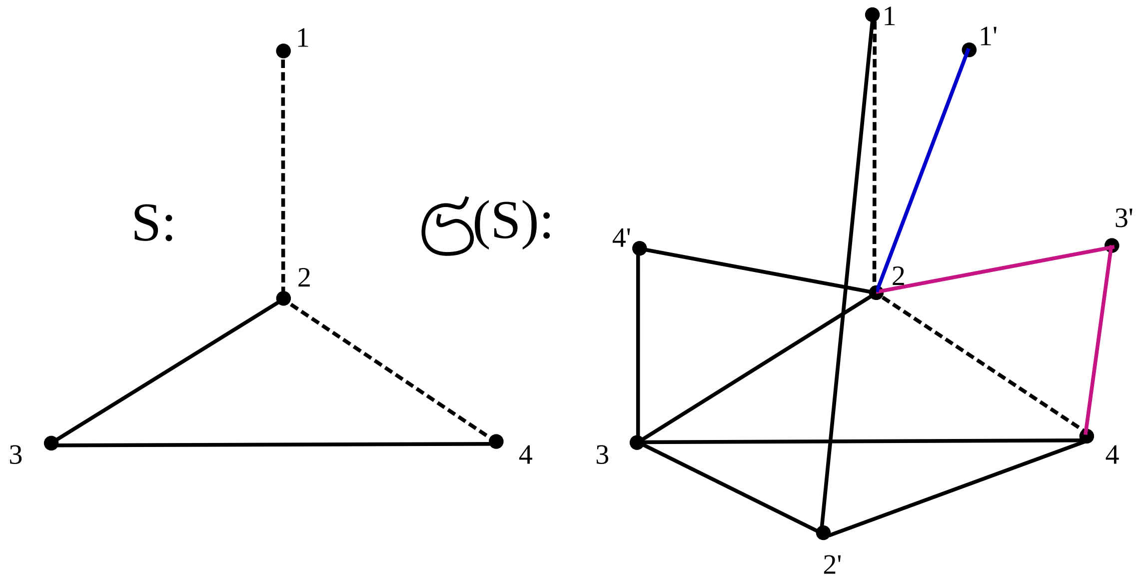

Figure 6.2: A signed graph S and its $\mathcal{C}$-inconsistent $\mathfrak{S}(S)$

6.3 Balanceness of $\mathfrak{S}(S)$

Theorem 6.3.1. $\mathfrak{S}(S)$ *is balanced if and only if the following conditions hold in* S*:*

(i) S *is balanced and;*

(ii) S *does not contain a homogeneous path* P_3 *of marking* $+,-,-$ *and the marking of a heterogeneous path* P_3 *is* $+,-,-$ *only.*

Proof. Necessity:

Let $\mathfrak{S}(S)$ be balanced. Since S is subsignedgraph of $\mathfrak{S}(S)$, S is balanced. The marking of path $P_3 = (u, v, w)$ may be one of the following:

1. $+, +, +$ 2. $-, -, -$ 3. $+, -, +$

4. $-, +, -$ 5. $-, +, +$ 6. $+, -, -$

By the definition of $\mathfrak{S}(S)$, a path $P_3 = (u, v, w)$ of S induces a cycle $C_4 = (u, v, w, v')$ in $\mathfrak{S}(S)$. This cycle C_4 is positive (negative) if P_3 is homogeneous (heterogeneous) path having the marking given in 1-5 and C_4 is positive (negative) if P_3 is heterogeneous (homogeneous) path having the marking given in 6. Since $\mathfrak{S}(S)$ is balanced; that is, C_4 is positive, S does not contain a homogeneous path P_3 of marking $+, -, -$ and the marking of a heterogeneous path P_3 is $+, -, -$ only.

Sufficiency:

Suppose conditions hold then it can be easily seen that all cycles in $\mathfrak{S}(S)$ are positive. Therefore, by Lemma 1.2.1, $\mathfrak{S}(S)$ is balanced. Hence the result follows. □

Signed graph S shown in **Figure 6.3** does not satisfy conditions **(i)** and **(ii)** of Theorem 6.3.1 and thus $\mathfrak{S}(S)$ is unbalanced.

Signed graph S shown in **Figure 6.4** satisfies conditions **(i)** and **(ii)**

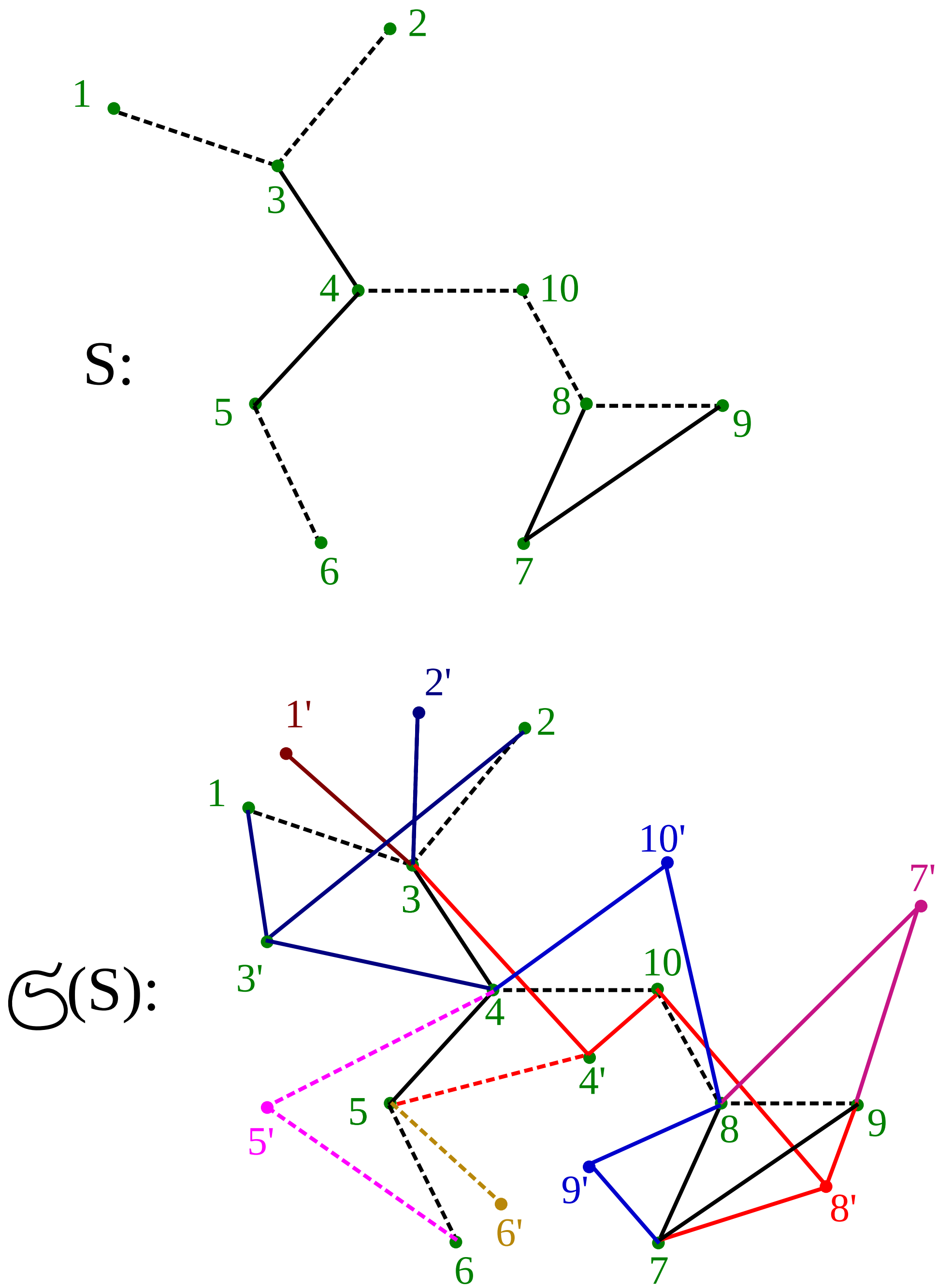

Figure 6.3: A signed graph S and its unbalanced $\mathfrak{S}(S)$

of Theorem 6.3.1 and thus $\mathfrak{S}(S)$ is balanced.

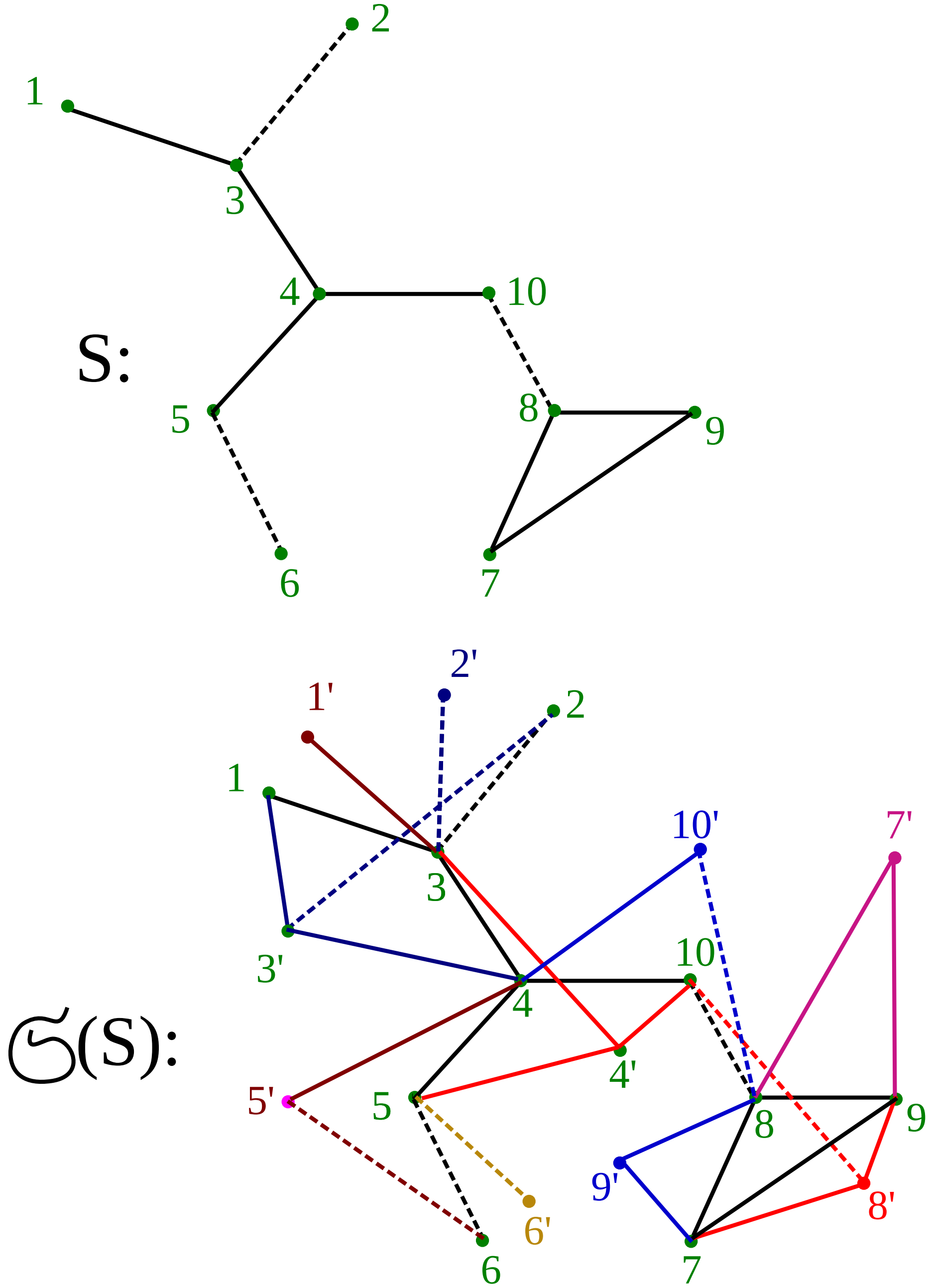

Figure 6.4: A signed graph S and its balanced $\mathfrak{S}(S)$

6.4 $\mathcal{C}$-consistency of $\mathfrak{S}(S)$

Theorem 6.4.1. $\mathfrak{S}(S)$ *is* $\mathcal{C}$*-consistent if and only if the following conditions hold in* S*:*

(i) *All vertices of a cycle are positive and;*

(ii) *for a path* $P_3 = (u, v, w)$ *of* S*,*

- *its marking is not* $+, -, +$*;*
- *for marking* $-, -, -$*,* $N(u)$ *and* $N(w)$ *simultaneously do not contain even (odd) negative vertices;*
- *for marking* $-, +, -$*,* $N(u)$ *and* $N(w)$ *simultaneously contain even (odd) negative vertices;*
- *for marking* $-, -, +$*,* $N(u)$ *contains even negative vertices;*
- *for marking* $-, +, +$*,* $N(u)$ *contains odd negative vertices.*

Proof. The necessity of **(i)** follows from Proposition 6.2.1. Now, we prove the necessity of **(ii)**. The marking of path $P_3 = (u, v, w)$ may be one of the following:

1. $+, -, +$ 2. $-, -, -$ 3. $-, +, -$
4. $-, -, +$ 5. $-, +, +$ 6. $+, +, +$

By Theorem 6.2.1, a negative vertex v of S having even (odd) negative vertices in $N(v)$ is negative (positive) in $\mathfrak{S}(S)$ and v' is of opposite sign to v and by the definition of $\mathfrak{S}(S)$, a path $P_3 = (u, v, w)$ of S induces a cycle $C_4 = (u, v, w, v')$ in $\mathfrak{S}(S)$. Hence, the following cases arise:

- For marking $+$, $-$, $+$, cycle C_4 will be $\mathcal{C}$-inconsistent. Hence, marking of P_3 can not be $+$, $-$, $+$.
- For marking $-, -, -$, one of the vertices v and v' will be negative in $\mathfrak{S}(S)$. Hence, for $\mathcal{C}$-consistency of C_4, u and w simultaneously

can not be positive (negative) in $\mathfrak{S}(S)$; that is, $N(u)$ and $N(w)$ simultaneously do not contain even (odd) negative vertices.

- For marking $-, +, -$, both the vertices v and v' will be positive in $\mathfrak{S}(S)$. Hence, for $\mathcal{C}$-consistency of C_4, u and w simultaneously must be positive (negative) in $\mathfrak{S}(S)$; that is, $N(u)$ and $N(w)$ simultaneously contain even (odd) negative vertices.

- For marking $-, -, +$, one of the vertices v and v' will be negative in $\mathfrak{S}(S)$. Hence, for $\mathcal{C}$-consistency of C_4, u must be negative in $\mathfrak{S}(S)$; that is, $N(u)$ contains even negative vertices.

- For marking $-, +, +$, both the vertices v and v' will be positive in $\mathfrak{S}(S)$. Hence, for $\mathcal{C}$-consistency of C_4, u must be positive in $\mathfrak{S}(S)$; that is, $N(u)$ contains odd negative vertices.

Thus the necessity follows.

Sufficiency:

Suppose conditions hold in S then it can be easily seen that all the cycles in $\mathfrak{S}(S)$ are $\mathcal{C}$-consistent. Therefore, $\mathfrak{S}(S)$ is $\mathcal{C}$-consistent. Hence the result follows. □

Signed graphs S_1 and S_2 shown in **Figure 6.5** do not satisfy conditions **(i)** and **(ii)** of Theorem 6.4.1, therefore, $\mathfrak{S}(S_1)$ and $\mathfrak{S}(S_2)$ are $\mathcal{C}$-inconsistent.

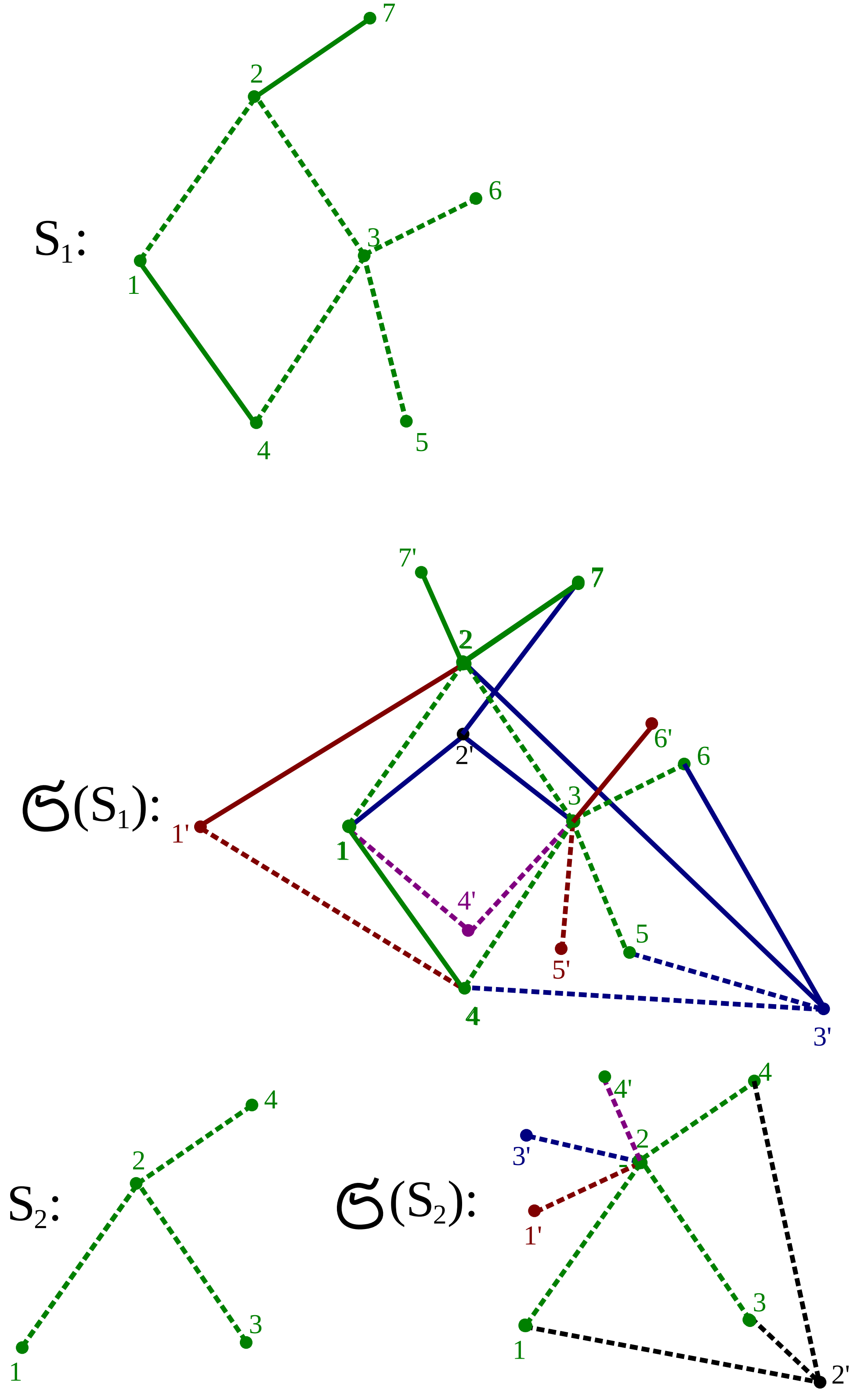

Figure 6.5: Signed graphs S_1, S_2 and their $\mathcal{C}$-inconsistent $\mathfrak{S}(S_1)$ and $\mathfrak{S}(S_2)$

6.5 Characterization of $\mathfrak{S}$-splitting signed graph

Theorem 6.5.1. *A signed graph S is a $\mathfrak{S}$-splitting signed graph if and only if the following conditions hold:*

(i) *$V(S)$ can be partitioned into two sets V_1 and V_2 such that there exists a bijection $v \to v'$ from V_1 to V_2 and $N(v') = N(v) \cap V_1$.*

(ii) *$\sigma(uv') = -$ whenever u, v are negative in induced subgraph $< V_1 >$ of S.*

Proof. Necessity:

Let a signed graph S be a $\mathfrak{S}$-splitting signed graph. Therefore, $S \cong \mathfrak{S}(T)$ for some signed graph T. For the construction of S from T, we insert a new vertex v' corresponding to each vertex v of T and join it to all vertices $u \in N(v)$. Clearly, the newly introduced vertex v' is different for all vertices of T. Let $V_1 = V(T)$ and V_2 be the set containing such type of vertices v'. Thus, **(i)** follows.

Further, by the definition of $\mathfrak{S}(T)$,

$$\sigma(uv') = \begin{cases} - & \text{if } u, v \in V(T) \text{ are negative;} \\ + & \text{otherwise.} \end{cases}$$

Since T is induced subgraph $< V_1 >$ of S, **(ii)** follows.

Sufficiency:

Suppose conditions hold for a signed graph S. Let T be the subsigned-graph of S induced by the vertices of V_1. It can be easily verified that $S \cong \mathfrak{S}(T)$. Hence the result follows. □

In following sections, we establish structural characterizations of signed graph S for which $\mathfrak{S}(S)$ and $\Gamma(S)$ are isomorphic and $\mathcal{C}$-cycle compatible. These results have been reported in [2].

6.6 Isomorphism of $\mathfrak{S}(S)$ and $\Gamma(S)$

Theorem 6.6.1. *For a signed graph S, $\mathfrak{S}(S) \cong \Gamma(S)$ if and only if S is any one of the following:*

(i) *All-positive or;*

(ii) *All-negative in which degree of each vertex is odd or;*

(iii) *heterogeneous in which end vertices of every negative (positive) edge are (are not) negative.*

Proof. Necessity:

Let for a signed graph S, $\mathfrak{S}(S) \cong \Gamma(S)$. Since S is a subsignedgraph of $\mathfrak{S}(S)$ and $\Gamma(S)$, we concentrate our attention only on the sign of edge uv' in $\mathfrak{S}(S)$ and $\Gamma(S)$. By the definition of $\mathfrak{S}(S)$, $uv' \in E^-(\mathfrak{S}(S))$ if and only if $u, v \in V(S)$ are negative and by the definition of $\Gamma(S)$, $uv' \in E^-(\Gamma(S))$ if and only if $uv \in E^-(S)$. Therefore, we have following three possible cases:

Case I: If $\mathfrak{S}(S) \cong \Gamma(S)$ and both $\mathfrak{S}(S)$ and $\Gamma(S)$ are all-positive then no edge of S will be negative. Hence, **(i)** follows.

Case II: If $\mathfrak{S}(S) \cong \Gamma(S)$ and both are all-negative then every edge and every vertex of S will be negative. Hence, **(ii)** follows.

Case III: If $\mathfrak{S}(S) \cong \Gamma(S)$ and both are heterogeneous then S will be heterogeneous and edge uv' in both $\mathfrak{S}(S)$ and $\Gamma(S)$ must be of the same sign. This implies that end vertices of every negative (positive) edge of S are (are not) negative. Hence, **(iii)** follows.

Thus, the necessity follows.

Sufficiency:

Suppose S is any one of the following:

(i) All-positive or;

(ii) All-negative in which degree of each vertex is odd or;

(iii) heterogeneous in which end vertices of every negative (positive) edge are (are not) negative.

then by the definitions $\mathfrak{S}$ - and Γ- splitting signed graphs, we have the following cases:

Case I: If S is all-positive then $\mathfrak{S}(S)$ and $\Gamma(S)$ will be all-positive and $\mathfrak{S}(S) \cong \Gamma(S)$.

Case II: If S is all-negative in which degree of each vertex is odd then $\mathfrak{S}(S)$ and $\Gamma(S)$ will be all-negative and $\mathfrak{S}(S) \cong \Gamma(S)$.

Case III: If S is heterogeneous in which end vertices of every negative (positive) edge are (are not) negative then $\mathfrak{S}(S)$ and $\Gamma(S)$ will be heterogeneous as S is a subsignedgraph of $\mathfrak{S}(S)$ and $\Gamma(S)$ and edge uv' in both $\mathfrak{S}(S)$ and $\Gamma(S)$ will be of the same sign. Hence, $\mathfrak{S}(S) \cong \Gamma(S)$.

This completes the proof. □

Corollary 6.6.1. *For a signed graph S, $\mathfrak{S}(S) \cong \Gamma(S)$ if and only if S is $\mathcal{C}$-sign compatible.*

6.7 $\mathcal{C}$-cycle compatibility of $\mathfrak{S}(S)$

Theorem 6.7.1. *$\mathfrak{S}(S)$ is $\mathcal{C}$-cycle compatible if and only if the following conditions hold in S:*

(i) *if Z is a positive (negative) cycle then an even (odd) number of negative vertices of cycle Z contain even numbers of negative vertices in their neighbourhoods and;*

(ii) *for a path $P_3 = (u, v, w)$, any one condition holds:*

- *it is homogeneous of marking $+,\ +,\ +$;*
- *it is heterogeneous of marking $+, -, +$;*
- *it is homogeneous (heterogeneous) of marking $-, +, +$ or $-, -, +$ and $N(u)$ contains an odd (even) number of negative vertices;*
- *it is homogeneous (heterogeneous) of marking $-, +, -$ and vertices u, w are (are not) of same parity (i.e., $N(u)$ and $N(w)$ contain even number of negative vertices or odd number of negative vertices);*
- *it is homogeneous (heterogeneous) of marking $-, -, -$ and vertices u, w are not (are) of same parity.*

Proof. Necessity:

Let $\mathfrak{S}(S)$ be $\mathcal{C}$-cycle compatible. Therefore, every cycle in $\mathfrak{S}(S)$ is either positive and $\mathcal{C}$-consistent or negative and $\mathcal{C}$-inconsistent. By Theorem 6.2.1, every positive vertex of S is positive in $\mathfrak{S}(S)$ and every negative vertex of S having an even (odd) number of negative vertices in its neighbourhood is negative (positive) in $\mathfrak{S}(S)$. Since S is sub-signedgraph of $\mathfrak{S}(S)$, if Z is a positive (negative) cycle of S then Z must be $\mathcal{C}$-consistent ($\mathcal{C}$-inconsistent) in $\mathfrak{S}(S)$, i.e., an even (odd) number of negative vertices of cycle Z must contain an even number of negative vertices in their neighbourhoods. Thus, **(i)** follows.

By the definition of $\mathfrak{S}(S)$, a path $P_3 = (u, v, w)$ of S induces a cycle $C_4 = (u, v, w, v')$ in $\mathfrak{S}(S)$. The marking of path $P_3 = (u, v, w)$ may be one of the following:

1. $+, +, +$ 2. $+, -, +$ 3. $-, +, +$
4. $-, -, +$ 5. $-, +, -$ 6. $-, -, -$

Hence, the following cases arise:

- if marking of path $P_3 = (u, v, w)$ is $+, +, +$ then by Theorem 6.2.1, vertices u, v, w, v' have signs $+, +, +, +$ respectively in $\mathfrak{S}(S)$. Thus, path P_3 induces a $\mathcal{C}$-consistent cycle C_4 in $\mathfrak{S}(S)$. By Theorem 6.3.1, this cycle C_4 is positive (negative) if and only if P_3 is homogeneous (heterogeneous). Since $\mathfrak{S}(S)$ is $\mathcal{C}$-cycle compatible, P_3 will be homogeneous.

- if marking of path $P_3 = (u, v, w)$ is $+, -, +$ then by Theorem 6.2.1, vertices u, v, w, v' have signs $+, -, +, +$ or $+, +, +, -$ respectively in $\mathfrak{S}(S)$. Thus, path P_3 induces a $\mathcal{C}$-inconsistent cycle C_4 in $\mathfrak{S}(S)$. By Theorem 6.3.1, this cycle C_4 is positive (negative) if and only if P_3 is homogeneous (heterogeneous). Since $\mathfrak{S}(S)$ is $\mathcal{C}$-cycle compatible, P_3 will be heterogeneous.

- if marking of path $P_3 = (u, v, w)$ is $-, +, +$ and $N(u)$ contains an odd (even) number of negative vertices then by Theorem 6.2.1, vertices u, v, w, v' have signs $+, +, +, +(-, +, +, +)$ respectively in $\mathfrak{S}(S)$. Thus, path P_3 induces a $\mathcal{C}$-consistent ($\mathcal{C}$-inconsistent) cycle C_4 in $\mathfrak{S}(S)$. By Theorem 6.3.1, this cycle C_4 is positive (negative) if and only if P_3 is homogeneous (heterogeneous). Since $\mathfrak{S}(S)$ is $\mathcal{C}$-cycle compatible, for homogeneous (heterogeneous) P_3, $N(u)$ must contain an odd (even) number of negative vertices.

  Similarly, if marking of path $P_3 = (u, v, w)$ is $-, -, +$ and $N(u)$ contains an even (odd) number of negative vertices then by Theorem 6.2.1, vertices u, v, w, v' have signs $-, -, +, +$ or $-, +, +, -$ ($+, -, +, +$ or $+, +, +, -$) respectively in $\mathfrak{S}(S)$. Thus, path P_3 induces a $\mathcal{C}$-consistent ($\mathcal{C}$-inconsistent) cycle C_4 in $\mathfrak{S}(S)$. By Theorem 6.3.1, this cycle C_4 is positive (negative) if and only if P_3 is heterogeneous (homogeneous). Since $\mathfrak{S}(S)$ is $\mathcal{C}$-cycle compatible, for heterogeneous (homogeneous) P_3, $N(u)$ will contain an even (odd) number of negative vertices.

- if marking of path $P_3 = (u, v, w)$ is $-, +, -$ and vertices u and w are (are not) of the same parity, i.e., $N(u)$ and $N(w)$ contain even number of negative vertices or odd number of negative vertices, then by Theorem 6.2.1, vertices u, v, w, v' have signs $-, +, -, +$ or $+,\ +,\ +,\ +$ ($-, +, +, +$ or $+, +, -, +$) respectively in $\mathfrak{S}(S)$. Thus, path P_3 induces a $\mathcal{C}$-consistent ($\mathcal{C}$-inconsistent) cycle C_4 in $\mathfrak{S}(S)$. By Theorem 6.3.1, this cycle C_4 is positive (negative) if and only if P_3 is homogeneous (heterogeneous). Since $\mathfrak{S}(S)$ is $\mathcal{C}$-cycle compatible, for homogeneous (heterogeneous) P_3, vertices u and w will (will not) be of the same parity;

- if marking of path $P_3 = (u, v, w)$ is $-, -, -$ and vertices u and w are (are not) of same parity, i.e., $N(u)$ and $N(w)$ contain even number of negative vertices or odd number of negative vertices, then by Theorem 6.2.1, vertices u, v, w, v' have signs $-, -, -,\ +$; $-, +, -, -$ or $+,\ -,\ +,\ +$; $+,\ +,\ +,\ -$ ($-, -,\ +,\ +$; $-,\ +,\ +,$ $-$ or $+,\ -, -,\ +$; $+,\ +,\ -, -$) respectively in $\mathfrak{S}(S)$. Thus, path P_3 induces a $\mathcal{C}$-inconsistent ($\mathcal{C}$-consistent) cycle C_4 in $\mathfrak{S}(S)$. By Theorem 6.3.1, this cycle C_4 is positive (negative) if and only if P_3 is homogeneous (heterogeneous). Since $\mathfrak{S}(S)$ is $\mathcal{C}$-cycle compatible, for homogeneous (heterogeneous) P_3, vertices u and w will not (will) be of the same parity.

Thus, the necessity follows.

Sufficiency:

A cycle in $\mathfrak{S}(S)$ is induced due to a cycle or a path P_3 or their combinations in S. If conditions hold then it can be easily seen that every cycle in $\mathfrak{S}(S)$ is positive and $\mathcal{C}$-consistent or negative and $\mathcal{C}$-inconsistent, i.e, $\mathfrak{S}(S)$ is $\mathcal{C}$-cycle compatible. This completes the proof. □

Signed graph S shown in **Figure 6.6** does not satisfy conditions **(i)** and **(ii)** of Theorem 6.7.1. Therefore, $\mathfrak{S}(S)$ is $\mathcal{C}$-cycle incompatible.

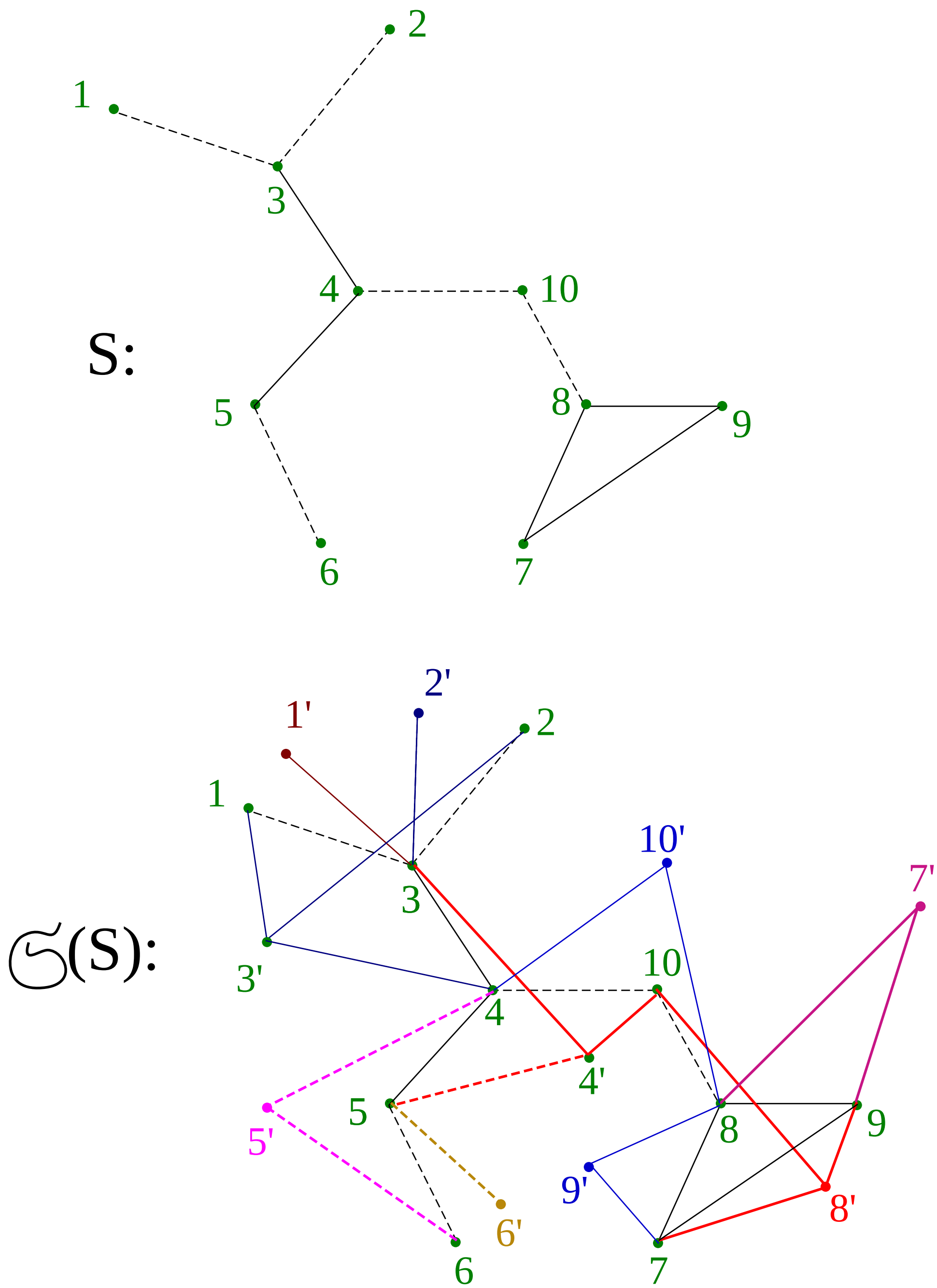

Figure 6.6: A signed graph S and its $\mathcal{C}$-cycle incompatible $\mathfrak{S}(S)$

Signed graph S shown in **Figure 6.7** satisfies conditions **(i)** and **(ii)** of Theorem 6.7.1, therefore, $\mathfrak{S}(S)$ is $\mathcal{C}$-cycle compatible.

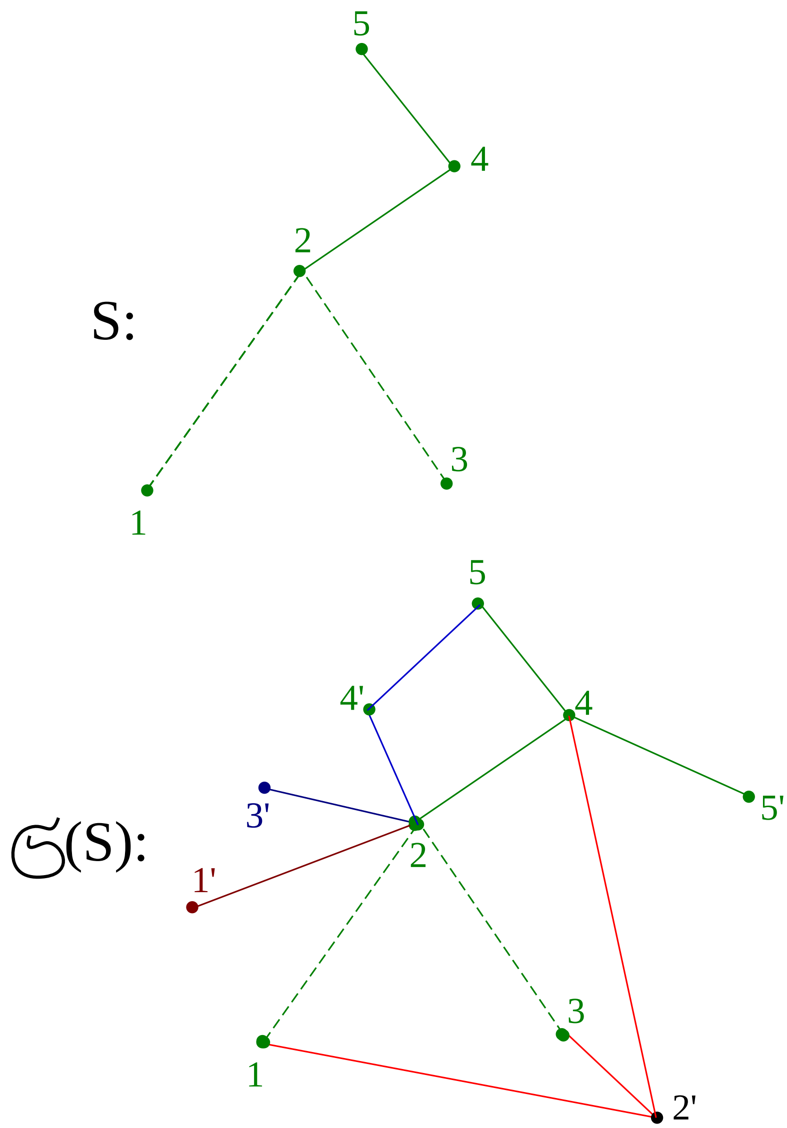

Figure 6.7: A signed graph S and its $\mathcal{C}$-cycle compatible $\mathfrak{S}(S)$

6.8 $\mathcal{C}$-cycle compatibility of $\Gamma(S)$

In order to carry out our investigations we need the following result:

Lemma 6.8.1. [5] *The following statements hold in* $\Gamma(S)$*:*

(i) *If* $v \in V(S)$ *is any vertex then* $v \in V(\Gamma(S))$ *is positive.*

(ii) *If* $v \in V(S)$ *is a negative vertex then* $v' \in V(\Gamma(S))$ *is negative.*

Theorem 6.8.1. *For a signed graph S, $\Gamma(S)$ is $\mathcal{C}$-cycle compatible if and only if the following conditions hold in S:*

(i) *S is balanced;*

(ii) *each non-pendant vertex of S is positive.*

Proof. Necessity:

Let $\Gamma(S)$ be $\mathcal{C}$-cycle-compatible, i.e., every cycle in $\Gamma(S)$ is either positive and $\mathcal{C}$-consistent or negative and $\mathcal{C}$-inconsistent. By Lemma 6.8.1, every vertex of S is a positive vertex of $\Gamma(S)$. Hence, every cycle Z of $\Gamma(S)$ that is due to a cycle Z of S is $\mathcal{C}$-consistent. Since $\Gamma(S)$ is $\mathcal{C}$-cycle-compatible, this cycle Z of S must be positive. Therefore, S will be balanced. Thus, **(i)** follows.

By the definition of $\Gamma(S)$, a path $P_3 = (u, v, w)$ of S induces a positive cycle $C_4 = (u, v, w, v')$ in $\Gamma(S)$ and by Lemma 6.8.1, vertices u, v, w, v' have signs $+, +, +, +$ $(+, +, +, -)$ in $\Gamma(S)$ if v is a positive (negative) vertex of S. Thus, this cycle C_4 is $\mathcal{C}$-consistent if $v \in V(S)$ is a positive vertex. Since $\Gamma(S)$ is $\mathcal{C}$-cycle compatible and cycle C_4 is positive, C_4 must be $\mathcal{C}$-consistent. Hence, every non-pendant vertex of S will be positive. Thus, the necessity follows.

Sufficiency:

A cycle in $\Gamma(S)$ is induced due to a cycle or a path P_3 or their combinations in S. If conditions hold then it can be easily seen that every cycle in $\Gamma(S)$ is positive and $\mathcal{C}$-consistent, i.e, $\Gamma(S)$ is $\mathcal{C}$-cycle-compatible. This completes the proof. □

Signed graph S shown in **Figure 6.8** satisfies conditions **(i)** and **(ii)** of Theorem 6.8.1. Therefore, $\Gamma(S)$ is $\mathcal{C}$-cycle compatible.

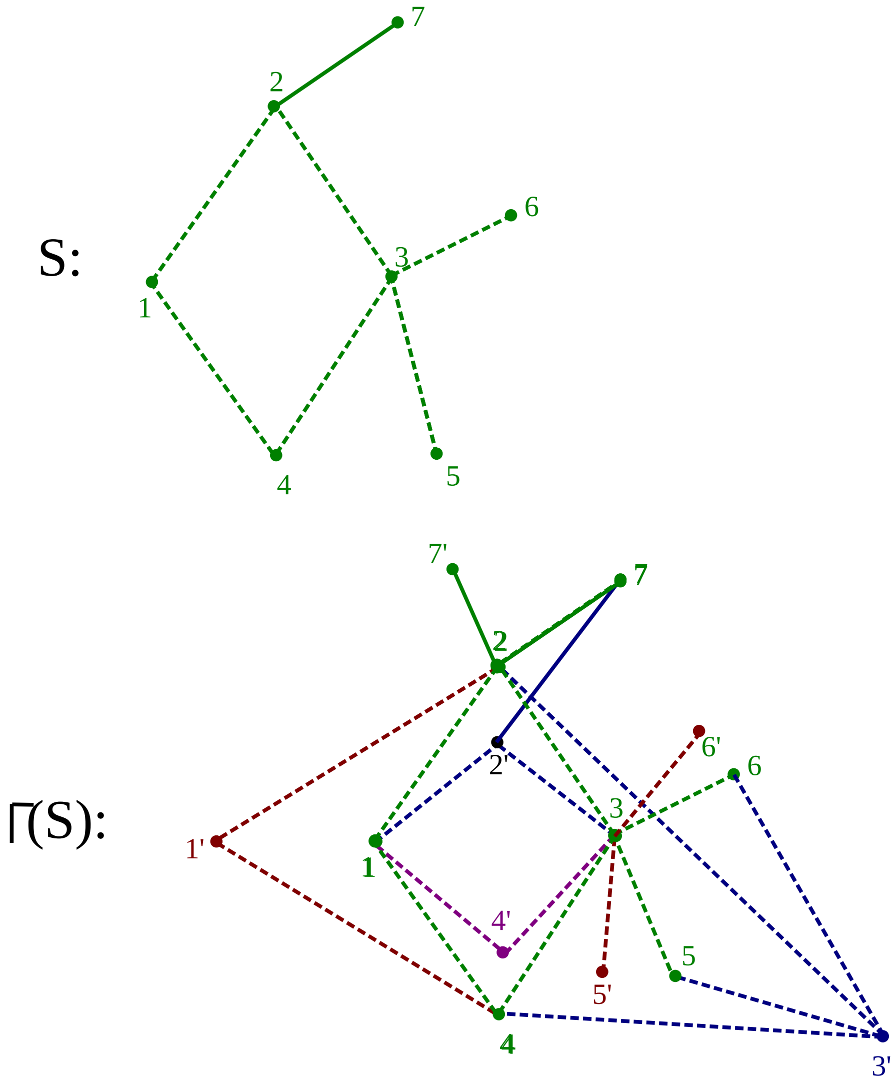

Figure 6.8: A signed graph S and its $\mathcal{C}$-cycle compatible $\Gamma(S)$

6.9 Conclusion and Scope

In this chapter, we have established structural characterizations of signed graph S for which $\mathfrak{S}(S)$ is balanced, $\mathcal{C}$-consistent, $\mathfrak{S}(S)$ and $\Gamma(S)$ are isomorphic and $\mathcal{C}$-cycle compatible. We also established a characterization of $\mathfrak{S}$-splitting signed graphs. Problems of characterizations of signed graph S for which $\mathfrak{S}(S)$ and $\Gamma(S)$ are $\mathcal{C}$-sign-compatible are still open.

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Chapter 7

VERTEX EQUITABLE LABELING IN SIGNED GRAPHS

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labelings were first introduced in the mid 1960s. In the intervening 55 years nearly 220 graph labelings techniques have been studied in over 2200 papers. Seenivasan and Lourdusamy [16] introduced a new type of graph labeling known as vertex equitable labeling. They studied the properties of this labeling and investigated vertex equitable behaviors of some standard graphs. They also proved that arbitrary supersubdivision of paths is vertex equitable and that every cycle C_n with $n \equiv 0$ or 3 (mod 4) has a vertex equitable superdivision. In this chapter, we initiate vertex equitable labeling of signed graphs and study a vertex equitable behavior of signed paths, signed stars, signed bistars and signed complete bipartite graphs $K_{2,n}$.

7.1 Introduction

Graph labelings, where the vertices are assigned values subject to certain conditions, have often been motivated by practical problems but also of interest in their own right. The labels of the vertices induce labels of edges under certain conditions and there is an enormous amount of literature build up on several kind of numerical labeling of graphs and an interested reader is referred to [3].

In literature many types of graph labelings exist, e.g., graceful labeling, multiplicative labeling, vertex equitable labeling, harmonious labeling, cordial labeling, set labeling and so on. Here we introduce vertex equitable labeling in the realm of signed graphs.

In this chapter, we are taking $|E^{+}(S)| = \mathfrak{m}$ and $|E^{-}(S)| = \mathfrak{n}$. By a (p, q)- signed graph, we mean a signed graph S with $|V(S)| = p$ and $|E(S)| = q$.

7.2 Vertex equitable labeling

In [16], Seenivasan and Lourdusamy introduced the idea of vertex equitable labeling of graphs as follows:

Suppose G is a (p, q)-graph and $\mathcal{A} = \{0, 1, 2, ..., \lceil \frac{q}{2} \rceil\}$. A vertex labeling $f : V(G) \longrightarrow \mathcal{A}$ which is onto, is said to be a *vertex equitable labeling* of G if it induces a bijective edge labeling $f^* : E(G) \longrightarrow \{1, 2, ..., q\}$ given by $f^*(uv) = f(u) + f(v)$ such that $|v_f(a) - v_f(b)| \leq 1$ $\forall a, b \in \mathcal{A}$, where $v_f(a)$ is the number of vertices with $f(v) = a$. Here $\lceil n \rceil$ denotes the smallest integer greater than or equal to n. A graph G is said to be *vertex equitable* if it admits a vertex equitable labeling.

They have shown that the graphs like paths, bistars $B(n, n)$, combs $P_n \odot K_1$, complete bipartite graphs $K_{2,n}$, friendship graphs $C_3^{(t)}$ for $t \geq 2$, quadrilateral snakes, $K_2 + mK_1, K_{1,n} \cup K_{1,n+k}$ if and only if $1 \leq k \leq 3$, ladder graphs $L_n = P_n \times K_2$, arbitrary super divisions of paths and cycles C_n with $n \equiv 0$ or 3 $(mod\ 4)$ are vertex equitable. Also they proved that the graphs $K_{1,n}$ if $n \geq 4$, Eulerian graphs with n

edges where $n \equiv 1$ or 2 $(mod\ 4)$, the wheels W_n, the complete graphs K_n if $n > 3$ and triangular cactuses with q edges where $q \equiv 0$ or 6 or 9 $(mod\ 12)$ are not vertex equitable. Moreover they proved that if G is a (p, q)-graph, q is even and $p < \lceil \frac{q}{2} \rceil + 2$ then G is not vertex equitable.

The *super subdivision graph* $S^*(G)$ of a graph G is the graph obtained from G by replacing every edge uv of G by $K_{2,m}$ (m may vary for each edge) and identifying u and v with the two vertices in $K_{2,m}$ that form the partite set with exactly two members. Jeyanthi et. al [14] proved that super subdivision graphs of $Pn \odot K_1$, bistars $B(n, n)$, $P_n \times P_2$ and quadrilateral snakes are vertex equitable.

For a graph H with vertices v_1, v_2, ...,v_n and n copies of a graph G, $H\widehat{o}G$ is a graph obtained by identifying a vertex u_i of the i^{th} copy of G with a vertex v_i of H for $1 \leq i \leq n$. The graph $H\widetilde{o}G$ is a graph obtained by joining a vertex u_i of the i^{th} copy of G with a vertex v_i of H by an edge for $1 \leq i \leq n$.

Jeyanthi and Maheswari [10] proved that the following graphs have vertex equitable labeling: the square of the bistar $B_{n,n}$, the splitting graph of the bistar $B_{n,n}$, C_4-snakes, connected graphs in which each block is a cycle of order divisible by 4 (they need not be of the same order) and whose block-cut point graph is a path, $C_m \odot P_n$, tadpoles, the one-point union of two cycles and the graph obtained by starting friendship graphs $C_{n_1}^{(2)}$, $C_{n_2}^{(2)}$,...,$C_{n_k}^{(2)}$, where each $n_i \equiv 0$ (mod 4) and joining the center of $C_{n_i}^{(2)}$ to the center $C_{i+1}^{(2)}$ with an edge for $i = 1, 2, ..., k-1$. In [5], Jeyanthi and Maheswari proved that T_p trees, bistars $B(n, n+1)$,

$C_n \odot K_m$, P_n^2, tadpoles, certain classes of caterpillars and $T \odot \overline{K_n}$, where T is a T_p tree with an even number of vertices are vertex equitable. Jeyanthi and Maheswari [6] gave vertex equitable labelings for graphs constructed from T_p trees by appending paths or cycles. Jeyanthi and Maheswari [4] proved that graphs obtained by duplicating an arbitrary vertex and an arbitrary edge of a cycle, total graphs of a paths, splitting graphs of paths and the graphs obtained identifying an edge of one cycle with an edge of another cycle are vertex equitable. Jeyanthi et. al proved that the graphs $L_m \widehat{O} n C_4$, $L_m \widetilde{O} n C_4$, $C_m \widetilde{O} n C_4$ and $P_m \widetilde{O} n C_4$ are vertex equitable graphs in [11] and they proved the graphs $S*(P_n.K_1)$, $S*(B(n,n))$, $S*(P_n \times P_2)$ and $S*(Q_n)$ of the quadrilateral snake are vertex equitable in [15].

In [9], Jeyanthi and Maheswari proved the double alternate triangular snake $DA(T_n)$ obtained from a path $u_1, u_2, ..., u_n$ by joining u_i and u_{i+1} (alternatively) to two new vertices v_i and w_i is vertex equitable, the double alternate quadrilateral snake $DA(Q_n)$ obtained from a path $u_1, u_2, ..., u_n$ by joining u_i and u_{i+1} (alternatively) to two new vertices v_i, x_i and w_i, y_i respectively and then joining v_i, w_i and x_i, y_i is vertex equitable and $NQ(m)$, the n^{th} quadrilateral snake obtained from the path $u_1, u_2, ..., u_m$ by joining u_i and u_{i+1} with $2n$ new vertices v_j^i and w_j^i, $1 \leq i \leq m-1$, $1 \leq j \leq n$ is vertex equitable. Jeyanthi and Maheswari [12] proved $DA(T_n) \odot K_1$, $DA(T_n) \odot 2K_1$, $DA(T_n)$, $DA(Q_n) \odot K_1$, $DA(Q_n) \odot 2K_1$ and $DA(Q_n)$ are vertex equitable.

In [4, 7, 8] Jeyanthi and Maheswari have shown a number of families of graphs that have vertex equitable labelings. Their results include armed crowns $C_m \oplus P_n$, shadow graphs $D_2(K_1, n)$, the graph $C_m * C_n$ obtained by identifying a single vertex of a cycle graph C_m with a single vertex of a cycle graph C_n if and only if $m + n \equiv 0, 3 \pmod 4$, the graphs $[P_m, C_n^{(2)}]$ when $n \equiv 0 \pmod 4$, the graph obtained from m copies of $C_n * C_n$ and P_m by joining each vertex of P_m with the cut vertex in one copy of $C_n * C_n$ and graphs obtained by duplicating an arbitrary vertex and an arbitrary edge of a cycle, the total graph of P_n, the splitting graph of P_n and the fusion of two edges of C_n.

Jeyanthi et. al [13] proved the following graphs are vertex equitable: jewel graphs J_n with vertex set $\{u, v, x, y, u_i : 1 \leq i \leq n\}$ and edge set $\{ux, uy, xy, xv, yv, uu_i, vu_i : 1 \leq i \leq n\}$, jelly fish graphs $(JF)_n$ with vertex set $\{u, v, u_i, v_j : 1 \leq i \leq n, 1 \leq j \leq n-2\}$ and edge set $\{uu_i : 1 \leq i \leq n\} \cup \{vv_j : 1 \leq i \leq n-2\} \cup \{u_{n-1}u_n, vu_n, vu_{n-1}\}$, lobsters constructed from the path $a_1, a_2, ..., a_n$ with vertices a_{i1} and a_{i2} adjacent to a_i for $1 \leq i \leq n$ and pendant vertices a_{ij}^1, a_{ij}^2, ...,a_{ij}^k joining a_{ij} for $1 \leq i \leq n$ and $j = 1, 2$; $L_n \odot \overline{K_m}$ and the graph obtained from ladder a L_n and $2n$ copies of $K_{1,m}$ by identifying a non-central vertex of i^{th} copy of $K_{1,m}$ with i^{th} vertex of L_n.

We extend the definition of a vertex equitable graph to the realm of signed graphs as follows:

Let S be a (p, q)-signed graph with $q = \mathfrak{m} + \mathfrak{n}$, where $\mathfrak{m}(\mathfrak{n})$ is the number of positive(negative) edges in S and $\mathcal{A} = \{0, 1, 2, ..., \lceil \frac{q}{2} \rceil\}$. A vertex labeling $f : V(S) \longrightarrow \mathcal{A}$ which is onto, is said to be a *vertex eq-*

uitable labeling of S if it induces a bijective edge labeling $f^* : E(S) \longrightarrow \{1, 2, ..., \mathfrak{m}, -1, -2, ..., -\mathfrak{n}\}$ defined by $f^*(uv) = \sigma(uv)(f(u)+f(v))$ such that $|v_f(a) - v_f(b)| \leq 1$, $\forall a, b \in \mathcal{A}$, where $v_f(a)$ is the number of vertices with $f(v) = a$. A signed graph S is said to be *vertex equitable* if it admits a vertex equitable labeling. In **Figure 7.1**, we show a vertex equitable signed graph.

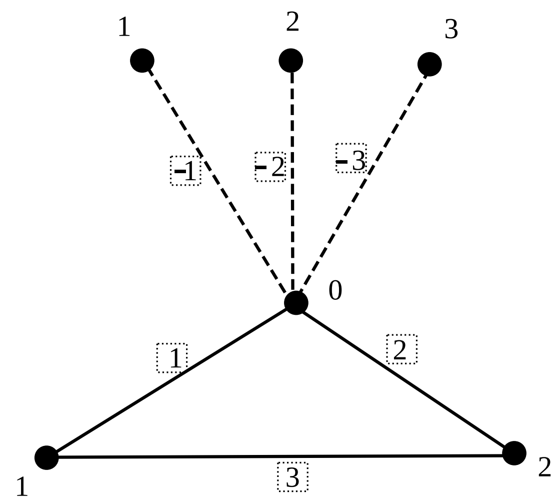

Figure 7.1: A vertex equitable signed graph

A bistar $B(m, n)$ is obtained from $K_2 = \{u, v\}$ by adding m pendent edges to u and n pendent edges to v. In this chapter, we define the signed bistar $B^+(\mathfrak{m}, \mathfrak{n})$, which is obtained by taking uv as positive edge, $\mathfrak{m}$ positive edges incident to u and $\mathfrak{n}$ negative edges incident to v. Similarly, the signed bistar $B^-(\mathfrak{m}, \mathfrak{n})$ is obtained by taking uv as negative edge, $\mathfrak{m}$ positive edges incident to u and $\mathfrak{n}$ negative edges incident to v.

7.3 Results on vertex equitable signed graphs

In this section, we give results on vertex equitable labeling of signed graphs and study vertex equitable behavior of signed paths, signed stars and signed complete bipartite graphs $K_{2,n}$. These results have been reported in [1].

For our investigations we need the following results:

Theorem 7.3.1 ([16]). *The path P_n is vertex equitable.*

Theorem 7.3.2 ([16]). *Complete bipartite graph $K_{2,n}$ is vertex equitable.*

Theorem 7.3.3. *If a signed graph S is vertex equitable then $\eta(S)$ is also vertex equitable.*

Proof. Let S be a vertex equitable signed graph. Therefore, there exists a vertex labeling $f : V(S) \longrightarrow \mathcal{A}$, where $\mathcal{A} = \{0, 1, 2, ..., \lceil \frac{q}{2} \rceil\}$ such that f induces an edge labeling f^* given by $f^*(uv) = \sigma(uv)(f(u)+f(v))$ and $|v_f(a)-v_f(b)| \leq 1, \forall a, b \in \mathcal{A}$, where $v_f(a)$ is the number of vertices of labeling a. $f^*(E) = \{1, 2, ..., \mathfrak{m}, -1, -2, ..., -\mathfrak{n}\}$. Since in $\eta(S)$, only signs of edges are opposite, i.e., in $\eta(S)$, $f^*(E) = \{-1, -2, ..., -\mathfrak{m}, 1, 2, ..., \mathfrak{n}\}$, this vertex labeling f is also a vertex equitable labeling for $\eta(S)$. Hence the result follows. □

Theorem 7.3.4. *Homogenous path P_n is vertex equitable.*

Proof. The result follows from Theorem 7.3.1 and Theorem 7.3.3. □

Remark 7.3.1. *Results reported in [4–16] for graphs hold for homogeneous signed graphs also.*

Now, a natural question is to determine heterogeneous vertex equitable paths. The following is a partial answer to this question:

Theorem 7.3.5. *A signed path P_n having a negative pendant edge and all other edges positive, is vertex equitable.*

Proof. Let $P_n = v_1v_2v_3...v_n$ be a signed path with n vertices and a negative pendant edge v_1v_2. We define $f : V(P_n) \longrightarrow \{0, 1, 2, ..., \lceil \frac{n-1}{2} \rceil\}$ as

$$f(v_1) = 0, f(v_2) = 1, f(v_3) = 0 \text{ and}$$

$$f(v_i) = f(v_{i-2}) + 1, \text{ for } 4 \leq i \leq n.$$

It is easy to see that f is a vertex equitable labeling. Thus the result follows. □

A vertex equitable labeling for signed path P_9 having a negative pendant edge and all other edges positive is shown in **Figure 7.2**.

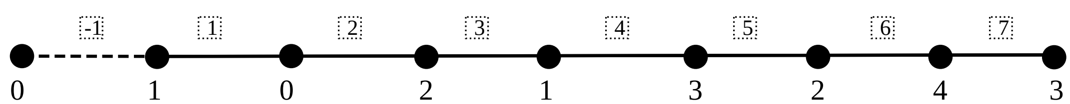

Figure 7.2: A vertex equitable signed path

Corollary 7.3.1. *A signed path P_n having a positive pendant edge and all other edges negative is vertex equitable.*

Now, in **Figure 7.3**, we give some vertex equitable heterogeneous signed paths.

In **Figure 7.3**, we have shown a signed path P_5 having one negative and

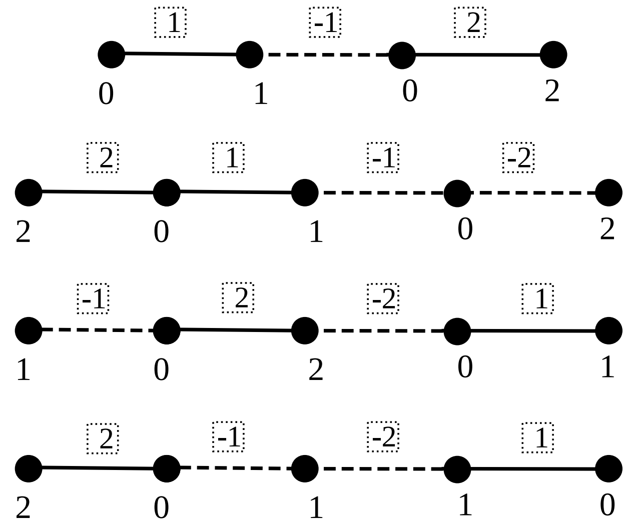

Figure 7.3: Some vertex equitable signed paths

one positive section each of length two which is vertex equitable. Note that the signed paths shown in **Figure 7.4** are not vertex equitable.

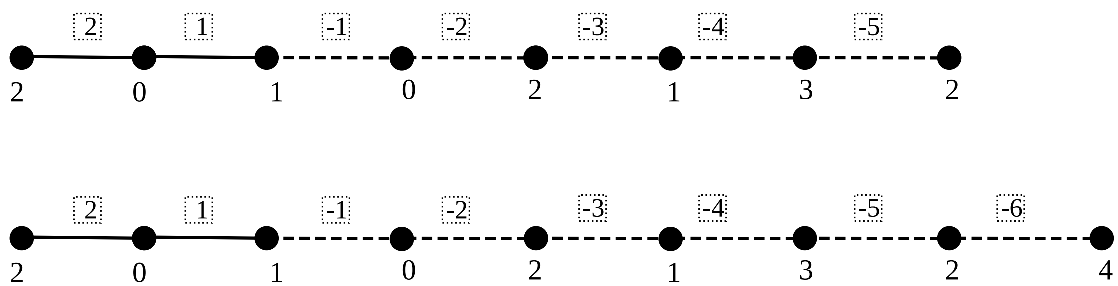

Figure 7.4: Signed paths which are not vertex equitable

Now, we pose the following conjecture:

Conjecture 7.3.1. *A signed path* P_n *having one positive section of length two having a pendant edge and one negative section of length greater than 2 and the negation of such type of path are not vertex equitable.*

Now naturally, we have the following problem:

Problem 7.3.1. *Find all non-isomorphic signed paths P_n which are vertex equitable.*

Theorem 7.3.6. *Star $K_{1,n}$ is vertex equitable if and only if $n \leq 3$.*

Proof. Let (V_1, V_2) be the bipartition of $K_{1,n}$ with $V_1 = \{u\}$ and $V_2 = \{v_1, v_2, v_3, ..., v_n\}$. Since $K_{1,n}$ has n edges, $\mathcal{A} = \{0, 1, 2, ..., \lceil \frac{n}{2} \rceil\}$. To get edge label 1, we must assign 0 and 1 labels to adjacent vertices. Therefore, $f(u) = 0$ or 1.

Assume that $K_{1,n}$ is vertex equitable.

Let $f(u) = 0$ then $f(v_i) = i$, $1 \leq i \leq n$.

If $n \geq 2$ then we can not assign this labelling as $f(v_n) = n$ and $n \notin \mathcal{A}$. Hence $f(u) \neq 0$.

For $f(u) = 1$, we have $f(v_i) = i - 1$ for $1 \leq i \leq n$. This is illustrated in **Figure 7.5**.

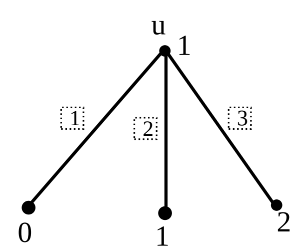

Figure 7.5: Vertex equitable star $K_{1,3}$

Since $K_{1,n}$ is vertex equitable, we must have

$$n - 1 \leq \lceil \tfrac{n}{2} \rceil.$$

Clearly, this holds for $n = 1, 2, 3$. Further we can easily see that when $n = 1$ or 2 or 3, $K_{1,n}$ admits a vertex equitable labeling. Thus, the star $K_{1,n}$ is vertex equitable if and only if $n \leq 3$. This completes the proof. $\square$

Figure 7.6 illustrates $K_{1,4}$ which is not vertex equitable.

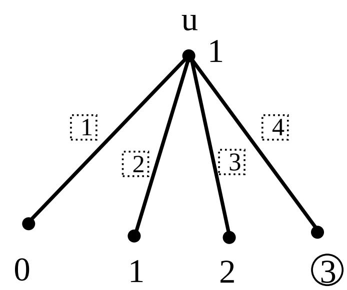

Figure 7.6: $K_{1,4}$ is not vertex equitable

But we have signed stars $K_{1,n}$, for $n \geq 4$, which are vertex equitable as follows from the following result:

Theorem 7.3.7. *A signed star $K_{1,n}$ where $n = \mathfrak{m}+\mathfrak{n}$, is vertex equitable if and only if*

(i) $|\mathfrak{m} - \mathfrak{n}| \leq 1$
or

(ii) *If $\mathfrak{m} = 1$ then $\mathfrak{n} \leq 4$ and if $\mathfrak{n} = 1$ then $\mathfrak{m} \leq 4$.*

Proof. Let (V_1, V_2) be the bipartition of $K_{1,n}$ with $V_1 = \{u\}$ and $V_2 = \{u_1, u_2, u_3, ..., u_{\mathfrak{m}}, v_1, v_2, v_3, ..., v_{\mathfrak{n}}\}$, where u_i and v_j are the vertices adjacent to u through positive and negative edges respectively. As $K_{1,n}$ has $\mathfrak{m} + \mathfrak{n}$ edges, $\mathcal{A} = \{0, 1, 2, ..., \lceil \frac{\mathfrak{m}+\mathfrak{n}}{2} \rceil\}$.

Necessity:

To get the edge labels 1 and -1, we must assign 0 and 1 labels to adjacent vertices. Therefore, $f(u) = 0$ or 1. Now, we have the following two cases:

Case I: If $f(u) = 0$ then to find edge labels of pendant edges, we assign

$$f(u_i) = i \text{ for } 1 \leq i \leq \mathfrak{m} \text{ and}$$
$$f(v_j) = j \text{ for } 1 \leq j \leq \mathfrak{n},$$

This is illustrated in **Figure 7.7**.

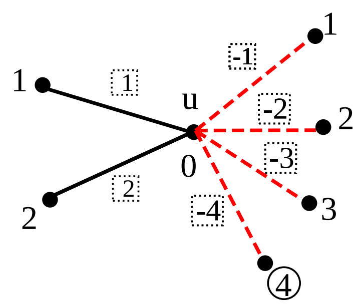

Figure 7.7: Signed star $K_{1,6}$ which is not vertex equitable

As $\mathcal{A} = \{0, 1, 2, ..., \lceil \frac{\mathfrak{m}+\mathfrak{n}}{2} \rceil\}$, we must have

$\mathfrak{m} \leq \lceil \frac{\mathfrak{m}+\mathfrak{n}}{2} \rceil$ and $\mathfrak{n} \leq \lceil \frac{\mathfrak{m}+\mathfrak{n}}{2} \rceil$

- If $\mathfrak{m}$ and $\mathfrak{n}$ both are even or odd then we must have

  $\mathfrak{m} \leq \frac{\mathfrak{m}+\mathfrak{n}}{2}$ and $\mathfrak{n} \leq \frac{\mathfrak{m}+\mathfrak{n}}{2}$

  Clearly, this holds for $\mathfrak{n} = \mathfrak{m}$

- If $\mathfrak{m}$ is even and $\mathfrak{n}$ is odd or $\mathfrak{n}$ is even and $\mathfrak{m}$ is odd then

  $\mathfrak{m} \leq \frac{\mathfrak{m}+\mathfrak{n}+1}{2}$ and $\mathfrak{n} \leq \frac{\mathfrak{m}+\mathfrak{n}+1}{2}$. That is

  $2\mathfrak{m} \leq \mathfrak{m} + \mathfrak{n} + 1$ and $2\mathfrak{n} \leq \mathfrak{m} + \mathfrak{n} + 1$

  $\mathfrak{m} \leq \mathfrak{n} + 1$ and $\mathfrak{n} \leq \mathfrak{m} + 1$

  $\mathfrak{m} - \mathfrak{n} \leq 1$ and $\mathfrak{n} - \mathfrak{m} \leq 1$

  This holds only when $|\mathfrak{m} - \mathfrak{n}| \leq 1$. Thus, **(i)** follows.

Case II: Suppose $f(u) = 1$ then we must assign $f(u_1) = f(v_1) = 0$.

This is illustrated in **Figure 7.8**.

We can not assign $f(u_2) = f(v_2) = 1$ as f will not be a vertex

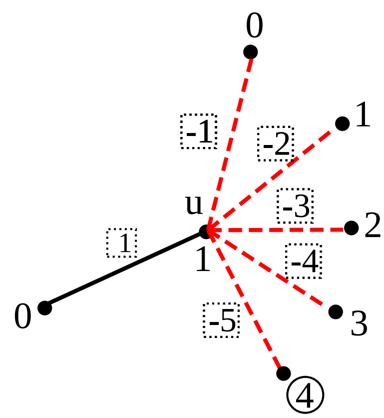

Figure 7.8: Signed star $K_{1,6}$ which is not vertex equitable

equitable labeling. Hence $\mathfrak{m} = 1$ or $\mathfrak{n} = 1$.

- If $\mathfrak{m} = 1$ then vertex labeling is

$$f(u_1) = 0,\ f(v_i) = i - 1,\ 1 \leq i \leq \mathfrak{n}.$$

Thus, $\mathfrak{n} - 1 \leq \lceil \frac{\mathfrak{n}+1}{2} \rceil$. This holds only for $\mathfrak{n} \leq 4$.

On the other hand

- If $\mathfrak{n} = 1$ then similarly we have $\mathfrak{m} - 1 \leq \lceil \frac{\mathfrak{m}+1}{2} \rceil$ and this too holds only for $\mathfrak{m} \leq 4$. Thus, **(ii)** follows.

Hence, the necessity follows.

Sufficiency:
Suppose conditions hold. We define the vertex labeling $f : V(K_{1,n}) \longrightarrow \mathcal{A}$ as

- If $|\mathfrak{m} - \mathfrak{n}| \leq 1$ then

$$f(u) = 0,\ f(u_i) = i,\ 1 \leq i \leq \mathfrak{m} \text{ and}$$
$$f(v_j) = j,\ 1 \leq j \leq \mathfrak{n}.$$

This is illustrated in **Figure 7.9**.

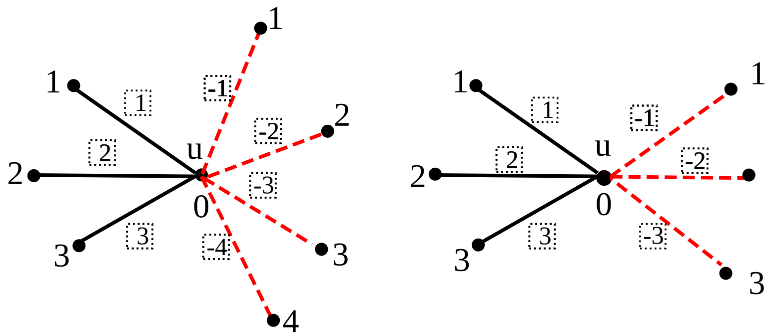

Figure 7.9: Vertex equitable signed stars

- If $\mathfrak{m} = 1$ and $\mathfrak{n} = 3$ or 4 then

$$f(u) = 1,\ f(u_1) = 0 \text{ and } f(v_j) = j - 1,\ 1 \leq j \leq \mathfrak{n}.$$

This is illustrated in **Figure 7.10**.

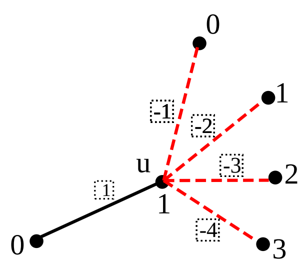

Figure 7.10: Vertex equitable signed star $K_{1,5}$

- If $\mathfrak{n} = 1$ and $\mathfrak{m} = 3$ or 4 then

$$f(u) = 1,\ f(v_1) = 0 \text{ and } f(u_i) = i - 1,\ 1 \leq i \leq \mathfrak{m}.$$

Note that this signed star is the negation of the previous signed star. It is easy to see that f is a vertex equitable labeling. This completes the proof.

□

7.4 Results on vertex equitable signed bistars

In this section, we give results on vertex equitable labeling of signed bistars. These results have been reported in [2].

Theorem 7.4.1. *Bistar* $B(n, n+k)$ *is vertex equitable if and only if* $k \leq 2$.

Proof. Let for each $i = 1, 2, ..., n$ and $j = 1, 2, ..., n+k$; u_i and v_j be the vertices adjacent to u and v respectively. Since $B(n, n+k)$ has $2n+k+1$ edges, $\mathcal{A} = \{0, 1, 2, ..., n + \lceil \frac{k+1}{2} \rceil\}$. To find edge labels, we must assign

$$f(u) = 0,\ f(u_i) = i,\ f(v) = n+1 \text{ and } f(v_j) = j.$$

This is illustrated in **Figure 7.11**.

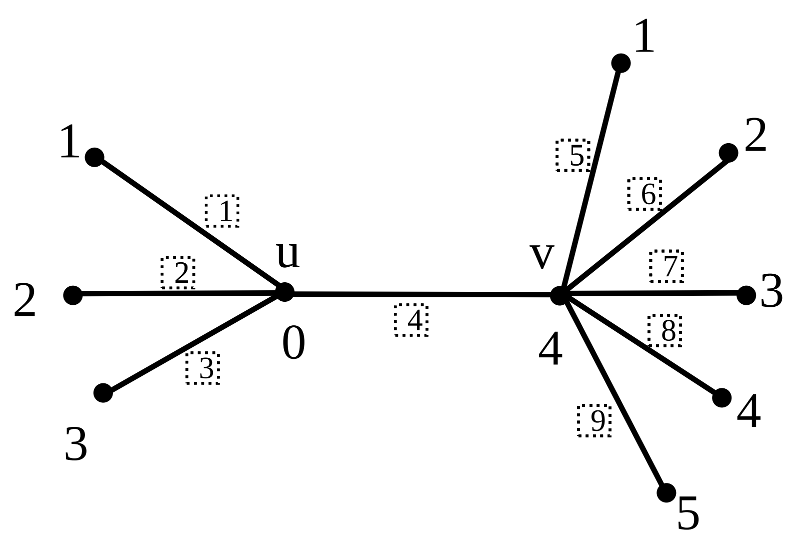

Figure 7.11: Vertex equitable bistar $B(3, 5)$

Clearly, f is a vertex equitable labeling if and only if

$$\mathfrak{n} + k \leq \mathfrak{n} + \lceil \tfrac{k+1}{2} \rceil \text{ that is}$$

$k \leq \lceil \frac{k+1}{2} \rceil.$

This holds only if $k = 1$ or 2.

For $k \leq 2$, we have edge labeling

$$f^*(uu_i) = i, \ 1 \leq i \leq n$$
$$f^*(uv) = n + 1$$
$$f^*(vv_j) = n + 1 + j, \ 1 \leq j \leq n + k.$$

Thus, bistar $B(n, n + k)$ is vertex equitable if and only if $k \leq 2$. □

Theorem 7.4.2. *Signed bistar $B^+(\mathfrak{m}, \mathfrak{n})$ is vertex equitable if and only if*

(a) $\mathfrak{m} = \mathfrak{n}$ *or* $\mathfrak{n} - 1$ *or* $\mathfrak{n} - 3$ *or* $\mathfrak{n} - 4$
or

(b) $\mathfrak{n} = 1$ *and* $\mathfrak{m} \leq 3$

Proof. Let $u_1, u_2, u_3, ..., u_\mathfrak{m}$ and $v_1, v_2, v_3, ..., v_\mathfrak{n}$ be the vertices adjacent to u and v respectively. Since $B^+(\mathfrak{m}, \mathfrak{n})$ has $\mathfrak{m} + \mathfrak{n} + 1$ edges, $\mathcal{A} = \{0, 1, 2, ..., \lceil \frac{\mathfrak{m}+\mathfrak{n}+1}{2} \rceil\}$.

Necessity:

To get the edge labels 1 and -1, we must assign 0 and 1 labels to adjacent vertices. Therefore, $f(u) = 0$ or 1 and $f(v) = 0$ or 1. Thus, 0,1; 1,0 and 1,1 are three possible labels for vertices u and v respectively. Now, we have the following three cases:

Case I: If $f(u) = 0$ and $f(v) = 1$ then to find edge labels of pendant edges, we must assign

$$f(u_i) = i + 1 \text{ for } 1 \leq i \leq \mathfrak{m} \text{ and}$$

$$f(v_j) = j - 1 \text{ for } 1 \leq j \leq \mathfrak{n}.$$

This is illustrated in **Figure 7.12**.

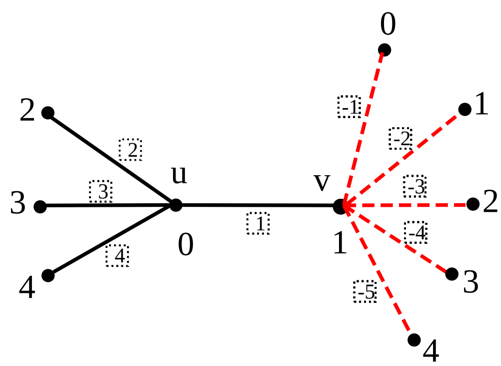

Figure 7.12: Bistar $B^+(3, 5)$ is not vertex equitable

Since $\mathcal{A} = \{0, 1, 2, ..., \lceil\frac{\mathfrak{m}+\mathfrak{n}+1}{2}\rceil\}$, we must have $\mathfrak{m} + 1 = \lceil\frac{\mathfrak{m}+\mathfrak{n}+1}{2}\rceil$ and $\mathfrak{n} - 1 = \lceil\frac{\mathfrak{m}+\mathfrak{n}+1}{2}\rceil$.

- If $\mathfrak{m}$ and $\mathfrak{n}$ both are even or odd then

  $\mathfrak{m} + 1 = \frac{\mathfrak{m}+\mathfrak{n}+2}{2}$ and $\mathfrak{n} - 1 = \frac{\mathfrak{m}+\mathfrak{n}+2}{2}$

  $2\mathfrak{m} + 2 = \mathfrak{m} + \mathfrak{n} + 2$ and $2\mathfrak{n} - 2 = \mathfrak{m} + \mathfrak{n} + 2$. Thus

  $\mathfrak{m} = \mathfrak{n}$ and $\mathfrak{m} = \mathfrak{n} - 4$

- If $\mathfrak{m}$ is even and $\mathfrak{n}$ is odd or $\mathfrak{n}$ is even and $\mathfrak{m}$ is odd then

  $\mathfrak{m} + 1 = \frac{\mathfrak{m}+\mathfrak{n}+1}{2}$ and $\mathfrak{n} - 1 = \frac{\mathfrak{m}+\mathfrak{n}+1}{2}$, that is

  $2\mathfrak{m} + 2 = \mathfrak{m} + \mathfrak{n} + 1$ and $2\mathfrak{n} - 2 = \mathfrak{m} + \mathfrak{n} + 1$. Thus

  $\mathfrak{m} = \mathfrak{n} - 1$ and $\mathfrak{m} = \mathfrak{n} - 3$

Therefore, signed bistar $B^+(\mathfrak{m}, \mathfrak{n})$ is vertex equitable if $\mathfrak{m} = \mathfrak{n}$ or $\mathfrak{n} - 1$ or $\mathfrak{n} - 3$ or $\mathfrak{n} - 4$, i.e., **(a)** follows.

Case II: Suppose $f(u) = 1$ and $f(v) = 0$ then to find edge labels 2 and -1, we must assign $f(u_1) = f(v_1) = 1$. Then we have $v_f(1) = 3$ and $v_f(0) = 1$, i.e., condition $|v_f(a) - v_f(b)| \leq 1$, $\forall a, b \in \mathcal{A}$, fails. This is illustrated in **Figure 7.13**.

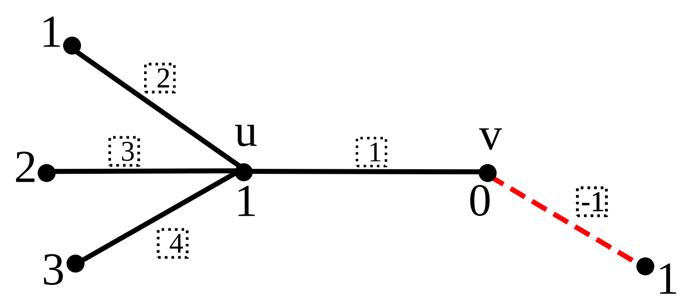

Figure 7.13: Signed bistar $B^+(3, 1)$ is not vertex equitable

Hence $f(u) = 1$ and $f(v) = 0$ is not possible.

Case III: Suppose $f(u) = f(v) = 1$ then we must assign $f(u_1) = f(v_1) = 0$ to get edge labels 1 and -1. This is illustrated in **Figure 7.14**.

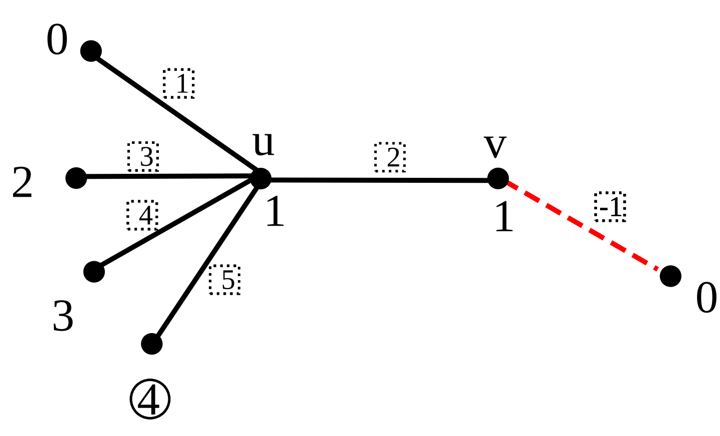

Figure 7.14: Signed bistar $B^+(4, 1)$ which is not vertex equitable

We can not assign $f(v_2) = 1$ as f will not be a vertex equitable labeling.

Hence $\mathfrak{n} = 1$. For $\mathfrak{n} = 1$, $\mathfrak{m} = \lceil \frac{\mathfrak{m}+1+1}{2} \rceil$

$\mathfrak{m} = \lceil \frac{\mathfrak{m}}{2} \rceil + 1$. Further

$\mathfrak{m} = \frac{\mathfrak{m}}{2} + 1$, if $\mathfrak{m}$ is even and we get

$\mathfrak{m} = 2$. Further

$\mathfrak{m} = \frac{\mathfrak{m}+1}{2} + 1$, if $\mathfrak{m}$ is odd and we get

$\mathfrak{m} = 3$.

Note that for $\mathfrak{m} = 1$ and $\mathfrak{n} = 1$, we get a vertex equitable signed path P_4 having a pendant negative edge. Therefore, **(b)** follows.

Thus, the necessity follows.

<u>Sufficiency</u>:

Suppose conditions hold. We define the vertex labeling $f : V(B^+(\mathfrak{m}, \mathfrak{n})) \longrightarrow \mathcal{A}$ as

- If $\mathfrak{m} = \mathfrak{n}$ or $\mathfrak{n} - 1$ or $\mathfrak{n} - 3$ or $\mathfrak{n} - 4$ then
  $f(u) = 0$, $f(v) = 1$, $f(u_i) = i + 1$ and $f(v_j) = j - 1$. This is illustrated in **Figure 7.15**.

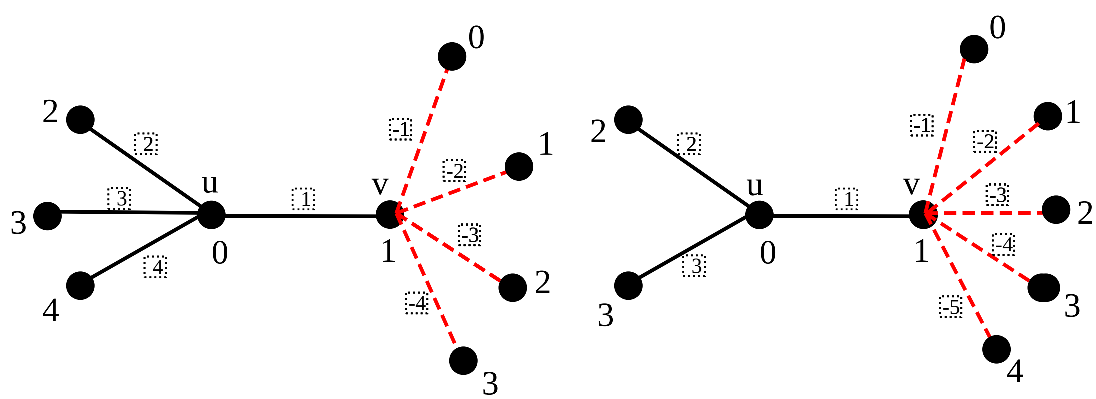

Figure 7.15: Vertex equitable signed bistars $B^+(3, 4)$ and $B^+(2, 5)$

- If $\mathfrak{n} = 1$ and $\mathfrak{m} = 2$ or 3 then

$$f(u) = f(v) = 1,\ f(u_1) = f(v_1) = 0 \text{ and}$$
$$f(u_i) = i \text{ for } 2 \leq i \leq \mathfrak{m}.$$

This is illustrated in **Figure 7.16**.

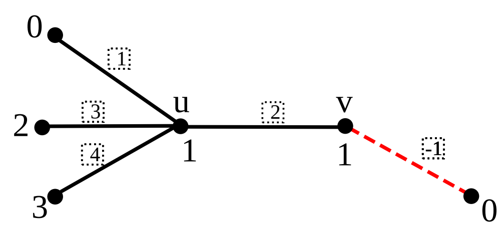

Figure 7.16: Vertex equitable bistar $B^{+}(3, 1)$

It is easy to observe that f is a vertex equitable labeling. This completes the proof. □

Theorem 7.4.3. *Signed bistar $B^{-}(\mathfrak{m}, \mathfrak{n})$ is vertex equitable if and only if*

(a) $\mathfrak{n} = \mathfrak{m}$ *or* $\mathfrak{m} - 1$ *or* $\mathfrak{m} - 3$ *or* $\mathfrak{m} - 4$

or

(b) $\mathfrak{m} = 1$ *and* $\mathfrak{n} \leq 3$

Proof. Since $B^{-}(\mathfrak{m}, \mathfrak{n})$ is negation of $B^{+}(\mathfrak{n}, \mathfrak{m})$, result follows from Theorem 7.3.3 and 7.4.2. □

Now, we consider a signed bistar $B(m, n)$ which is not $B^{+}(\mathfrak{m}, \mathfrak{n})$ and $B^{-}(\mathfrak{m}, \mathfrak{n})$.

Theorem 7.4.4. *If signed bistar $B(m, n)$ which is not $B^{+}(\mathfrak{m}, \mathfrak{n})$ and $B^{-}(\mathfrak{m}, \mathfrak{n})$, is vertex equitable then $| \mathfrak{m} - \mathfrak{n} | \leq 3$.*

Proof. Suppose a signed bistar which is not $B^+(\mathfrak{m}, \mathfrak{n})$ and $B^-(\mathfrak{m}, \mathfrak{n})$, is vertex equitable. To get the edge labels 1 and -1, we must assign 0 and 1 labels to adjacent vertices. Therefore, $f(u) = 0$ or 1 and $f(v) = 0$ or 1. Now, $\mathcal{A} = \{0, 1, 2, ..., \lceil\frac{\mathfrak{m}+\mathfrak{n}+1}{2}\rceil\}$, hence the number of positive edges can not exceed $(\lceil\frac{\mathfrak{m}+\mathfrak{n}}{2}\rceil + 1)$, i.e., $\mathfrak{m} \leq \lceil\frac{\mathfrak{m}+\mathfrak{n}}{2}\rceil + 1$.

$\mathfrak{m} \leq \lceil\frac{\mathfrak{m}+\mathfrak{n}}{2}\rceil + 1$, i.e., $\mathfrak{m} \leq \frac{\mathfrak{m}+\mathfrak{n}}{2} + 1$, if $\mathfrak{m}$ and $\mathfrak{n}$ both are even or odd.

Further,

$$2\mathfrak{m} \leq \mathfrak{m} + \mathfrak{n} + 2 \text{ implies that}$$

$$\mathfrak{m} \leq \mathfrak{n} + 2.$$

$\mathfrak{m} \leq \lceil\frac{\mathfrak{m}+\mathfrak{n}}{2}\rceil + 1$, i.e., $\mathfrak{m} \leq \frac{\mathfrak{m}+\mathfrak{n}+1}{2} + 1$, if $\mathfrak{m}$ or $\mathfrak{n}$ is even. We get

$$2\mathfrak{m} \leq \mathfrak{m} + \mathfrak{n} + 3, \text{ i.e.,}$$

$$\mathfrak{m} \leq \mathfrak{n} + 3.$$

Thus,

$$\mathfrak{m} - \mathfrak{n} \leq 3. \tag{7.1}$$

Similarly, the number of negative edges can not exceed $(\lceil\frac{\mathfrak{m}+\mathfrak{n}}{2}\rceil + 1)$, i.e., $\mathfrak{n} \leq \lceil\frac{\mathfrak{m}+\mathfrak{n}}{2}\rceil + 1$. Therefore, as discussed above,

$$\mathfrak{n} \leq \lceil\frac{\mathfrak{m}+\mathfrak{n}}{2}\rceil + 1 \text{ gives}$$

$$\mathfrak{n} - \mathfrak{m} \leq 3. \tag{7.2}$$

Therefore, from equations (7.1) and (7.2),

$$| \mathfrak{m} - \mathfrak{n} | \leq 3.$$

This completes the proof. □

There is not any general vertex equitable labeling for signed bistars $B(m,n)$ for which $| \mathfrak{m} - \mathfrak{n} | \leq 3$, but in **Figure 7.17**, we give some vertex equitable signed bistars for which $| \mathfrak{m} - \mathfrak{n} | \leq 3$.

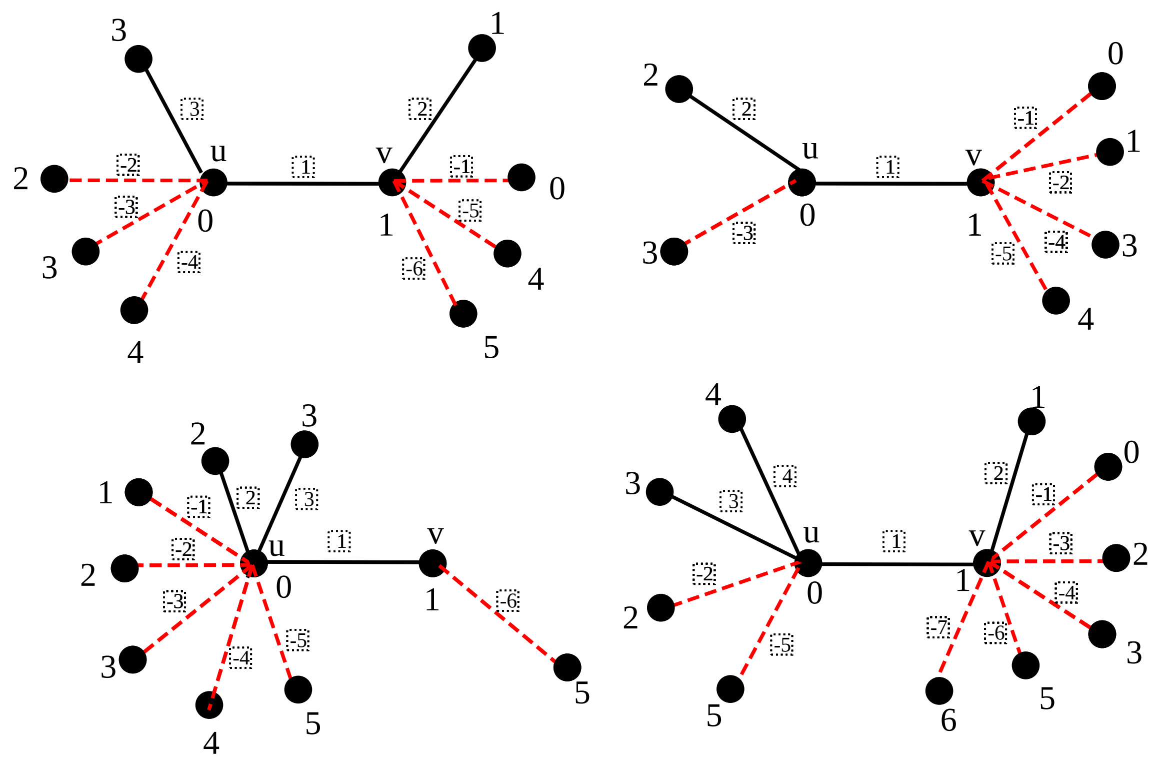

Figure 7.17: Some vertex equitable signed bistars

Theorem 7.4.5. *Signed complete bipartite graphs $K_{2,n}$ are vertex equitable if and only if it is any one of the following:*

1. *homogeneous*

or

2. *heterogeneous, in which for each vertex v_i of degree 2, $d^+(v_i) = d^-(v_i) = 1$.*

Proof. Let (V_1, V_2) be the bipartition of signed $K_{2,n}$ with $V_1 = \{u, v\}$ and $V_2 = \{v_1, v_2, v_3, ..., v_n\}$.

Necessity:

We prove the necessity by contrapositive, i.e., we prove that heterogeneous signed complete bipartite graphs $K_{2,n}$, in which for at least one vertex v_i, $d^+(v_i) \neq d^-(v_i)$, is not vertex equitable.

To get the edge labels 1 and -1, we must assign 0 and 1 labels to adjacent vertices. Therefore, $f(u) = 0$ or 1 and $f(v) = 0$ or 1. Now, in all the cases some edge labels are repeated. Hence, signed $K_{2,n}$ are not vertex equitable. This is illustrated in **Figure 7.18**.

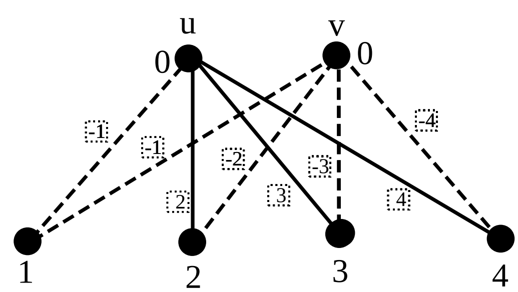

Figure 7.18: Signed $K_{2,4}$ is not vertex equitable

Thus, the necessity follows.

Sufficiency:

Suppose conditions hold. Now, we have the following two cases:

Case I: If $K_{2,n}$ are homogeneous then by Theorem 7.3.2 and Theorem 7.3.3, these are vertex equitable signed graphs.

Case II: If $K_{2,n}$ are heterogeneous, in which for each vertex v_i of degree

2, $d^+(v_i) = d^-(v_i) = 1$, we define $f : V(K_{2,n}) \longrightarrow \{0, 1, 2, ..., n\}$ as

$$f(u) = f(v) = 0 \text{ and } f(v_i) = i \text{ for } 1 \leq i \leq n.$$

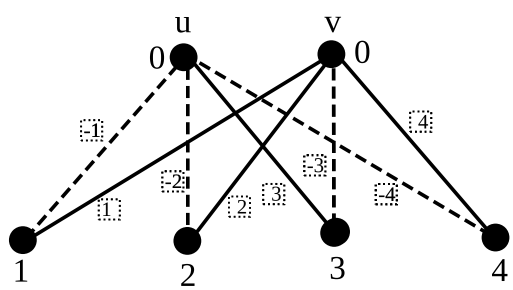

Figure 7.19: Vertex equitable signed $K_{2,4}$

It can be easily seen that f is a vertex equitable labeling, as shown in **Figure 7.19**. This completes the proof. □

7.5 Conclusion and Scope

In this chapter, we have initiated a vertex equitable labeling of signed graphs and established some results on vertex equitable behavior of signed paths, signed stars, signed bistars and signed complete bipartite graph $K_{2,n}$. All work reported on vertex equitable graphs by Seenivasan and Lourdusamy, and P. Jeyanthi et. al discussed in this chapter is still open in the realm of signed graphs. All their results hold for homogeneous signed graphs. Open problems are to determine vertex equitable heterogeneous signed graphs.

*****

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Chapter 8

SCOPE FOR FURTHER RESEARCH

In this chapter, we have put together some problems that looked stray in the main course of our research work reported so far, for, on second thought we felt they might be of independent interest to investigate and perhaps may get linked up eventually to some of the directions of research reported in this thesis. We have preferred presenting them with some illustrative examples, just to attract attention of an inquisitive researcher.

8.1 Litact graph

Recall that the *litact graph* [15] of a graph $G = (V, E)$, denoted here by $L_{ct}(G)$, is the graph having vertex set $E(G) \cup C(G)$ in which its two vertices are adjacent if the corresponding members of G are adjacent or incident. A graph G and its $L_{ct}(G)$ are shown in **Figure 8.1**.

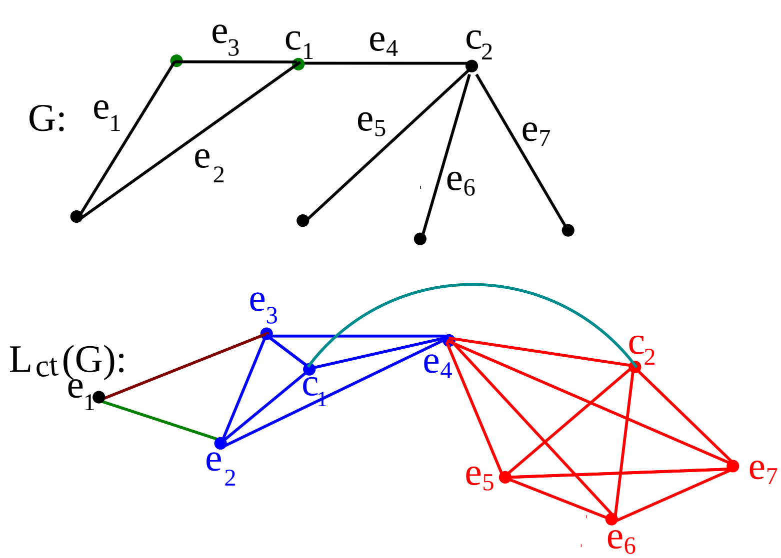

Figure 8.1: A graph G and its litact graph $L_{ct}(G)$

In Chapter 2, we have established a characterization of lict graphs. The problem of characterizing litact graph is still an open problem.

Problem 8.1.1. *Characterize litact graphs.*

8.2 Litact signed graphs

In Chapter 3, we have established characterizations of lict signed graphs $L_c(S)$, $L_{\times_c}(S)$, $L_{\bullet_c}(S)$ and also for line signed graphs $L_\times(S)$ and $L_\bullet(S)$. Litact signed graphs yet to be defined. Extension of litact graph in the realm of signed graph had not been taken up. Anyone can define and characterize various types of litact signed graphs.

Problem 8.2.1. *Define and characterize litact signed graphs.*

8.3 Directed graph and signed directed graph

A *directed graph* (in short, *digraph*) is $D = (V, \mathcal{A})$, where V is the set whose elements are called vertices, nodes or points and $\mathcal{A}$ is the set of ordered pairs of vertices called arrows, directed edges (or lines or arcs).

An ordered pair $S = (D, \sigma)$ is called a *signed digraph* with underlying digraph $D = (V, \mathcal{A})$, if S is obtained from D by designating each of its arc as positive or negative.

8.4 Line directed graph and lict directed graph

Aigner [5] defined the 'line digraph' of a given digraph as follows. The *line-digraph* $L(D)$ of a given digraph $D = (V, \mathcal{A})$ has $\mathcal{A}$ for its vertex set and (e, f) is an arc in $L(D)$ whenever the arcs e and f have a vertex in common which is the *head* of e and tail of f, as shown in **Figure 8.2**.

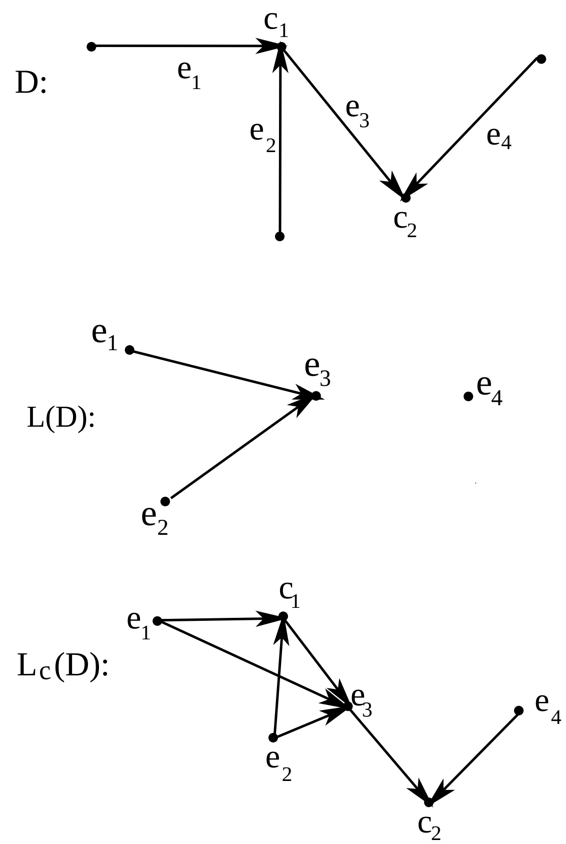

Figure 8.2: A digraph D, its line digraph $L(D)$ and lict digraph $L_c(D)$

A given digraph H is called a *line digraph* if there exists a digraph D such that $L(D)$ is isomorphic to H, written as $L(D) \cong H$. Harary and Norman [6] gave a characterization of line digraphs.

Nagesh and Chandrasekhar introduced the concept of lict digraph in [16] as follows:

The *lict digraph* denote here by $L_c(D)$ of a given digraph $D = (V, \mathcal{A})$ has $\mathcal{A}(D) \cup C(D)$ as its vertex set and (e, f) is an arc in $L_c(D)$ whenever the arcs e and f have a vertex in common which is the *head* of e and tail of f or $e \in C(D)$ and f is the out going arc from e and (f, e) is an arc in $L_c(D)$ if $e \in C(D)$ and f is the in-coming arc to e.

A given digraph H is called a lict digraph if there exists a digraph D such that $L_c(D)$ is isomorphic to H, written $L_c(D) \cong H$. Line digraph and lict digraph of a digraph D are shown in **Figure 8.2**.

Observe that the in-degree and out-degree of a cut-vertex of D remains the same in $L_c(D)$.

8.5 Signed line and lict digraphs

The definition of line digraph was extended to the realm of signed line digraphs by Acharya and Sinha [4] as given below and they also characterized signed line digraphs.

Given any signed digraph $S = (D, \sigma)$, *signed line digraph* $L(S)$ is a signed digraph with $L(D)$ as its underlying digraph in which an arc (e, f) is defined negative if and only if e and f are negative arcs in S.

One can think about defining signed line digraphs as well as lict signed digraphs as par line signed graphs and lict signed graphs as treated in this thesis. Such a study is still open.

8.6 $\mathcal{C}$-consistency and $\mathcal{S}$-consistency

We have following problems concerning $\mathcal{C}$-consistency:

Problem 8.6.1. *Characterize signed graph S such that $L_\times(S)$ is $\mathcal{C}$-consistent.*

Problem 8.6.2. *Characterize signed graph S such that its lict signed graphs $L_c(S)$, $L_{\times_c}(S)$ and $L_{\bullet_c}(S)$ are $\mathcal{C}$-consistent.*

Acharya et. al in [3] have defined $\mathcal{S}$-consistency of line signed graph $L(S)$ as follows:
If to each vertex e of $L(S)$, which is an edge of $S = (S^u, \sigma)$, if one assigns the sign $\sigma(e)$ and the resulting marked signed graph $L(S)$ is consistent, then $L(S)$ is said to be $\mathcal{S}$-consistent.

Problem 8.6.3. *Characterize signed graph S such that $L_\times(S)$ and $L_\bullet(S)$ are $\mathcal{S}$-consistent.*

We have defined lict signed graphs. In order to study $\mathcal{S}$-consistency for lict signed graphs, we can give appropriate sign to the vertices of $L_c(S)$, $L_{\times_c}(S)$ and $L_{\bullet_c}(S)$. This study has not been taken so far. Hence, we have the following problem:

Problem 8.6.4. *Characterize signed graph S such that its lict signed graphs $L_c(S)$, $L_{\times_c}(S)$ and $L_{\bullet_c}(S)$ are $\mathcal{S}$-consistent.*

8.7 Cycle-compatibility, $\mathcal{C}$-cycle-compatibility and $\mathcal{S}$-cycle-compatibility

A signed graph S is called *cycle-compatible* [2] if there exists a marking μ such that for any cycle Z in S_μ,

$$\prod_{e\in E(Z)} \sigma(e) = \prod_{u\in V(Z)} \mu(u).$$

In Chapter 5, we have defined $\mathcal{C}$-cycle-compatibility. So, we have following problems:

Problem 8.7.1. *Characterize signed graph S such that $L(S)$ and $L_\times(S)$ are $\mathcal{C}$-cycle-compatible.*

Problem 8.7.2. *Characterize signed graph S such that its lict signed graphs $L_c(S)$, $L_{\times_c}(S)$ and $L_{\bullet_c}(S)$ are $\mathcal{C}$-cycle-compatible.*

If to each vertex e of $L(S)$, which is an edge of $S = (S^u, \sigma)$, if one assigns the sign $\sigma(e)$ and the resulting marked signed graph $L(S)$ is cycle-compatible, then $L(S)$ is said to be $\mathcal{S}$-cycle-compatible [8].

Acharya et. al in [2] have characterized $\mathcal{S}$-cycle-compatible signed line graphs $L(S)$.

Problem 8.7.3. *Characterize signed graph S such that $L_\times(S)$ and $L_\bullet(S)$ are $\mathcal{S}$-cycle-compatible.*

Problem 8.7.4. *Characterize signed graph S such that its lict signed graphs $L_c(S)$, $L_{\times_c}(S)$ and $L_{\bullet_c}(S)$ are $\mathcal{S}$-cycle-compatible.*

8.8 Sign-compatibility and $\mathcal{C}$-sign-compatibility

While looking into the structure of the line signed graphs, Acharya and Sinha [1] observed that there are forbidden subsigned graphs for a signed graph to be a line signed graph. In providing a characterization of line sigraphs, they suggested that the vertices of a line signed graph can be marked by positive and negative signs so that both the ends of every negative edge receive negative signs and no positive edge receives negative signs at both of its ends.

A signed graph S is called *sign-compatible* [8] if there exists a marking μ of its vertices such that the end vertices of every negative edge receive '$-$' signs in μ and no positive edge in S has both of its ends assigned '$-$' signs in μ. A sign-compatible signed graph is shown in **Figure 8.3**.

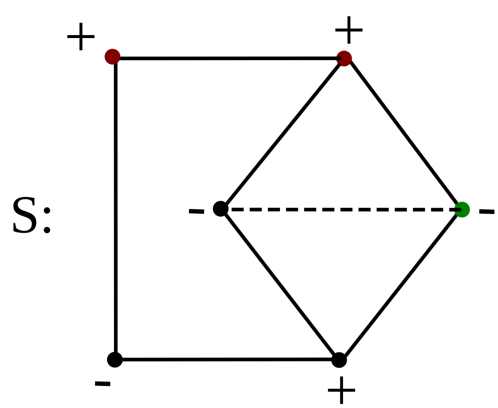

Figure 8.3: A sign-compatible signed graph S

A canonically marked signed graph S, is said to be *canonically sign-compatible* (or *$\mathcal{C}$-sign-compatible* in short), if the end vertices of every negative edge receive '$-$' signs and no positive edge has both of its ends assigned '$-$' under μ_σ. An example of $\mathcal{C}$-sign-compatible signed graph is exhibited in **Figure 8.4**.

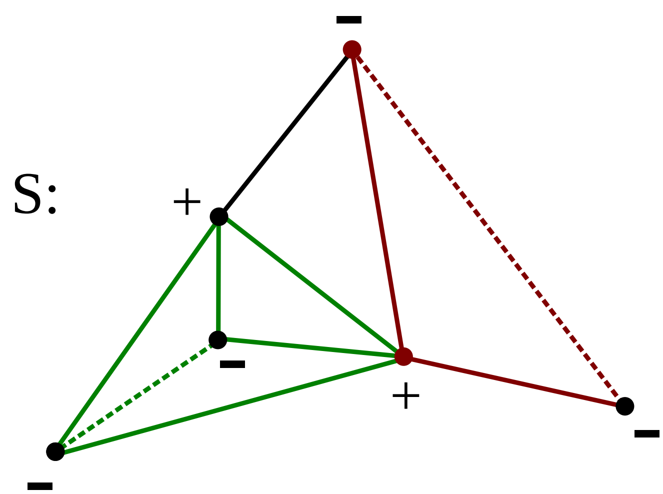

Figure 8.4: A $\mathcal{C}$-sign-compatible signed graph S

In [8], characterization of sign-compatible signed graphs has been given and it has been proved that every line signed graph is sign-compatible. In [9], Sinha and Dhama have characterized signed graphs whose derived signed graphs such as common-edge signed graphs, 2-path signed graphs, $\times$-line and dot line signed graphs, semi-total line signed graphs, semi-total point signed graphs and total signed graphs are sign-compatible. In [10], Sinha and Dhama have characterized signed graphs whose derived signed graphs such as line, $\times$-line and dot line signed graphs, splitting signed graphs, semi-total line signed graphs, semi-total point signed graphs and total signed graphs are $\mathcal{C}$-sign-compatible.

We have following problems:

Problem 8.8.1. *Characterize signed graph S such that its lict signed graphs $L_c(S)$, $L_{\times_c}(S)$ and $L_{\bullet_c}(S)$ are sign-compatible and $\mathcal{C}$-sign-compatible.*

Problem 8.8.2. *Characterize signed graph S such that its $\mathfrak{S}$-splitting signed graph $\mathfrak{S}(S)$ is sign-compatible and $\mathcal{C}$-sign-compatible.*

Note that in sign-compatibility, we do not bother about number of negative vertices in cycles, we can define terms like sign-compatible consistency and $\mathcal{C}$-sign-compatible consistency, in which we check sign compatibility as well as consistency. So, all problems of sign-compatibility and $\mathcal{C}$-sign-compatibility are open for sign-compatible consistency and $\mathcal{C}$-sign-compatible consistency.

8.9 Graph equations

Graph equations are equations in which the unknowns are graphs. Many problems and results in graph theory can be formulated in terms of graph equations.
Following problems are open:

Problem 8.9.1. *Characterize a signed graph S such that $L(S) \sim L_\times(S)$.*

Problem 8.9.2. *Characterize a signed graph S such that $L(S) \sim L_\bullet(S)$.*

Problem 8.9.3. *Characterize a signed graph S such that $L_\bullet(S) \sim L_\times(S)$.*

Problem 8.9.4. *Characterize a signed graph S such that $L(S) \sim L(\eta(S))$.*

Problem 8.9.5. *Characterize a signed graph S such that $L_\times(S) \sim L(\eta(S))$.*

Problem 8.9.6. *Characterize a signed graph S such that $L_\bullet(S) \sim L(\eta(S))$.*

These problems also can be extended for lict signed graphs.

In Chapter 4, we have established many results on $\bullet$-lict signed graphs, lict signed graphs and also for $\bullet$-line signed graphs and line signed graphs. Mathad and Narayanker in [7], characterize signed graphs S and S' for which $L_\times(S) \sim L_{\times c}(S')$, $J(S) \sim L_{\times c}(S')$ and

$T_1(S) \sim L_{\times c}(S')$, where $J(S)$ and $T_1(S)$ are jump signed graph and semitotal line signed graph of S respectively. This study related to dot-lict signed graphs, lict signed graphs and litact signed graphs is yet open.

In Chapter 7, we have initiated a vertex equitable labeling of signed graphs and established some results on vertex equitable behavior of signed paths, signed stars, signed bistars and signed complete bipartite graphs $K_{2,n}$. All work reported on vertex equitable graphs by Seenivasan and Lourdusamy, and P. Jeyanthi et. al discussed in this chapter is still open in the realm of signed graphs.

Algorithmic approach to detect balancing, $\mathcal{C}$-sign-compatibility of a given graph, $\mathcal{C}$-consistency and $\mathcal{S}$-consistency for $\times$-line signed graph and dot line signed graphs and also for lict signed graphs are open problems. For such a study we refer to [11–14].

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LIST OF RESEARCH PAPERS OF THE SCHOLAR

Published papers:

(1) Mukti Acharya, Rashmi Jain and Sangita Kansal; **Characterization of line-cut graphs**, *Graph Theory Notes of New York*, LXVII (2014), 43-46.

(2) Mukti Acharya, Rashmi Jain and Sangita Kansal; **Some results on the splitting signed graphs $\mathfrak{S}(S)$**, *Journal of Combinatorics, Information and System Sciences*, 39(1-2) (2014), 23-32.

(3) Rashmi Jain, Sangita Kansal and Mukti Acharya; **$\mathcal{C}$-cycle Compatible Splitting Signed Graphs $\mathfrak{S}(S)$ and $\Gamma(S)$**, *European Journal of Pure and Applied Mathematics*, 8(4) (2015), 469-477.

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Accepted papers:

(1) Rashmi Jain, Mukti Acharya and Sangita Kansal; **$\mathcal{C}$-consistent and $\mathcal{C}$-cycle compatible •-line signed graphs**

(2) Mukti Acharya, Rashmi Jain and Sangita Kansal; **Vertex equitable labeling in signed graphs**

Communicatted papers:

(1) Rashmi Jain, Mukti Acharya and Sangita Kansal; **Characterizations of line-cut signed graphs**

(2) Rashmi Jain, Mukti Acharya and Sangita Kansal; **Characterizations of product and dot line-cut signed graphs**

(3) Mukti Acharya, Rashmi Jain and Sangita Kansal; **Vertex equitable labeling of signed bistars**

Papers presented in Conferences:

(1) Paper entitled **Results on lict signed graphs-I** at 8^{th} International Conference on Discrete Mathematics (ICDM-2013) held at Karnatak University, Dharwad, during June 10-14, 2013.

(2) Paper entitled **Some results on the splitting signed graphs** in a National Conference on Discrete Mathematics (ADMA-2014) held at Reva University, Bangalore, during June 10-13, 2014 and was awarded the best paper presentation award.

(3) Paper entitled **$\mathcal{C}$-consistent and $\mathcal{C}$-cycle compatible $\bullet$-line signed graphs** at International Conference on Current Trends in Graph Theory and Computation (CTGTC-2016) held at South Asian University, Delhi, during Sep 17-19, 2016.

(4) Paper entitled **Vertex equitable labeling of signed graphs** International Conference on Current Trends in Graph Theory and Computation (CTGTC-2016) held at South Asian University, Delhi, during Sep 17-19, 2016.

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